Astrophysics Oral Exam

Wenxuan Jia

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Tips:

1. Study from Colloquia/talks/meetings. The first 15 minutes are usually on background that you should know everything about it. Other popular topics
2. 180 questions to know. Study qualification exam guide from other universities.
3. Alan Levine usually hosts some meetings. Join them.
4. Ask fellow students who are taking the exams and ask them 180 questions.
5. Need to study more cosmology stuff. 8.902 only covers half of it.
6. Past psets. Midterms of 8.901, 8.902

🚀 🌌 🌞 Bullets 🪐 🌍 🌑

1 *Radiation*

- 🌌 Blackbody Radiation
  - 🌞 Planck's law
    - 🪐 Energy density per frequency interval:

$$u_\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/k_BT}-1}$$

    - 🪐 Intensity per frequency interval:

$$I_\nu = B_\nu = c\frac{du_\nu}{d\Omega} = \frac{c}{4\pi}u_\nu$$

      you can add a factor of 2 for two polarizations of light.
    - 🪐 Energy flux from a hole of a blackbody box:

$$f_\nu = \frac{c}{4}u_\nu$$

      The factor 1/4 is calculated from integrating the phase space density with hemisphere at the hole that includes all particles leaked out from the hole. Integrating all frequency:

$$f = \int_0^\infty d\nu f_\nu = \frac{2\pi h}{c^2}\int_0^\infty d\nu\frac{\nu^3}{e^{h\nu/k_BT}-1} = \underbrace{\frac{2\pi^5}{15}\frac{k_B^4}{c^2h^3}}_{\sigma}T^4$$

  - 🌞 Wien's law:

$$\lambda_{max}T = 0.29 \text{ cm K} \qquad \text{and} \qquad h\nu_{max} = 2.8k_BT$$

    CMB spectrum has peak wavelength 1.063 mm (microwave) → 2.7 K temperature.
- 🌌 Radiative transfer

  The radiative transfer eqution is

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

  where j_ν is the emission coefficient and α_ν is the extinction coefficient. Define optical depth $d\tau_\nu = \alpha_\nu ds$ and source function $S_\nu = j_\nu/\alpha_\nu$, we can rewrite RTE

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

  Multiply each side with $e^{-\tau_\nu}$ to solve it

$$I_\nu(\tau - \nu) = I_\nu(0)\,e^{-\tau_\nu} + \int_0^{\tau_\nu} d\tau_\nu'\,S_\nu(\tau_\nu')\,e^{-\tau_\nu-\tau_\nu'}$$

  - 🌞 Eddington approximation

    Define $\mu = \cos\theta$, the intensity can be approximated as

$$I_\nu(\mu, z) = a(z) + \mu b(z)$$

- Spectral Types
  - Hydrogen lines
    - Lyman series: transitions to the $n = 1$ ground level.
      - Lyα $(2 \to 1)$: 122 nm
      - Lyβ $(3 \to 1)$: 103 nm ... up until the Lyman continuum
      - LyC $(\infty \to 1)$: 91 nm

      All of them are in UV region.
    - Balmer series: transitions to the $n = 2$ level.
      - Hα $(3 \to 2)$: 656 nm
      - Hβ $(4 \to 2)$: 486 nm ... up until the Balmer continuum
      - BaC $(\infty \to 2)$: 365 nm

      Balmer series are of interest because Earth atmosphere transmits these frequencies.
  - Spectral class: Oh Boy A Fine Girl Kissed Me Let's Talk (OBAFGKMLT, decreasing temperature)
- Einstein Coefficients

  For a bounded atomic gas with two energy levels $E_1 < E_2$ and density n_1 at E_1 and n_2 at E_2, there are three Einstein coefficients: the coefficient of

  - spontaneous emission A
  - stimulated emission B
  - absorption B'

  These can be calculated from perturbative quantum field theory, but we can still infer something. The change of n_2 due to emission and absorption is

$$dn_2 = -An_2 - BI(\omega)n_2 + B'I(\omega)n_1$$

  where $\omega = (E_2 - E_1)/\hbar$ is the photon with intensity $I(\omega)$ and right frequency to trigger stimulated emission and absorption. The spontaneous emission doesn't depend on the photons in the environment. Conservation of numbers give

$$dn_1 + dn_2 = 0$$

  Assume gas at equilibrium ($dn_1 = dn_2 = 0$) and follow Boltzmann distribution ($n \propto e^{-\beta E}$), we have

$$I(\omega) = \frac{An_2}{B'n_1 - Bn_2} = \frac{A}{B'e^{\beta\hbar\omega} - B}$$

  we know the Planck's law (two polarizations)

$$I(\omega) = 2 \times \frac{c}{4\pi}u(\omega) = \frac{\hbar\omega^3}{\pi^2c^2}\frac{1}{e^{\beta\hbar\omega} - 1} = \frac{A}{B'}\frac{1}{e^{\beta\hbar\omega} - B/B'}$$

  So we have

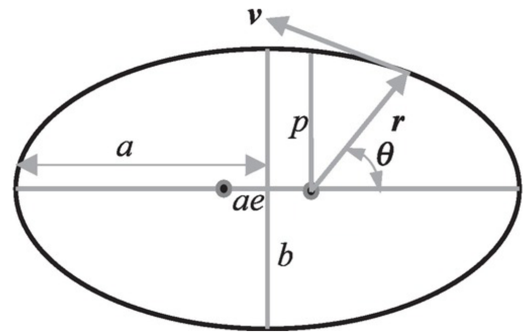
$$B = B' \quad \text{and} \quad \frac{A}{B} = \frac{\hbar\omega^3}{\pi^2c^2}$$

  The stimulated emission coefficient equals to the absorption.

2 Binary Systems

Kepler Problem

Conic section



Ellipse ($e < 1$)

Polar parametrization relative to focus

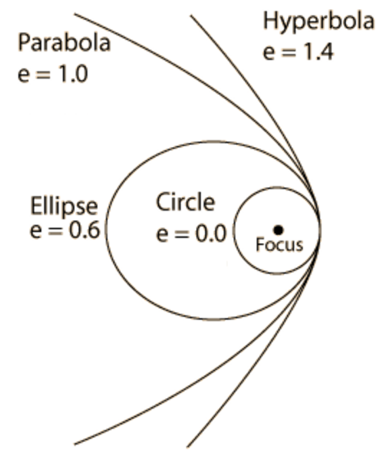
$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$

where eccentricity $e = c/a$ and $b^2 = a^2 - c^2 = a^2(1-e^2)$.

The other standard parametrization is

$$x^2/a^2 + y^2/b^2 = 1$$

The point closest to focus is called **perigee**, and the farthest is **apogee**.



Parabola ($e = 1$) and hyperbola ($e > 1$)

The polar parametrization is

$$r(\theta) = \frac{p}{1+e\cos\theta}$$

Orbit shape (Kepler's first law)

A classic two-body problem with a central force $\vec{f}(r)$. If we pick CM frame ($m_1\vec{r}_1 + m_2\vec{r}_2 = 0$) and assume no external force ($m_1\ddot{r}_1 + m_2\ddot{r}_2 = 0$), the two masses are always aligned on a single line with origin. So the problem can be reduced to one-body problem

$$\mu\ddot{r} = \vec{f}(r)$$

where $\mu = m_1m_2/(m_1+m_2)$ is symmetric the reduced mass, and $r_1 = rm_2/(m_1+m_2)$, $r_2 = -rm_1/(m_1+m_2)$. The Lagrangian for gravitational attraction $f(r) = -Gm_1m_2/r^2$ is

$$\mathscr{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \left(-\frac{Gm_1m_2}{r}\right)$$

Apply Euler-Lagrange equation

$$\begin{cases} \mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{Gm_1m_2}{r^2} \\ \frac{d}{dt}(\mu r^2\dot{\theta}) = \frac{d}{dt}L = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{r} - \frac{L^2}{\mu^2 r^3} + \frac{Gm_1m_2}{\mu r^2} = 0 \\ \frac{d}{dt}L = 0 \end{cases}$$

To solve it, use the trick of replacing time derivative $\partial_t = \dot{\theta}\partial_\theta = (L/\mu r^2)\partial_\theta$, and solve $1/r$ instead:

$$\dot{r} = \frac{L}{\mu r^2}\frac{dr}{d\theta} = -\frac{L}{\mu}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad , \quad \ddot{r} = \frac{L}{\mu r^2}\frac{d}{d\theta}\dot{r} = -\frac{L^2}{\mu^2 r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

The solution with $r(\theta = 0) = r_p$ at perigee is

$$r(\theta) = \frac{L^2/G\mu m_1m_2}{1 + A(L^2/G\mu m_1m_2)\cos\theta} = \frac{L^2/GM\mu^2}{1 + A(L^2/GM\mu^2)\cos\theta}$$

So the eccentricity is naturally $e = AL^2/\mu G m_1 m_2$. If $e < 1$, the orbit is bounded and $L^2/\mu G m_1 m_2 = a(1-e^2)$, corresponding to energy $E = T + V < 0$ (easy to check).

- Kepler's second law

  The differential area swept is

$$\frac{dA}{dt} = \frac{d}{dt}\frac{1}{2}r \times r\dot{\theta} \propto \frac{d}{dt}L = 0$$

  So the area swept at each constant time interval is constant.

- Orbit dynamics (Kepler's third law)

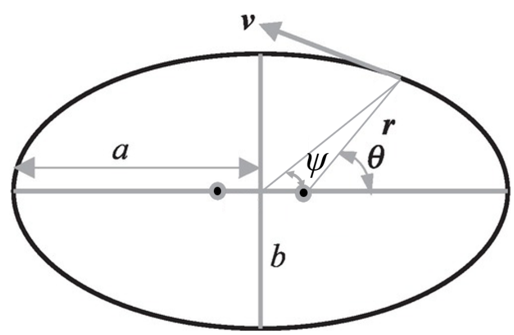
  Conservation of energy implies

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = \frac{1}{2}\mu\dot{r}^2 + Gm_1m_2\left(\frac{a(1-e^2)}{2r^2} - \frac{1}{r}\right) = Gm_1m_2\left(\frac{a(1-e^2)}{2r_p^2} - \frac{1}{r_p}\right)$$

$$\Rightarrow \dot{r} = \sqrt{\frac{2Gm_1m_2}{\mu}\left(-\frac{1}{2a} + \frac{1}{r} - \frac{a(1-e^2)}{2r^2}\right)} = \sqrt{\frac{2G(m_1+m_2)}{r^2}\left(-\frac{r^2}{2a} + r - \frac{a(1-e^2)}{2}\right)}$$

  where r_p is still the perigee distance $a(1-e)$ at $t = 0$. Use the trick by representing $r = a(1 - e\cos\psi)$, where angle ψ is measured from origin instead of from focus. r is the same r from point on orbit to focus. The differential equation can be simplified

$$d\psi(1 - e\cos\psi) = dt\sqrt{\frac{G(m_1+m_2)}{a^3}}$$



  The angular frequency is simply $\omega = \sqrt{G(m_1+m_2)/a^3}$ by integrating both sides for a full circle. The solution is

$$\psi - e\sin\psi = \omega t$$

  The angle ψ is called **eccentric anomaly**.

Classifications

- Visual binary: both members are resolved individually
- Astrometric binary: only one member is seen wobbling around a center
- Eclipsing binary: inclined orbital plane. One eclipsing the other.
- Spectroscopic binary: spatially unresolved pair. High radial velocity and period shifts of lines.

Accretion Disks

- Eddington limit

  The Eddington luminosity is the max limit of what the accretion disk can achieve. It's a balance of radiation pressure force and the gravitational force. Consider the luminosity of accretion is L_ν at distance r and frequency ν, the photon density is

$$L_\nu = nc(4\pi r^2)h\nu \quad \Rightarrow \quad n = \frac{L_\nu}{4\pi r^2 ch\nu}$$

  The Thomson scattering cross section is σ_T, so the radiation force is the rate of momentum transfer due to scattering

$$F_{rad} = n\sigma_T c\frac{h\nu}{c} = n\sigma_T h\nu = \frac{L_\nu \sigma_T}{4\pi r^2 c}$$

This has to be smaller than the gravitational force on electron (equivalently on proton because they are bounded tightly by Coulomb force) to maintain accretion flow

$$L_{Edd} = \frac{4\pi c G M m}{\sigma_T}$$

This is the Eddington luminosity.

- Roche lobe
  - Consider a binary stars M_1 and M_2 separated by r. There's a particle at Δr away from M_1 center. The gravitational force in the rotating frame is

$$F_{grav} = \frac{GM_1 m}{\Delta r^2}$$

and it's tidal force caused by M_2 is

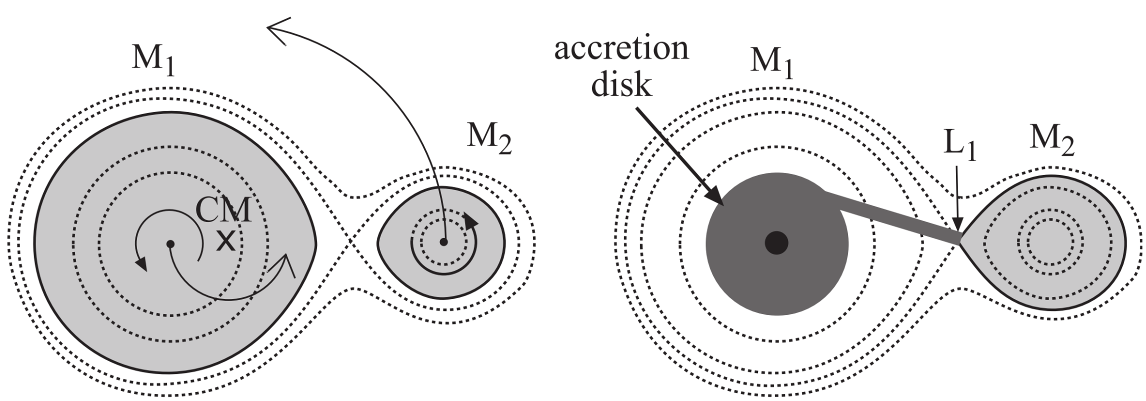
$$F_{tide} = GM_2 m\left(\frac{1}{(r-\Delta r)^2} - \frac{1}{r^2}\right) \approx \frac{2GM_2 m\Delta r}{r^3}$$

So

$$\frac{F_{tide}}{F_{grav}} = \frac{2M_2}{M_1}\frac{\Delta r^3}{r^3}$$

As long as the stars are not tidally locked (i.e., synchronized and circularized), energy is lost to friction while the different parts of each star are deformed. Once tidal locking is achieved, everything appears stationary in a reference frame rotating at the binary frequency, and the system achieves its minimum energy.

  - The equipotential surface of the close binary looks like ∞. This boundary is called the Roche lobe. At the **first Lagrange point** L_1, the gravitational forces due to the two stars, and the centrifugal force in the rotating frame due to rotation about the center ofmass, all sum to zero. In any star, surfaces of constant gas density and pressure will be parallel to surfaces of constant potential. If a star fills its Roche lobe, its matter will be transferred to another star.



  - Accretion disk
    The luminosity is

$$L = \frac{1}{2}\frac{GM\dot{M}}{r_{in}}$$

and the efficiency is

$$\eta = \frac{L}{\dot{M}c^2} = \frac{GM}{2c^2 r_{in}}$$

which is 8-10%, higher than nuclear reaction of 0.7%.

- If the receiving star is
  - Main sequence star: Algol-type binary ssytem
  - White dwarf: cataclysmic variable or novae
  - Black hole: X-ray binary because r_{in} of black hole is so small that the temperature peaks at X-ray.

3 *Stellar Physics*

- Time Scales
  - Free-falling timescale:

$$\tau_{ff} = \sqrt{\frac{3\pi}{32G\bar{\rho}}}$$

    For sun, the free-falling time is 1800 s.
  - Kevin-Helmholtz timescale:

$$\tau_{kh} = \frac{|E_{gr} - E_{th}|}{L} \sim \frac{1}{2}\frac{GM^2}{rL}$$

    shows star's lifetime if all of its luminosity is provided by gravitational energy. For sun, the Kevin-Helmholtz timescale is 16 million years.
- Virial theorem
  - By integrating $\int dr 4\pi r^3$ on both sides of hydrostatic equation, without any additional assumption we get

$$\bar{P} = -\frac{1}{3}\frac{E_{gr}}{V}$$

    For monoatomic nonrelativistic ideal gas, we have $E_{th} = \frac{3}{2}Nk_BT$:

$$E_{gr} = -2E_{th} \Rightarrow E_{tot} = E_{gr} + E_{th} = -E_{th}$$

  - Another way of writing Virial theorem is

$$2T + U = 0$$

    where T is kinetic energy and U is potential energy.
- Seven Equations of Stellar Structure for seven unknowns: $\rho, T, M, L, P, \kappa, \epsilon$ as functions of r.
  - Equation of hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

  - Mass continuity equation

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

  - Energy continuity equation

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

- Equation of energy transport
  - Radiative-dominant
$$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 16\sigma T^3(r)}$$
  - Convective-dominant
$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma}\frac{T(r)}{P(r)}\frac{dP}{dr}$$
    Convection is more efficient than radiation.
- Equation of state
$$P = P_{matter} + P_{radiation} = \frac{\rho k_B T}{\bar{m}} + \frac{1}{3}u$$
  where 1/3 is a factor from integrating up solid angles with $\cos^2$.
- Opacity equation
  - Thomson scattering:
$$\kappa = \frac{n_e \sigma_T}{\rho} = \frac{\sigma_T}{2m_H}(1+X)$$
    where photon is scattered by free electron.
  - Kramers opacity law:
$$\bar{\kappa} \sim \frac{\rho}{T^{3.5}}$$
    it's the mean opacity due to both bound-free absorption and free-free absorption, averaged over all wavelengths. It happens at lower temperature where electrons are still bound to their atoms.
  - Rosseland mean opacity
$$\frac{1}{\kappa} = \frac{\int_0^\infty \kappa_\nu^{-1} u(\nu, T) d\nu}{\int_0^\infty u(\nu, T) d\nu}$$
    The photon mean free path is $\lambda_\nu = (\kappa_\nu \rho)^{-1}$. The Rosseland opacity is derived in the diffusion approximation to the radiative transport equation. It is valid whenever the radiation field is isotropic over distances comparable to or less than a radiation mean free path, such as in local thermal equilibrium.
- Nuclear burning equation
$$\epsilon = \frac{2^{5/3}\sqrt{2}}{\sqrt{3}} \frac{\rho X_A X_B}{m_H^2 A_A A_B \sqrt{\mu}} Q S_0 \frac{E_G^{1/6}}{(k_B T)^{2/3}} \exp\left[-3\left(\frac{E_G}{4k_B T}\right)^{1/3}\right]$$

Scaling relations

- Low-mass stars:
  From hydrostatic equation and matter-dominated EoS:
$$T \sim \frac{P}{\rho} \sim \frac{M}{r} \sim 1$$
  because internal temperature is comparable in all main-sequence stars. So lower mass, smaller star, higher density. Stars like the Sun (low mass stars) have temperatures in their outer envelopes which are low enough that hydrogen is not ionized. So higher energy photons from the interior of the star are easily absorbed by the hydrogen – the outer portions of low mass stars have high opacity, and are thus convective. Low mass stars operate via the p-p chain, which has a relatively weak temperature

dependence ($E \sim T^4$). Radiative transport can handle the energy flux, so low mass stars have radiative cores.

$$L \sim \frac{T^4 r}{\kappa\rho} \sim \frac{r}{\rho^2} \sim M^5$$

the opacity is dominated by bound-free and free-free scattering due to high density, so Kramers opacity law is applied. Low mass stars: radiative cores, convective envelopes.

- Moderately massive stars:
  The matter-dominated EoS still holds:

$$T \sim \frac{P}{\rho} \sim \frac{M}{r} \sim 1$$

From radiative transport and Thomson scattering:

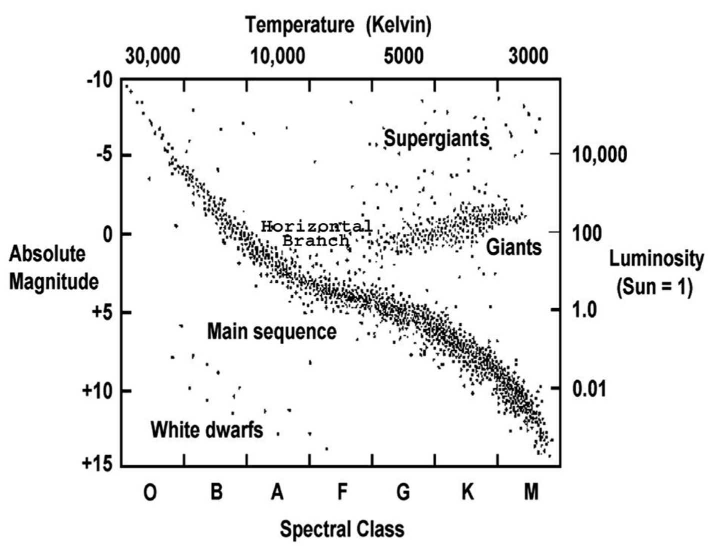
$$L \sim \frac{T^4 r}{\kappa\rho} \sim \frac{r}{\rho} \sim M^3$$

- High-mass stars:
  In high mass stars, the temperature is high enough that the hydrogen stays ionized, so radiation is not so easily absorbed – high mass stars have radiative envelopes. High mass stars operate via the CNO cycle which has a much stronger temperature dependence ($E \sim T^{20}$). This is too strong for radiative transport, so high mass stars have convective cores. Radiation pressure is dominant:

$$L \sim \frac{T^4 r}{\kappa\rho} \sim \frac{Pr}{\rho} \sim M$$

a flattening of mass-luminosity relationship. High mass stars: convective cores, radiative envelopes.

- Hertzsprung-Russell (HR) Diagram



The luminosity scales with effective temperature as $L \sim T_E^8$ from the HR diagram. So

$$L \sim r^2 T_E^4 \sim M^2 \sqrt{L} \Rightarrow L \sim M^4$$

revealing an intermediate slope between low-mass and medium-mass stars.

Nuclear Burning

Proton-proton chain (p-p chain): produces 99% of sun's energy

- A proton wanders for 20 billion years before reacting with another proton

$$p + p \to d + e^+ + \nu_e$$

- Within 1 s,

$$p + d \to {}^3\text{He} + \gamma$$

- With 400 years, one of the branches is

$$^3\text{He} + {}^3\text{He} \to {}^4\text{He} + p + p$$

Overall, 4 protons are converted to a ^{4}He and release 25.7 MeV. Compared with rest energy, the conversion coefficient is only 0.7 %.

Nucleaer reaction rates

The kinetic energy of nucleus in sun's core ($E_k = 1.5k_B \times 10$ million K ~ 1 keV) can't classically overcome the Coulomb potential to trigger nuclear reaction. But quantum tunneling allows it to happen.

- Gamow factor

  Solving the radial part of Schrodinger equation using Coulomb potential, we have

$$\psi \propto \frac{e^{r\sqrt{\mu E_k}/\hbar}}{r} \quad \text{and} \quad \int_0^\infty dr r^2 |\psi(r)|^2 = 1$$

  A $E_k = 1$ keV nucleus can get as close as $r(E_k) = Z_A Z_B e^2/E_k$. The probability of it tunneling to a nuclear reaction distance $r_0 \ll r$ is approximately

$$P_G(E_k) \sim \frac{r_0^2|\psi(r_0)|^2}{r(E_k)^2|\psi(r(E_k))|^2} = \exp\left[-2(r(E_k) - r_0)\sqrt{\mu E_k}/\hbar\right] \sim \exp\left[-2Z_A Z_B e^2\sqrt{\mu}/(\sqrt{E_k}\hbar)\right]$$

  where μ is the reduced mass between two colliding particles. The full calculation gives

$$P_G(E_k) = e^{-\sqrt{E_G/E_k}} \quad \text{with} \quad E_G = \left(\pi\sqrt{2\mu}\frac{Z_A Z_B e^2}{\hbar}\right)^2$$

  where P_G is Gamow factor and E_G is Gamow energy. For proton, $E_G = 500$ keV, giving $P_G \sim e^{-22} \sim 10^{-10}$.

- Cross section

  The cross section of nuclear reaction is

$$\sigma_{AB}(E) \sim \frac{P_G(E)}{E}$$

  The reaction rate per volume is naturally $n_A n_B \sigma_{AB} v$; that's all the rate can depend on. Convert it to energy generation rate per unit mass

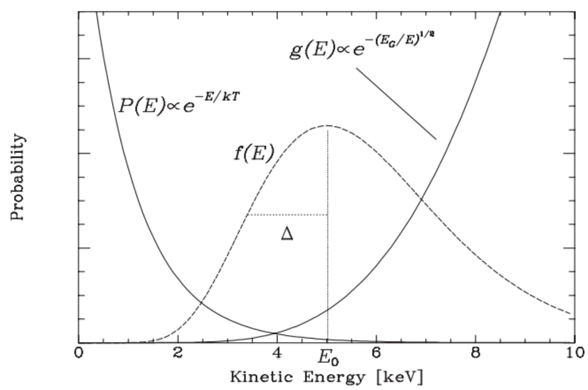
$$\begin{aligned}
\epsilon &= (n_A n_B \sigma_{AB} v)(Q/\rho) \\
&\sim \left(\frac{\rho X_A}{A_A m_H}\right)\left(\frac{\rho X_B}{A_B m_H}\right)\frac{Q}{\rho}\langle\sigma_{AB} v\rangle \\
&\sim \left(\frac{\rho X_A X_B Q}{A_A A_B m_H^2}\right)\langle\sigma_{AB} v\rangle
\end{aligned}$$

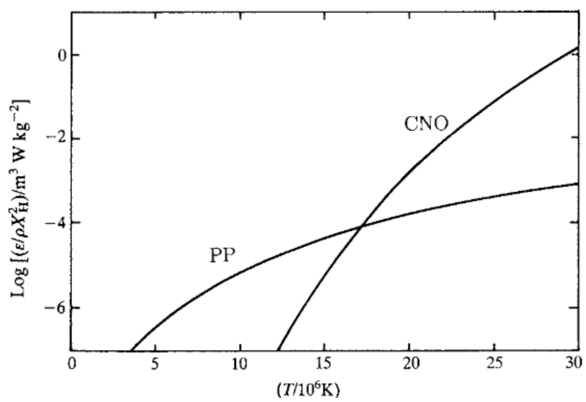
Invoking the Maxwell-Boltzmann distribution

$$P(v) = \frac{4}{\sqrt{\pi}} \left(\frac{\mu}{2k_B T}\right)^{3/2} v^2 \exp\left[-\frac{\mu}{2k_B T} v^2\right] \qquad \text{with} \qquad \int_0^\infty dv P(v) = 1$$

Switch variable v to E, we have

$$\langle \sigma v \rangle \sim \int_0^\infty dE e^{-E/k_B T - \sqrt{E_G/E}}$$





Only the central part contributes to the integral. The detailed calculation gives

$$\epsilon \sim \frac{\rho X_A X_B Q}{m_H^2 A_A A_B \sqrt{\mu}} \frac{E_G^{1/6}}{(k_B T)^{2/3}} \exp\left[-3\left(\frac{E_G}{4k_B T}\right)^{1/3}\right]$$

This process works as a "thermostat" for stars. If T is too high, $\epsilon \uparrow$, $L \uparrow$, and opcaity causes $E_{tot} = E_{gr}/2 = -E_{th} \uparrow$ and $r \uparrow$. So the star expands and cools until equilibrium is reached.

CNO cycle

Stars more massive than Sun have slightly higher core temperature to support this reaction. C, N and O act as catalysts in H-to-He burning. It's more steeply dependent on temperature as seen in figure.

4 *Stellar Evolution*

Journey of Life

Lifetime

From the scaling relation of main-sequence star: $L \sim M^\alpha$, where $\alpha = 5, 3, 1$ for low, medium and high mass star. The lifetime is

$$t_{ms} \sim \frac{M}{L} \sim M^{1-\alpha}$$

So the more massive a star, the shorter its hydrogen-burning phase on the main sequence. Typically,

$$\begin{aligned} 0.5M_\odot &\to \sim 50 \text{ billion yr;} \\ 1.0M_\odot &\to \sim 10 \text{ billion yr;} \\ 10M_\odot &\to \sim 20 \text{ million yr;} \\ > 30M_\odot &\to \sim 3 \text{ million yr;} \end{aligned}$$

Our Sun is halfway done. Very massive star has $\alpha = 1$ and doesn't depend on mass anymore.

- Phases
  - Protostar - need $0.07M_\odot$ to ignite hydrogen burning
  - Main sequence phase - hydrogen burning, studied above
  - Red-giant phase - core hydrogen is burned out
    - After hydrogen in the core is fully converted to helium, $\epsilon \downarrow$, $L \downarrow \Rightarrow E_{tot} \downarrow$ and $r \downarrow$, the core contracts. Or you can think the radiation pressure can't support gravitation so it starts to collapse.
    - The outer shell containing hydrogen starts burning $\Rightarrow$ expansion of outer layers. The shell's luminosity increases, but effective temperature decreases because radius increases to the squared. Star moves up and right on HR diagram and becomes a red giant.
    - Red-giant phase takes only 10% of the time of main-sequence.
  - Horizontal branch - helium burning starts
    - Triple-alpha reaction

$$^4\text{He} + {}^4\text{He} + {}^4\text{He} \to {}^{12}\text{C} + \gamma \ (7.3\text{ MeV})$$

      This happens when the core contracts and reaches 10^8 K. No stable elements exist between atomic number 5-8, but a small amount intermediate ^{8}Be is at equilibrium. A *resonance* of excited state of ^{12}C equivalent to energy sum of ^{4}He and ^{8}Be makes triple-alpha possible.
    - The core temperature rises due to helium burning, but the extra luminosity is absorbed by the plasma above. This phase lasts only 1% of main-sequence time.
  - Asymptotic giant branch - helium out, double-shell burning
    - Repeat of red-giant phase, but has both hydrogen and helium shell burning after the core is mainly carbon and oxygen.
    - AGB phase has mass losses, primarily due to **thermal pulses** - flashes of enhanced helium shell burning.
    - Giants are very convective, leading to **dredge-up** of metals to stellar winds. They form essentially all metals in the Universe.
  - Remnant - carbon, oxygen, silicon all burned out to iron
    - The heavy-element burning is triggered repeatedly until the contraction can't ignite the next.

$$\begin{aligned} \text{H} &\sim 5 \text{ million yr} \\ \text{He} &\sim 0.5 \text{ million yr} \\ \text{C} &\sim 500 \text{ yr} \\ \text{Ne} &\sim 1 \text{ yr} \\ \text{Si} &\sim 1 \text{ day} \end{aligned}$$

      The stellar remnant depends on the mass:

$$(0.07, 1.4)M_\odot \Rightarrow \text{White dwarf}$$

- Stellar Objects
  - EoS of degenerate matter
    - Fermi-Dirac distribution:

$$dN = \frac{2s+1}{\exp\left[\frac{E-\mu(T)}{k_B T}\right]+1} \frac{d^3p d^3x}{h^3}$$

where s is spin of fermion and μ is chemical potential. When $k_B T \ll \mu$, the gas is **degenerate** and F-D distribution looks more step-like. The number density in phase space becomes

$$n(p)dp = \frac{dN}{dV} \approx \frac{(2s+1)}{h^3} 4\pi p^2 dp \qquad \text{for } p \in [0, \sqrt{2m\mu}]$$

In the limit $\mu(T \to 0) = E_F$, aka Fermi energy, the upper bound of momentum is $\sqrt{2m\mu} = p_F$, **Fermi momentum**.

$$n = \int_0^{p_F} n(p)dp = \frac{4\pi}{3} \frac{2s+1}{h^3} p_F^3$$

- Pressure

  The pressure in one direction exerted by kinematic collision of particles are

$$\frac{dF_x}{dA} = \left(\frac{2p_x}{dtdA}\right)\left(ndAv_x dt\right) = 2np_x v_x$$

  The isotropic total pressure is

$$P = \langle npv \rangle = \frac{1}{3}\int_0^\infty dpn(p)pv$$

  where we divide 2 because only half of all particles are going in one direction. Plug in Maxwell-Boltzmann distribution recovers classical EoS $P = nk_B T$.

- EoS of nonrelativistic degenerate gas

  Nonrelativistic gas has $v = p/m$

$$P = \frac{1}{3}\int_0^\infty dpn(p)pv = \frac{4\pi}{3m}\frac{(2s+1)}{h^3}\int_0^{p_F} dpp^4 \sim p_F^5 \sim n^{5/3} \sim \rho^{5/3}$$

  Note that the degenerate pressure doesn't depend on temperature anymore.

- EoS of ultrarelativistic degenerate gas

  Ultrarelativistic gas has $v = c$

$$P = \frac{1}{3}\int_0^\infty dpn(p)pv = \frac{4\pi}{3}\frac{(2s+1)c}{h^3}\int_0^{p_F} dpp^3 \sim p_F^4 \sim n^{4/3} \sim \rho^{4/3}$$

  Plug in $n = Z\rho/(Am_p)$, the full EoS is

$$P = \frac{1}{4}\left(\frac{3}{4\pi(2s+1)}\right)^{1/3} hc \left(\frac{Z\rho}{Am_p}\right)^{4/3}$$

- Chandrasekhar mass

  This is a mass limit of a star that can be supported by degenerate electron pressure. Estimate it with Virial theorem:

$$\bar{P}V = \frac{1}{4}\left(\frac{3}{4\pi(2s+1)}\right)^{1/3} hc \left(\frac{Z\rho}{Am_p}\right)^{4/3} V = -\frac{1}{3}E_{gr} = \frac{GM^2}{3r}$$

$$M_{ch} \sim \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_p^2}$$

  For a helium core with $Z/A = 1/2$, the Chandrasekhar mass from degenerate electron pressure is $M_{ch} \sim 1.4 M_\odot$

- White dwarf
  - Scaling relation

$$P \sim \frac{GM\rho}{r} \sim b\rho^{5/3}$$

$$r \sim \frac{b}{G}M^{-1/3}$$

    Larger mass gives smaller white dwarfs until the Chandrasekhar limit of $1.4M_{\odot}$

$$E_{th} = \frac{3}{2}Nk_BT = -\frac{1}{2}E_{gr} = \frac{GM^2}{2r}$$

$$k_BT \sim M^{4/3}$$

  - White dwarf cooling
    Young white dwarf is very hot (10 million K) and loses energy by radiation peaked at X-rays. There is a thin nondegenerate insulating surface layer that lowers the effective temperature. So the white dwarf cooling is very slow, at the scale of age of Universe.
- Brown dwarf
  If the protostar is small enough that the gravitational contraction is stopped by degenerate gas pressure before $T_{ig} = 10^7$ K, the hydrogen burning can't be ignited and it fails to be a star. The mass limit for brown dwarf is $M_{min} = 0.07M_{\odot}$.
- Neutron star
  - After burning out of all heavy elements, the remnant is like an onion with shells of burning metals, from heaviest iron core to lightest hydrogen outermost layer. When the iron core continues to grow and approaches M_{ch}, two things happen:
    - *Nuclear photodisintegration*
      Two reactions happen with energetic photon: iron absorbs photon and converts to 13 helium nuclei, and helium absorbs photon and converts to 2 proton and 2 neutron. The whole process is endothermic and lowers the thermal pressure.
    - *Neutronization*
      Large densities in the core lead to processes that convert electron + nucleus to neutrons and neutrinos. It lowers electron pressure and carries off energy.

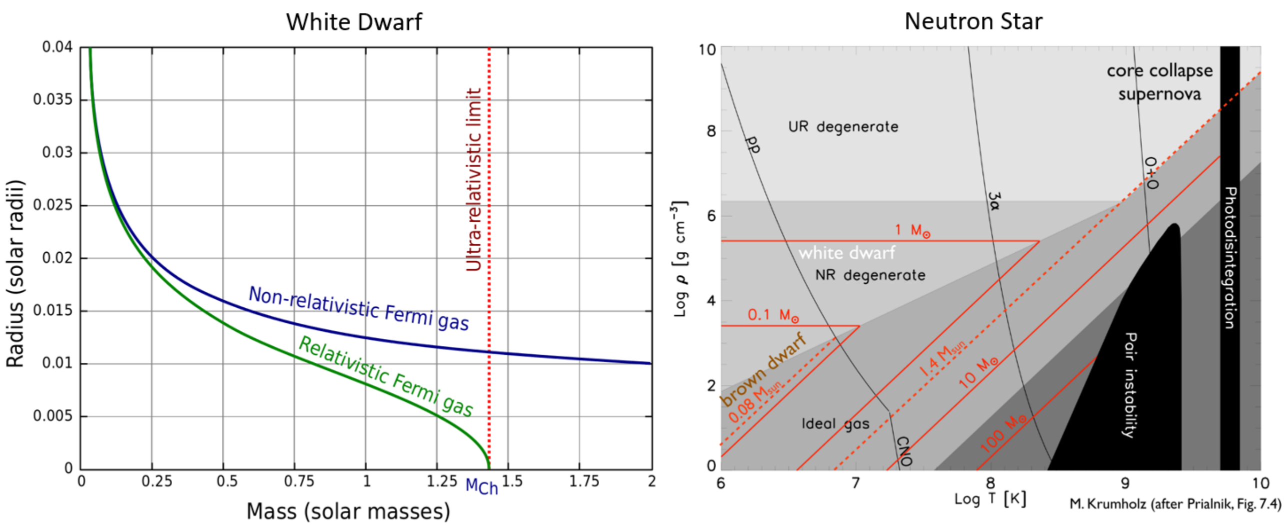
    In a few seconds, the **core collapse** happens and $e^- + p \to n + \nu_e$ occurs. A neutron star forms.
  - The Chandrasekhar mass due to degenerate neutron pressure is $M_{ch} = 4 \times 1.4M_{\odot} = 5.6M_{\odot}$, the extra factor of 4 comes from $Z/A = 1$ in neutron star (where $Z/A = 1/2$ in white dwarf). This is very approximate because of our poor understanding of EoS of nuclear matter. A full calculation, including strong interaction gives $1.5 - 3M_{\odot}$. The evolution fo a star depends on its mass loss, which increases for more massive stars and for higher content of metals in the stellar plasma. Initial mass of 9 $M_{\odot}$ would evolve to a neutron star.
  - Supernova
    The fall of layers of matter onto the surface of neutron star sets off a shockwave that blows off outer shell, aka supernova explosion.
    - The luminosity is on the order of billions of $L_{\odot}$, comparable to the entire galaxy of stars. The radiation only takes 1% of kinetic energy of particles. It lasts for a month.
    - The total amount of energy (radiation + kinetic) is only 0.1% of release of gravitational energy of core collapse, which are mostly carried away by neutrino-antineutrino pairs.
  - Pulsars
    Pulsars are fast-spinning neutron stars. It spins so fast that only neutron stars have high enough density to balance centrifugal force with gravity of the particles on the star's surface.

- Crab nebula
  A remnant caused by event SN1054, observed by people in Song Dynasty. The center is a pulsar with period of 33 ms, so it's spinning extremely fast.
- The period of a pulsar increases steadily (spinning slower) due to loss of kinetic energy to radiation. It sometimes has glitches because the solid core of neutron star changes shape abruptly to become more flattened at the pole.
- Spinning magnetic dipole
  The energy loss rate of spinning magnetic dipole is

$$\dot{E} = -\frac{\mu_0}{6\pi c^3}|\ddot{\mathbf{m}}|^2 = -\frac{\mu_0 \omega^4 \sin^2\alpha}{6\pi c^3}|\mathbf{m}|^2 = -\frac{\mu_0 \omega^4 \sin^2\alpha}{6\pi c^3}\left|\frac{2\pi B R^3}{\mu_0}\right|^2 = I\dot{\omega}$$

  where α is the inclination angle of the magnetic field axis to the rotation axis. Given the change of period, the magnetic field can be calculated. For the Crab pulsar, $B = 5 \times 10^8$ T, which is very high due to the flux freezing theorem.

- Neutron star cooling
  Given long cooling times of white dwarf, neutron star is smaller by five orders of magnitude and cools even slower. It's just stuck at temperature of 10^5 K.



- Black hole
  Initial mass > 25 $M_\odot$ would evolve to a black hole. Famous Schwartzchild radius

$$r_s = \frac{2GM}{c^2}$$

  and **Schwartzchild metric**:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

  It takes a chapter to talk about black hole, so we'll do it later.

5 *Plasma Physics*

Plasma is just a gas of ions and free electrons

Basic Fluid Dynamics

Essentially three dynamic variables: $\rho(x,t), T(x,t)$ and $v^i(x,t)$.

Two derivatives

- Eulerian/partial derivative: rate of change of something at fixed point in spacetime, aka $\partial/\partial t$.
- Lagrangian/material derivative: rate of change of something following a moving element with speed v

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}$$

Mass continuity equation

$$\partial_t \rho + \partial_i(\rho v^i) = 0$$

The rate of change of local density is -1 times divergence of moving particles.

Energy continuity equation

Assuming ideal gas under adiabatic condition (no heat transfer in particular direction, cosmological principle), the energy of a moving element is conserved. From the adiabatic process:

$$\frac{d}{dt}(PV^\gamma) = \frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0$$

Equation of motion (Euler equation)

Assuming ideal fluid with no viscosity, Newton's 2nd law states

$$\rho \delta V \frac{dv^i}{dt} = \delta F^i_{body} + \delta F^i_{surface} = \rho \delta V g^i - \partial^i P \delta V$$

where g^i is for example a gravitational field that acts on all particles of the body, and P is the pressure. Rearrage this to get **Euler equation**

$$\frac{dv^i}{dt} = (\partial_t + v^j \partial_j) v^i = g^i - \frac{\partial^i P}{\rho}$$

If we include viscosity, we will have Navier-Stokes equation instead.

Jeans Instability

A battle between acoustic waves and gravitational attraction.

Static condition

When the gas is at static equilibrium, its property ρ_0, P_0, and v^i_0 have zero time derivative. The two continuity equations are trivial, with equation of motion be

$$\nabla P_0 = \rho_0 g_0 = -\rho_0 \nabla \Phi_0$$

But the Poisson equation of gravity states

$$\nabla^2 \Phi_0 = 4\pi G \rho_0$$

So a uniform infinite gas (constant P_0, ρ_0) doesn't satisfy these two equations at the same time. Still, **Jeans swindle** calculates the perturbation from these two equations.

Perturbation

The first order perturbations are ρ_1, P_1, v_1, and Φ_1. Solve energy continuity equation, we have

$$\frac{P_0 + P_1}{P_0} = \left(\frac{\rho_0 + \rho_1}{\rho_0}\right)^\gamma \quad \Rightarrow \quad P_1 \approx P_0 \gamma \frac{\rho_1}{\rho_0} = c_s^2 \rho_1$$

Plug it in the other two equations:

$$\partial_t \rho_1 + \rho_0 \partial_i (v_1^i) = 0$$
$$\partial_t v_1^i = (g_0 + g_1)^i - \frac{\partial^i (P_0 + P_1)}{(\rho_0 + \rho_1)} = -\partial^i (\Phi_1 + P_1/\rho_0)$$

with another one $\partial_i \partial^i \Phi_1 = 4\pi G \rho_1$. Assume negligible enhanced gravity $\Phi_1 \approx 0$ to solve. These two equations above can combine to

$$\partial_t^2 \rho_1 - \partial_i \partial^i (\Phi_1 \rho_0 + c_s^2 \rho_1) \approx (\partial_t^2 - c_s^2 \partial_i \partial^i) \rho_1 = 0$$

Clearly a wave equation (the sound/acoustic wave). The general solution is the integration of all Fourier modes. Pick one mode $e^{i(kx - \omega t)}$ and plug in without ignoring Φ_1

$$(-i\omega)^2 - (4\pi G \rho_0 + c_s^2 (ik)^2) = -\omega^2 - (4\pi G \rho_0 - c_s^2 |k|^2) = -\omega^2 + c_s^2 (|k|^2 - |k_J|^2) = 0$$

Apparently, the frequency would be imaginary if we have $|k| < |k_J| = \sqrt{4\pi G \rho_0}/c_s$. The corresponding length scale is **Jeans length**

$$\lambda_J = \frac{2\pi}{k_J} = \frac{2\pi c_s}{\sqrt{4\pi G \rho_0}} = \sqrt{\frac{\pi \gamma k_B T}{G \rho_0 \bar{m}}}$$

If things happen at scales larger than λ_J, the enhanced gravitation can dominate over acoustic waves. So the corresponding **Jeans mass** is

$$M_J = \frac{4\pi}{3} \rho_0 \lambda_J^3$$

For example, $n_0 = 10^2\ /\mathrm{cm}^{-3}$ and $T = 100$ K, the Jeans mass is $10^5 M_\odot$. A more complex yet realistic theory is done by Spitzer.

Magneto-hydrodynamics

The actual plasma is fluid dynamics + electrical properties.

Only magnetic field B is important because the charges of plasma remain equilibrium, so no strong electric field can build up. we can also solve for E directly by Maxwell's equation

$$\nabla \times B = \mu_0 j + \frac{1}{c^2} \frac{\partial E}{\partial t} \approx \mu_0 j = \mu_0 \sigma (E + v \times B)$$

where σ is the electrical conductivity. The E can be solved.

Induction equation

Plug E into the dynamic part of B in Maxwell's equation

$$\partial_t B = -\nabla \times E = \nabla \times (v \times B) + \frac{\nabla^2 B}{\mu_0 \sigma}$$

- Equation of motion
  we have an extra magnetic force

$$\frac{\delta F_{mag}}{\delta V} = \rho_c v \times B = j \times B = \frac{1}{\mu_0}(\nabla \times B) \times B = \frac{1}{\mu_0}(B \cdot \nabla)B - \frac{1}{\mu_0}B \cdot \nabla B \ (``\nabla|B|^2/2")$$

  Plug it in equation of motion:

$$\frac{dv^i}{dt} = g^i - \frac{\partial^i}{\rho}\left(P + \frac{|B|^2}{2\mu_0}\right) + \frac{1}{\mu_0 \rho}B^j \partial_j B^i$$

  So magnetic field introduces an additional pressure ($|B|^2/2\mu_0$) and a tension force along magnetic field lines ($B^j \partial_j B^i/\mu_0 \rho$).

Phenomenology

- Alfven's Theorem of Flux Freezing
  - Magnetic Reynolds number
    The contributions of two terms in the induction equation can be roughly estimated as

$$R_M = \frac{vB/L}{B/(L^2 \mu_0 \sigma)} = vL\mu_0\sigma$$

    where L is the typical length where B varies significantly. In lab, L is very small so $R_M \ll 1$, the second term $\nabla^2 B/\mu_0\sigma$ dominates. The induction equation is now a wave equation with lots of damping. In astrophysical system, it is the opposite.
  - Flux freezing
    In astrophysics system, the flux of a surface inside a plasma is constant

$$\frac{d}{dt}\int_S B \cdot dS = \int_S \frac{\partial B}{\partial t} \cdot dS + \int_S B \cdot \frac{d}{dt}dS = \int_S dS \left[\partial_t B - \nabla \times (v \times B)\right] = 0$$

    So the flux of magnetic field is constant throughout the time. This can explain why the neutron star has very high magnetic field (10^8 T) as it is collapsed from a star.
- Sunspots
  - The sunspot is a region of concentrated magnetic field bundles. The study of magneto-convection simulates that the strong magnetic fields create tensions that inhibit convection. So less heat is convected at sunspot and it looks cooler or darker.
  - Magnetic buoyancy
    Imagine a magnetic bundle exists under Sun's surface. The external pressure is larger than inside by the magnetic pressure term $|B|^2/2\mu_0$. So the bundle has less density and floats up the surface. It pierces the surface twice when coming out and back in, so it causes bipolar sunspots.
  - Parker instability
    Interstellar medium is not a uniform cloud but formed in clumps. Parker's instability theory explains why it is unstable to be uniform. A battle between gravitation and magnetic buoyancy.

6 ***Interstellar Medium***

Matter and radiation between stars in a galaxy

Star Formation

- Molecular clouds

  Molecular clouds are the largest, most massive, gravitationally bound objects in the ISM.

  - Highest density in ISM, but still extremely rarefied: $\sim 10^2 - 10^4 \text{ cm}^{-3}$, $\sim 10^{-13}$ Pa.

- Jeans instability

  Jeans instability occurs if the cloud is unstable to gravitational collapse. This is the start of star formation. A simple argument can be made: if the change in gravitational energy is greater than the rise of thermal energy, the thermal pressure would not be able to support the star, so it collapses.

$$|dE_{gr}| > dE_{th}$$

$$\frac{GM^2}{r^2}dr > PdV = nk_BT4\pi r^2 dr = 3k_BT\frac{M}{\bar{n}}\frac{dr}{r}$$

  So the **Jeans mass** is

$$M_J = \frac{3k_BT}{G\bar{m}}r$$

  Reverse the equation to get **Jeans radius**

$$r_J = \frac{G\bar{m}}{3k_BT}M$$

  and **Jeans density**

$$\rho_J = \frac{M}{4\pi r_J^3/3} = \frac{3}{4\pi M^2}\left(\frac{3k_BT}{G\bar{m}}\right)^3$$

  So cloud with larger mass M is easier to collapse. Note that we did not consider other pressure sources such as turbulence, magnetic fields, etc.

- Cloud collapse

  Cloud collapse is facilitated if the gravitational energy is released in non-pressure-producing form, instead of thermal energy. Two avenues are possible:

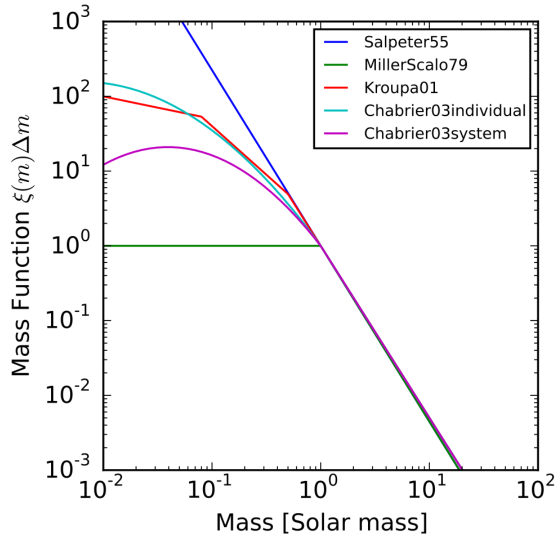
  - Dissociation of H_2: takes 4.5 eV per molecule
  - Ionization of hydrogen: takes 13.6 eV per atom, higher than photodissociation.

  These two can consume enough energy to bring a solar-mass protostar down to dimensions of order 100 times of the Sun.

- Initial mass function

$$\frac{dN}{dm} \propto m^{-\alpha}$$

  $\alpha = 2.35$ is called Salpeter initial mass function.



Thus, low-mass stars are much more common than high-mass stars.

ISM components

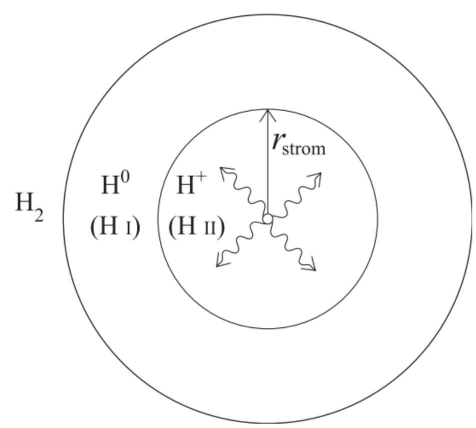
Astronomical terminology: H I is just H atom, H II is ionized H^+. H II region is part of molecular cloud that is ionized by radiations of young and hot stars.

Stromgren radius

A radius around a hot star in which photoionization and recombination is balanced. Inside the radius, the hydrogen is almost completely ionized, aka H II region. Beyond it is H I region because all photons capable of ionizing it has been absorbed. Then it's H_2 region, where photons capable of photodissociation has been absorbed.

$$Q = \frac{4}{3}\pi r_{strom}^3 n_p n_e \langle \sigma_{rec} v \rangle$$

where Q is the number of ionizing photons rate and σ_{rec} is the cross section of recombination.



H I region

- Cold-neutral medium
- Warm-neutral medium
- 21-cm hyperfine splitting

Shocks

A discontinuity in the thermodynamic variables v, T, P, ρ. The mass, momentum, and energy continuity equation across the shock gives these relations

$$\begin{cases} \frac{\rho_1}{\rho_0} = \frac{v_0}{v_1} = \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2} \\ \frac{P_1}{P_0} = 1 + \frac{2\gamma}{\gamma+1}(M^2-1) \\ \frac{P_1}{\rho_1 T_1} = \frac{P_0}{\rho_0 T_0} \end{cases}$$

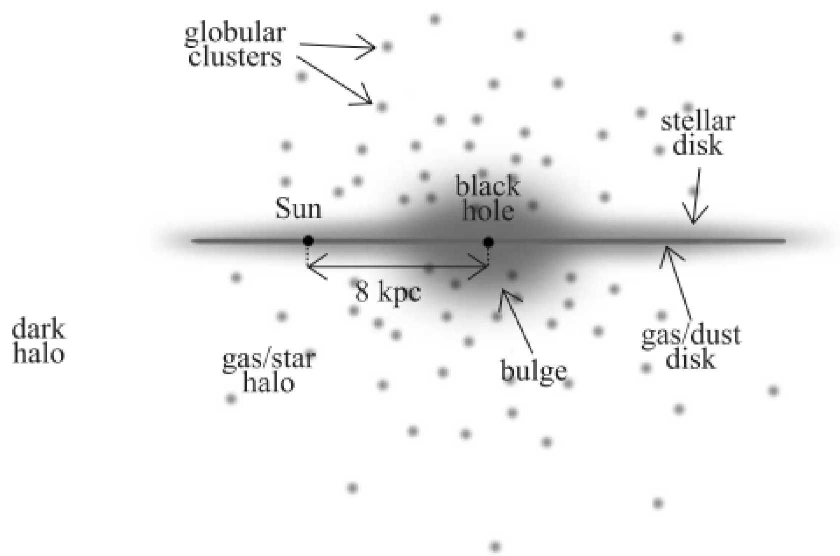
M is the Mach number of the shock front. Just remember that the density change across shock won't be too high (1-2), but the pressure and temperature can be much higher.

7 Exoplanets

- Planet
  - Unlike stars supported by thermal and radiation pressure, or stellar remnants supported by degeneracy pressure, the planet has small enough masses to be supported by electrostatic force.
  - Borderline between planet and brown dwarf - 13 Jupiter mass, which ignites deuterium burning.
- Detection Methods
  - Doppler (radial-velocity) method
    - Similar to the spectroscopic binary, the unseen planet can still be deduced from the periodic radial-velocity Doppler shifts in the spectral lines of the primary star, due to the motion of common center of mass.
    - A Jupiter-sized planet causes very tiny Doppler shift: $\Delta\lambda/\lambda = v/c = 31/3e8 = 10^{-7}$. Our sensitive spectroscopic technique can detect as low as a few ppb.
  - Planetary transits
    - Similar to eclipsing binaries, planet transits across the face of star.
    - Probability of observing a transit is $(r_* + r_p)/a$, where a is distance between star(*) and planet(p).
  - Direct imaging
    - Very challenging because the contrast between light reflected by planet and stellar light is very low, around 10 ppb.
    - Use coronagraph to block some of the stellar light to enhance contrast.
  - Microlensing
    - Gravitational lensing magnifies the image and amplifies the image brightless.
    - It's called micro-lensing because the Einstein angle is ofter at an order of a micro-arcsecond.
    - The planet near the lensing star causes perturbation of the magnification. This perturbation can be easily detected during a microlensing event.
  - Timing
    - Some planets are discovered by their effects on the arrival times of signals from the system, such as radio pulses from a pulsar.
  - Astromotry
    - Gaia mission: measured 1 billion parallax of stars have been measured.

8 *Galaxy*

Galaxy Structure



- Disk
  - Mass distribution: falls exponentially from center and height

    $$\rho(r,z) = \rho_0 e^{-r/r_d} e^{-|z|/h_d}$$

    where scale of disk $r_d \sim 2$ kpc, so the Sun is in the outer region of galaxy (8 kpc). Stellar disk (lower-mass stars) has height 400 pc, and gas/dust disk has height of 130 pc.
  - Contains 10 billion stars. Number density is around 1 pc^{-3}. So mean distance between stars is around 1 pc. The mean free path is

    $$l = \frac{1}{n\sigma_{geom}} = \frac{1}{n\pi(2r_\odot)^2} \sim 10 \text{ Gpc}$$

    The actual collision time $l/20$ km/s is 10^8 longer than the age of Universe. So a head-on collision of our Sun to other stars is impossible.
  - Gravitational focusing
    The gravitational attraction of stars increases the effective cross section

    $$\sigma_{eff} = \sigma_{geom}\left(1 + \frac{v_{esc}^2}{v_{ran}^2}\right)$$

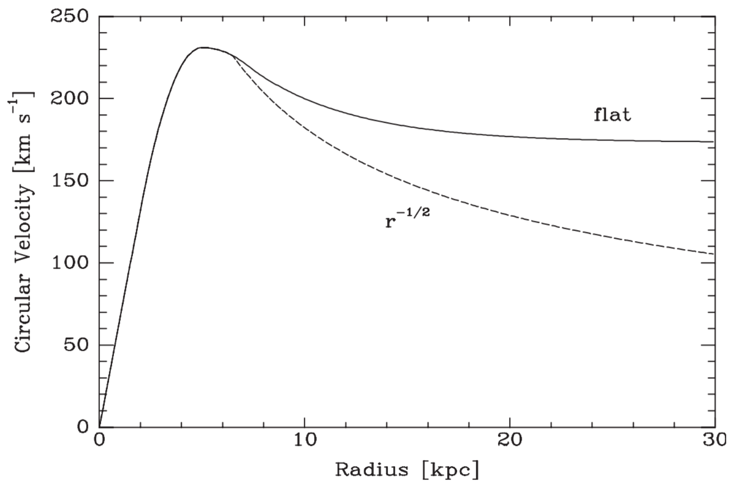
    The galactic center has higher density, so 1% of stars undergoes some disruption on its trajectory.
  - Spiral arms formed by **density waves**.
- The spheroid
  - Bulge
  - gas/star halo
  - globular clusters

Stars in the bulge and halo are very old. The spectra of their atmosphere don't show much metal, which typically from stellar winds of giants. So there were no giants when these stars initially formed. Very old.

- Supermassive black hole

  The orbits of stars near the galactic center show something of huge mass (millions of $M_\odot$, but its luminosity is small. So it might be a supermassive black hole.

- Dark halo
  - The evidence is kinematic, based on the measurements of **rotation curves**, aka circular velocity of particles around galactic center.
    - 21-cm emission line of H I region
    - Balmer Hα emission line from H II region.



  - Flat rotation curve

    Flat rotation curve at large enough radii reveals some additional mass:

$$M(r) \sim \frac{v^2 r}{G} \sim r$$

    So the majority of galaxy's mass is dark.

- Other evidences of dark matter
  - Gravitational lensing on the quasar.
  - CMB
  - Structure formation

Dark Matter - if it's made of:

- Gas
  - atomic gas - but it would radiate strongly at 21 cm.
  - molecular gas - but it would radiate from CO and H_2 emission.
  - ionized gas - but it has strong X-ray of thermal free-free radiation
- Dust - but it would emit IR radiation and be made of very heavy elements.
- Elementary particles
  - electron/protons - this is just ionized hydrogen that would emit X-ray.
  - neutron - but a free neutron decays to proton and electron (plus neutrinos) in 15 mins.
  - hot neutrinos - its so little mass means they must be very relativistic (hot), so they escape gravitational potential quickely.
  - cold dark matter - WIMPs, axions, etc.
- Massive Compact Halo Objects (MACHO) - including all gravitationally bound, star-like objects
  - main-sequence stars - but it's very visible
  - giants - even more luminous

    - neutron stars/black holes - accompanied by supernova explosion that produce large amounts of heavy elements, which are not observed.
    - white dwarf - but it would produce large amount of He, N, Ne, C, O (during giant stage) that are not observed.
    - brown dwarf/planets - they are leading candidates of MACHO-type dark matter, but are ruled out from microlensing experiment (The Large Magellanic Cloud Microlensing Experiments). Very few lensing events are observed at the edge of LMC, so dark matter can't be made of low-luminousity compact objects.
  - Navarro–Frenk–White (NFW) profile
    - The density of dark matter in galaxy as a function of radius is given by

$$\rho(r) = \frac{\rho_0}{\frac{r}{a}\left(1+\frac{r}{a}\right)^2}$$

      where ρ_0 and a are scale parameters.
    - It has two problems. One is Core-cusp problem:
      From observations of stellar dynamics, the inner profile of halos flattens to a slope □ 0 (core) instead of -1 (cusp).
    - Another problem is missing satellite galaxies

- Galaxy Demographics
  - Three types
    - Spiral galaxies: discussed above
    - Elliptical galaxies: looks like a bulge of spiral galaxy without a disk.
    - irregular galaxies: most common.
  - Luminosity function - **Schechter function**

$$\phi(L)dL \sim \phi(L_*)\left(\frac{L}{L_*}\right)^{-1} e^{-L/L_*}dL$$

    where $\phi(L)$ is galaxy number density with luminosity L. Usually, $L_* = 20$ billion $L_\odot$.

- Active Galactic Nuclei - "nuclei" of galaxy with large luminosity.
  - Caused by accretion disk of the supermassive black hole.
  - Most luminous AGN is called **quasar**, $10^9 L_\odot$. Our mil

- Galaxy Group/Clusters
  - Galaxies are often found in groups containing 10 galaxies or in clusters containing 100 galaxies. we are in **Local Group** with M31, M33 and some smaller galaxies.
  - Typical cluster has size 1 Mpc and velocity dispersion of 1000 km/s. So the typical **cluster-crossing timescale** is 1 Mpc/1000km/s = 1 billion year, more likely than star crossing.

- Satellite Galaxy
  - The Milky Way has a few satellite galaxy with the most notable Large Magellen Cloud (LMC). They have their dark matter halos, etc. LMC has around $10^{10} M_\odot$, roughly 1/100 of Milky Way.

9 *General Relativity*

Special relativity

Lorentz invariance

The Lorentz boost with velocity parameter v in x-direction is

$$dx^{\bar{\beta}} = \frac{dx^{\bar{\beta}}}{dx^{\alpha}} dx^{\alpha} = \Lambda^{\bar{\beta}}_{\alpha}(v) dx^{\alpha} = \begin{bmatrix} \gamma & -v\gamma & & \\ -v\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} dx^{\alpha}$$

where barred label means another reference system. The inverse transformation is $\Lambda^{\alpha}_{\bar{\beta}}(-v)$, and

$$\Lambda^{\bar{\beta}}_{\alpha}(v)\Lambda^{\nu}_{\bar{\beta}}(-v) = \delta^{\nu}_{\alpha}$$

sometimes the argument (v) is not written. So the Lorentz invariance can be immediately seen

$$dx^{\bar{\beta}} dx_{\bar{\beta}} = \Lambda^{\nu}_{\bar{\beta}}\Lambda^{\bar{\beta}}_{\alpha} dx_{\nu} dx^{\alpha} = \delta^{\nu}_{\alpha} dx_{\nu} dx^{\alpha} = dx_{\alpha} dx^{\alpha}$$

4-stuff

4-velocity

The particle is moving on a world line, a trajectory parametrized by its proper time τ in the space-time diagram. From an inertial reference frame $\mathcal{O}$, it has speed $v^i = dx^i/dt$. Since moving clocks tick slower, we have

$$d\tau = \frac{dt}{\gamma(v)}$$

So the 4-velocity is defined as

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma \frac{d}{dt}(t, x, y, z) = \gamma(1, v)$$

and $u^{\mu}u_{\mu} = -1$. The 3-velocity addition rule

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}$$

Massless particles (photon) don't have a defined 4-velocity.

4-acceleration

$$a^{\mu} = \frac{du^{\mu}}{d\tau}$$

and $a^{\mu}u_{\mu} = 0$.

4-momentum

4-momentum is intuitively

$$p^{\mu} = mu^{\mu} = m\gamma(1, v) = (E, \vec{p})$$

and $p^{\mu}p_{\mu} = m^2 u^{\mu}u_{\mu} = -m^2$. So we have $E^2 = \vec{p}^2 + m^2$, the usual relativistic energy of the particle measured in the reference frame $\mathcal{O}$. For another frame $\bar{\mathcal{O}}$ moving with $\bar{u}^{\alpha}$, the energy measured is

$$E = -p^{\mu}\bar{u}_{\mu}$$

At it's self frame, $E = -p^{\mu}u_{\mu} = m$, the rest mass energy. For massless particles, there's no self frame.

- Geometry
  - A geometrical object is frame-invariant. It is what it is, doesn't depend on which frame you observe it. A scalar is geometrical of course. Another example is a 4-vector

$$\vec{A} = A^{\mu}\vec{e}_{\mu} = \Lambda^{\mu}_{\bar{\beta}}A_{\bar{\beta}}\Lambda^{\bar{\alpha}}_{\mu}A_{\bar{\alpha}} = A^{\bar{\alpha}}A_{\bar{\alpha}}$$

    Note that both the component and the basis change with respect to frame, but the product doesn't. A tensor is also geometrical. The inner product of two vectors

$$\vec{A}\cdot\vec{B} = gA^{\mu}\vec{e}_{\mu}\otimes B^{\nu}\vec{e}_{\nu} = g\left(\vec{e}_{\mu},\vec{e}_{\nu}\right)A^{\mu}B^{\nu} = g_{\mu\nu}A^{\mu}B^{\nu}$$

    we use convention $g_{\mu\nu} = (-1,1,1,1)$ for flat spacetime.
  - n-form

    n-form is dual to n-vector. The product of them gives a scalar.

$$\tilde{\omega}(\vec{e}_{\alpha})\vec{e}_{\alpha} = \tilde{\omega}^{\alpha}\vec{e}_{\alpha} = \delta^{\alpha}_{\alpha}$$

    The differential forms also transform like a vector. The gradient of a scalar gives differential forms:t

$$\tilde{d} = \tilde{\omega}^{\alpha}\frac{\partial}{\partial x^{\alpha}}$$

    The form and vector can transform interchangeably

$$A_{\mu}\tilde{\omega}^{\mu} = g(A^{\alpha}\vec{e}_{\alpha}) = g_{\mu\nu}\tilde{\omega}^{\mu}\tilde{\omega}^{\nu}A^{\alpha}\vec{e}_{\alpha} = g_{\mu\nu}A^{\nu}\tilde{\omega}^{\mu}$$

    The metric has property

$$g_{\mu\nu}g^{\nu\alpha} = \delta^{\alpha}_{\mu}$$

    they are inverse to each other.
- Perfect fluid
  - Relativistic number-flux tensor

$$N^{\mu} = nu^{\mu} = (\gamma n, \gamma n v_x, \gamma n v_y, \gamma n v_z)$$

    where n is the number density in the rest frame. The first component is the relativistic number density due to length contraction, and the other three terms are the relativistic flux of particles.
  - Conservation of particles

$$\frac{\partial}{\partial x^{\alpha}}N^{\alpha} = N^{\alpha}_{,\alpha} = 0$$

  - Stress-energy tensor

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}$$

    It's a symmetric tensor.
  - Conservation of energy

$$T^{\mu\nu}_{,\nu} = 0$$

    It also implies the the entropy is conserved $dS/d\tau = 0$, so the system is adiabatic.
- Curvature
  - Equivalence principle

- Weak equivalence principle
  This is basically saying the inertial mass and gravitational mass are equivalent. If they are not, then Galileo would have found different mass falls differently.
  - Eotvos experiment: a balance of gravity acting on gravitational mass and centrifugal force on inertial mass (due to Earth spinning). If they unbalance over time, then the two masses are not equivalent.
- Einstein equivalence principle
  The physics law in free-falling frame is equivalent to those in SR (no gravity). There must be local inertial frame that is flat.

- Christoffel symbol

$$\frac{\partial \vec{e}_\alpha}{\partial x_\beta} = \Gamma^\mu_{\alpha\beta}\vec{e}_\mu$$

The covariant derivative is

$$\frac{\partial \vec{A}}{\partial x^\beta} = \left(\frac{\partial A^\alpha}{\partial x^\beta} + A^\mu \Gamma^\alpha_{\mu\beta}\right)\vec{e}_\alpha$$

or

$$A^\alpha_{;\beta} = A^\alpha_{,\beta} + A^\mu \Gamma^\alpha_{\mu\beta}$$

and

$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_\mu \Gamma^\mu_{\alpha\beta}$$

Note the minus sign because the differential form transforms covariantly.
The geometrical property of metric tensor states that

$$\nabla_\beta \tilde{A} = g(\nabla_\beta \vec{A}, \cdot) \quad \Rightarrow \quad A_{\alpha;\beta} = g_{\alpha\mu}A^\mu_{;\beta} = (g_{\alpha\mu}A^\mu)_{;\beta}$$

So

$$g_{\alpha\mu;\beta} = 0 \quad \Rightarrow \quad g_{\alpha\mu,\beta} - g_{\nu\beta}\Gamma^\nu_{\alpha\mu} - g_{\alpha\mu}\Gamma^\nu_{\beta\mu} = 0$$

The Christoffel symbol is also symmetric

$$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$$

So you can calculate Christoffel symbol from the metric

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\mu\gamma}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right)$$

- Local inertial frame
  The Einstein equivalence principle assumes a local SR frame at point P:

$$g_{\mu\nu}(P) = \eta_{\mu\nu} \quad \text{and} \quad g_{\mu\nu,\alpha}(P) = 0$$

The first derivative is also zero at this point, but the second derivative is not zero! That's the source of curvature.

- Proper volume element
  The proper volume element in the curved metric can be calculated from the flat metric:

$$d^4x = \sqrt{-g}\,d^4x'$$

where d^4x' is the curved differential volume.

Parallel transport

The definition of parallel-transport of $\vec{A}$ along $u(\vec{\lambda})$:

$$\frac{d}{d\lambda}\vec{A} = u^{\alpha} A^{\mu}_{;\alpha} = 0$$

Geodesics

Geodesics is analogy to the straight line in Euclidean space. It's the curve that parallel-transports its own tangent vector.

$$\frac{d}{d\lambda}u(\lambda) = \nabla_{\vec{u}}\vec{u} = u^{\alpha}u^{\mu}_{;\alpha} = 0$$

Or use momentum vector for both massive and massless particles

$$p^{\alpha}p_{\beta;\alpha} = p^{\alpha}p_{\beta,\alpha} - \Gamma^{\gamma}_{\beta\alpha}p^{\alpha}p^{\gamma} = 0$$

Plug in the Christoffel symbol and organize, we get a general geodesic equation

$$m\frac{dp_{\beta}}{d\tau} = \frac{1}{2}g_{\mu\nu,\beta}p^{\mu}p^{\nu}$$

Riemann curvature tensor

The curvature tensor has to be second-order:

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

and

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu - g_{\beta\nu,\alpha\mu}}\right)$$

The Riemann curvature tensor has following properties

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$$

It's antisymmetric on the first and second pair of indices but symmetric on both.

$$R^{\alpha}_{[\beta\mu\nu]} = 0$$

The fully symmetric form of three bottom indices is zero. If the manifold is flat, of course we have $R^{\alpha}_{\beta\mu\nu} = 0$.

Bianchi identity

From the symmetry of metric tensor, we have Bianchi identity

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$$

Ricci tensor and scalar

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} = R_{\beta\alpha}$$

$$R = g^{\alpha\beta}R_{\alpha\beta}$$

Einstein tensor

Contract Bianchi identity twice with metric tensor and we have Einstein tesor identity

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} \quad \text{and} \quad G^{\mu\nu}_{;\nu} = 0$$

It's a traceless tensor just like stress-energy tensor.

Geodesic deviations

The connecting vector ξ connecting to geodesics might not be constant in a curved spacetime. Consider two geodesics with tangent vector $V(\lambda)$ and $V'(\lambda)$, the equation of the separation is

$$\nabla_V \nabla_V \xi^\alpha = R^\alpha_{\mu\nu\beta} V^\mu V^\nu \xi^\beta$$

Of course, when space is flat, R tensor is 0 and ξ remains constant.

Linearized Einstein Field Equation

Einstein equation is the simplest and most aesthetic theory of gravity.

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

The cosmological constant term $\Lambda g_{\mu\nu}$ can also be inserted because $g^{\mu\nu}_{;\nu} = 0$. The equation can be approximated in weak gravitational setting.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \ \ |h_{\mu\nu}| \ll 1$$

Only the first order will be considered. It turns out that the perturbation $h_{\mu\nu}$ transforms as if it's a tensor in SR

$$h_{\bar{\alpha}\bar{\beta}} = \Lambda^\mu_{\bar{\alpha}} \Lambda^\nu_{\bar{\beta}} h_{\mu\nu}$$

Linearized theory

We have the freedom to choose the gauge and coordinate system we want to work with. Suppose we do

$$x^\mu \to x^\mu + \xi^\mu(x^\alpha) \quad \text{with} \ \ |\xi^\alpha_\beta| \ll 1$$

and the perturbation becomes

$$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

Note that the Riemann tensor is independent of the gauge we choose to first order

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})$$

It's easier to use trace-reversed h

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\alpha_\alpha$$

and we choose Lorentz gauge

$$\Box \xi^\mu = \bar{h}^{\mu\nu}_{,\nu}$$

such that the transformed h has

$$\bar{h}^{\mu\nu}_{,\nu} \to \bar{h}^{\mu\nu}_{,\nu} = 0$$

which simplifies the Einstein tensor to be $G^{\mu\nu} = (-1/2)\Box\bar{h}^{\mu\nu}$, and the field equation is linearized

$$\Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}$$

Newtonian limit

For static, non-relativistic dust, we have $T^{\mu\nu} = \rho u^\mu u^\nu$ with $u^\mu \approx (1,0,0,0)$. So only 00 component exist. In the non-relativistic limit $\Box \approx \nabla^2$, and we have

$$\Box \bar{h}^{00} = \nabla^2 \bar{h}^{00} = -16\pi G T^{00} = -16\pi G \rho$$

Recall the Newtonian equation of gravitational potential

$$\nabla^2\phi = 4\pi G\rho$$

So we get $\bar{h}^{00} = -4\phi$ and reverse the trace to get the original perturbation

$$h^{\mu\nu} = \begin{bmatrix} -2\phi & & & \\ & -2\phi & & \\ & & -2\phi & \\ & & & -2\phi \end{bmatrix} \approx h_{\mu\nu}$$

and the weak-gravity metric is

$$g_{\mu\nu} = (\eta^{\mu\nu} - h^{\mu\nu})^{-1} = \begin{bmatrix} -1-2\phi & & & \\ & 1-2\phi & & \\ & & 1-2\phi & \\ & & & 1-2\phi \end{bmatrix}$$

Note the minus sign in the parenthesis.

🌞 Newton's gravitational equation

Recall the general geodesic equation

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2}g_{\mu\nu,\beta}p^\mu p^\nu$$

In Newtonian (non-relativistic) setting, $p^\mu \approx m(1,0,0,0)$ and $d/d\tau = u^\alpha\partial/\partial x^\alpha \approx \partial/\partial t$

$$\frac{dp_\beta}{d\tau} = \frac{dp_\beta}{dt} = \frac{1}{2m}g_{00,\beta}p^0p^0 = \frac{1}{2}mh_{00,\beta} = -m\frac{\partial}{\partial x^\beta}\phi$$

aka $F = -m\nabla\phi$. The famous equation is recovered.

🌀 Gravitational Wave

🌞 Propagation in Vacuum

The wave equation of the metric perturbation

$$\Box\bar{h}^{\mu\nu} = 0$$

This is clearly a wave equation with solution

$$\bar{h}^{\mu\nu} = A^{\mu\nu}\exp ik^\alpha x_\alpha \quad \text{with} \quad k^\alpha k_\alpha = 0$$

The gauge condition implies

$$\bar{h}^{\mu\nu}_{,\nu} = A^{\mu\nu}k_\nu = 0$$

Since we still have a little bit more freedom in the Lorentz gauge, aka

$$\Box\xi^\mu \to \Box(\xi^\mu + \eta^\mu) = \bar{h}^{\mu\nu}_{,\nu}$$

is still satisfied if we have $\Box\eta^\mu = 0$. So we have four dof to choose. we will use them to constrain two following condition

$$A^\nu_\nu = 0 \quad \text{and} \quad A_{\mu\nu}u^\nu = 0$$

where the reference 4-velocity is chosen as $u = (1, 0, 0, 0)$ in the Lorentz frame. This is called the **transverse-traceless** gauge. Orient the wave vector $k = (\omega, 0, 0, \omega)$ in z direction. As the result, the perturbation is

$$h_{\mu\nu}^{(TT)} = \begin{bmatrix} 0 & & & \\ & h_{xx} & h_{xy} & \\ & h_{xy} & -h_{xx} & \\ & & & 0 \end{bmatrix} = h_+ \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{bmatrix} + h_\times \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 0 \end{bmatrix}$$

There are two types of polarizations.

Generation

The goal is to solve

$$\Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}$$

The Green's function for $\Box$ is well known

$$\Box G(t, x) = \delta^4(t, x) \quad \rightarrow \quad G(t, x) = -\frac{\delta(t - |x|)}{4\pi|x|}$$

So the solution is

$$\begin{aligned} \bar{h}^{\mu\nu}(t, x) &= -16\pi G \int d^4x' T^{\mu\nu}(x') G(x - x') \\ &= 16\pi G \int dt' d^3x' T^{\mu\nu}(t', x') \frac{\delta(t - |x| - (t' - |x'|))}{4\pi(|x| - |x'|)} \\ &= 4G \int d^3x' \frac{T^{\mu\nu}(t - (|x| - |x'|), x')}{|x| - |x'|} \\ &\approx \frac{4G}{|x|} \int d^3x' T^{\mu\nu}(t - |x|, x') \end{aligned}$$

where $|x|$ is the distance to Earth Mpc away and $|x'|$ is the distance integrated around the mass, $|x'| \ll |x|$. we invoke the tensor virial theorem:

$$\partial_t^2 \int d^3x T^{00} x^i x^j = 2 \int d^3x T^{ij}$$

The component $\bar{h}^{0\mu}$ is 0 due to energy conservation stuff. We only care about

$$\bar{h}^{ij} = \frac{2G}{r} \frac{d^2}{dt^2} \int d^3x T^{00} x^i x^j = \frac{2G}{r} \frac{d^2}{dt^2} I^{ij}$$

In the TT gauge, the equation becomes

$$h_{ij}^{TT}(t, x) = \frac{2G}{r} \frac{d^2}{dt^2} I^{kl} \left(P_{li} P_{kj} - \frac{1}{2} P_{lk} P_{ij} \right)$$

where $P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j$ is the projection tensor that projects anything perpendicular to the unit wave vector k. This is the **quadrupole formula**.

Detection

To detect GW, we measure the time difference between two lights traveled in the 4-km arm. The light is special because it travels in null geodesics

$$g_{\mu\nu} dx^\mu dx^\nu = 0$$

In the simplest case where the interferometer is aligned perfectly with h_+-polarized GW, the beam traveled in x and y arms have

$$-c^2dt^2 + (1+h_+)dx^2 = 0 \quad \rightarrow \quad t_x = \int_0^{2L} \frac{1}{c}\sqrt{1+h_+}dx = \frac{L}{c}(2+h_+)$$

$$-c^2dt^2 + (1-h_+)dy^2 = 0 \quad \rightarrow \quad t_y = \int_0^{2L} \frac{1}{c}\sqrt{1-h_+}dx = \frac{L}{c}(2-h_+)$$

So the time difference is

$$\delta t = t_x - t_y = \frac{2L}{c}h_+$$

LIGO can measure h_+ at the order of 10^{-23}.

- Black hole
  - Schwartzchild metric

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2\,dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}\,dr^2 + r^2d\Omega^2$$

where $r_s = 2GM/c^2$ is Schwartzchild radius. The Sun has 3 km Schwartzchild radius.

  - Equation of motion

In the spherical coordinate, the 4-momentum vector is $p = (p^0, p^r, p^\theta p^\phi)$. The geodesic equation says

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2}g_{\mu\nu,\beta}p^\mu p^\nu$$

Since the metric only has r-dependent terms, we have

Massive particles	**Massless particles**
$m\dfrac{dp_r}{d\tau}$	$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = 4H^2\tilde{\delta}$

10 *Cosmology*

- Olbers Paradox - Why is the night sky dark?
  - Flux from star reaching us is

$$f_{obs} = \frac{f_* 4\pi r_*^2}{4\pi d^2} = \frac{f_* d\Omega}{\pi}$$

where $d\Omega$ is the solid angle subtended on the sky by the star. Since all lines of sight eventually hit some stellar surface, the whole sky should radiate like a blackbody.

  - Resolution: the Universe must be finite in size or age.

- Cosmic Clocks

- Abundance ratios of isotopes

  Modeling of nuclear reactions tells initial ratio of isotopes, for example, ^{235}U and ^{238}U, when they were first produced in supernova explosion. Different isotopes have different lifetimes, so we know how much time has passed to reach current abundance ratio.

- White dwarf cooling

  Coolest white dwarf provides a lower limit on the age of Universe.

- Age dating of globular clusters

Big Bang Cosmology

- Cosmological principle

  The Universe is

  - Isotropic: from fluctuations of CMB.
  - Homogeneous: from Hubble's law and Copernican principle. No one is special in Universe.

  Note that these two are not mutually implied. A rotating homogeneous universe is not isotropic. A spherically symmetric universe is isotropic, but might not be homogeneous (radially changing density).

- Friemann-Lemaitre-Robert-Walker (FLRW) metric

  - Hypersphere

    A 2D positively curved surface can be embedded to a 2-sphere, aka a common sphere in 3D space.

    $$x^2 + y^2 + z^2 = a^2$$

    where a is the radius of curvature. Taking derivative gives $xdx + ydy + zdz = 0$. So the line element is

    $$dl^2 = dx^2 + dy^2 + dz^2 = dx^2 + dy^2 + \frac{(xdx + ydy)^2}{a^2 - x^2 - y^2} = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

    Only two parameters are needed. The actual space is finite and unbounded (without edges).

  - FLRW metric

    Similarly, our 3D world can be embedded to the surface of a 3-sphere

    $$\begin{aligned} dl^2 &= dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{a^2 - x^2 - y^2 - z^2} \\ &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 + \frac{r^2 dr^2}{a^2 - r^2} \\ &= \frac{dr^2}{1 - r^2/a^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \end{aligned}$$

    If we normalize distance parameters and add time part with a **curvature parameter** k:

    $$ds^2 = c^2dt^2 - dl^2 = c^2dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

    This is **Friemann-Lemaitre-Robert-Walker (FLRW) metric**. For $k = 1$, the space is positively curved and $k = 0$ means a flat space. So $k = -1$ describes a negatively curved space where each point is a saddle point. The coordinate (r, θ, ϕ) is **comoving**, meaning the galaxy at coordinates (r, θ, ϕ) remains at those coordinates. Only the scaling $a(t)$ has time dependence.

- Proper distance
  The instantaneous distance from us to galaxy at coordinate r, aka **proper distance** is

$$l = \int_0^r dl = a(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} a\sin^{-1} r & \text{for } k=+1\text{, closed} \\ ar & \text{for } k=0\text{, flat} \\ a\sinh^{-1} r & \text{for } k=-1\text{, open} \end{cases}$$

- Hubble velocity

$$v = \dot{l} = \dot{a}\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \frac{\dot{a}}{a} l = H(t) l$$

  where $H(t)$ is the **Hubble parameter**.

Friedmann equation

- From Einstein's theory of gravity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  with FLRW metric

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -\frac{a^2}{1-kr^2} & & \\ & & -a^2r^2 & \\ & & & -a^2r^2\sin^2\theta \end{bmatrix}$$

  After a fair amount of algebra, we have Einstein tensor components

$$G_{00} = \frac{3}{c^2a^2}\left(\dot{a}^2 + kc^2\right) \quad \text{and} \quad G_{11} = -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{c^2(1-kr^2)}$$

  The energy-momentumt tensor is

$$T_{\mu\nu} = (P + \rho c^2)\frac{v_\mu v_\nu}{c^2} - P g_{\mu\nu}$$

  The (0, 0) and (1, 1) components of Einstein equation give **Friedmann equations**:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

  The other two diagonal components are redundant due to the isotropy inherent to the FLRW metric. Given the EoS, the three unknown functions $a(t)$, $\rho(t)$, and $P(t)$ can be solved.

- Third Friedmann equation
  Combine these two equations above, we have

$$\dot{\rho}c^2 = -3\frac{\dot{a}}{a}(\rho c^2 + P)$$

  This is called **energy conservation equation**.

- Newtonian interpretation

- Consider a test mass m at the edge of a spherical region with mass M and radius a. The conservation of energy implies

$$\frac{1}{2}m\dot{a}^2 - \frac{GMm}{a} = E$$

E is some constant form of energy. Assume uniform density: $M = 4\pi a^3\rho/3$, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{ma^2}$$

Looks like the first Friedmann equation, which basically says the energy of Universe is locally conserved.

- The equation of motion of the test mass is

$$m\ddot{a} = -\frac{GMm}{a^2} = -\frac{4\pi G}{3}a\rho m \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho$$

Looks like the second Friedmann equation without pressure term.

- Under **adiabatic** expansion (because of cosmological principle, no particular direction of energy flow is allowed), the thermodynamics states

$$dU = -PdV$$

Substitute $U = \rho_{rad}c^2V$ with $dV = 4\pi a^2 da$, and take time derivative, we recover the third Friedmann equation

$$\dot{\rho}c^2 = -\frac{\dot{V}}{V}(\rho c^2 + P) = -3\frac{\dot{a}}{a}(\rho c^2 + P)$$

Evolution of Universe

Recall the EoS:

$$P = P_{matter} + P_{radiation} = \frac{\rho_{mat}k_BT}{\bar{m}} + \frac{1}{3}\rho_{rad}c^2$$

Matter-dominated	**Radiation-dominated**

- Early Universe

The third Friedmann equation under

matter-dominated Universe (negligible pressure $P \ll \rho c^2$) is

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho \propto a^{-3}$$

radiation-dominated Universe (pressure $P \approx \rho c^2/3$) is

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho \propto a^{-4}$$

Assume $a \ll 1/c$ at the very beginning, the first Friedmann equation with ρ is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \approx \frac{8\pi G}{3}\rho \propto a^{-3}$$

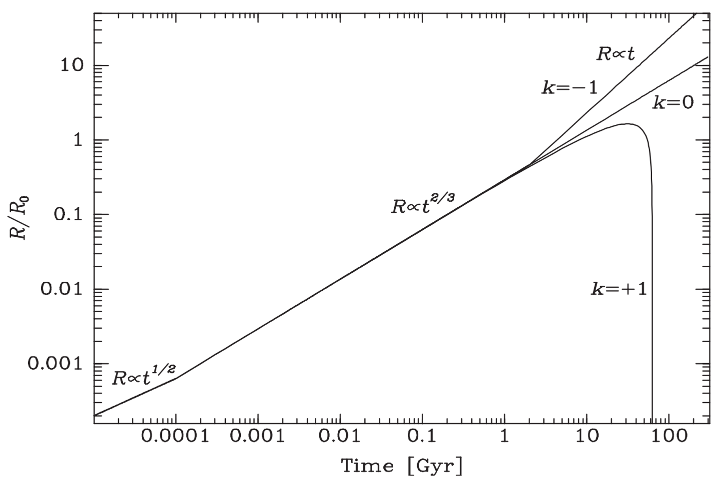
$$\Rightarrow \quad a \propto t^{2/3}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \approx \frac{8\pi G}{3}\rho \propto a^{-4}$$

$$\Rightarrow \quad a \propto t^{1/2}$$

This scaling justifies the assumption $\lim_{t\to 0}\rho(t) = 0$. No matter how the cosmological spacetime is curved, the singularity in density exists as $t \to 0$, aka the Big Bang. The coordinate t is universal, or cosmic. In principle, all observers could synchronize their clocks by agreeing that at $t \to t_0$, their local mean density reaches a particular value $\rho(t_0) = \rho_0$.

- Late Universe
  The curvature term can't be ignored as $a(t)$ increases in time.
  - Flat $k = 0$
    From the first equation:
    $$\rho_c = \frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2 = \frac{3H^2}{8\pi G}$$
    If the measured density is larger/smaller than ρ_c, it will show the Universe is closed/open. So ρ_c is called **critical density**. A flat Universe will keeps expanding.
  - Closed $k = +1$
    At time when $a(t)$ is large enough such that the right-hand side of the first equation becomes negative. Then $a(t)$ starts to decrease until the "Big Crunch" happens.
  - Open $k = -1$ It's easy to see that the term $-kc^2/a^2$ becomes dominant at later times and $\dot{a} = c$, or $a \propto t$. The Universe is expanding even faster.



- From radiation to matter dominant
  The matter density $\rho_m \propto a^{-3}$ and radiation density $\rho_r \propto a^{-4}$ evolve separately. Today's density can be measured
  $$\rho_{r,0} = \frac{4\sigma}{c}T_{CMB}^4 = 4.2 \times 10^{-13}\ \text{erg/cm}^3$$
  where T_{CMB} is the temperature of CMB at 2.73 K. The matter density can be estimated
  $$\rho_{m,0}c^2 \approx 0.3\rho_{c,0}c^2 = 2.5 \times 10^{-9}\ \text{erg/cm}^3$$
  So we are clearly at matter-dominated era. The transition occurs at
  $$\frac{\rho_r}{\rho_m} = \frac{1.7\rho_{r,0}a_0^4}{a^4}\frac{a^3}{\rho_{m,0}c^2a_0^3} = 1 \quad \Rightarrow \quad \frac{a_0}{a} \approx 3500$$
  where 1.7 comes from energy density due to **cosmic neutrino background**.
- Age of Universe The age of Universe until now can be roughly derived:
  $$H_0 = \frac{\dot{a}_0}{a_0} = \frac{2}{3t_0}$$
  so $t_0 = 2/3H_0$. For an extreme case of open universe, we have $a(t) = ct$ so $t_0 = 1/H_0$.

Structure Formation

The CMB reveals a density perturbation of $\delta = \delta\rho/\rho = 10^{-5}$, which is very different from the current galaxy $\delta = 10^6$. Need a theory to explain this.

Linear perturbation

Recall the fluid equation

$$\partial_t \rho_0 + \partial_i(\rho_0 v_0^i) = 0$$
$$\frac{dv_0^i}{dt} = (\partial_t + v_0^j \partial_j) v_0^i = \partial^i \left(\Phi_0 + \frac{P_0}{\rho} \right)$$
$$\nabla^2 \Phi_0 = 4\pi G \rho_0$$

Unlike what's happened in Jeans instability, the background density $\rho_0 = \rho_0(t)$ is not static but evolving with time. This is due to the cosmic expansion of FLRW metric $v = Hx$, where x is the physical coordinate or proper distance. Now we add perturbation $\delta\rho$, δv and keep only the first order:

$$\partial_t \delta\rho + \partial_i(\delta\rho v_0^i + \rho_0 \delta v^i) = 0$$
$$\partial_t \delta v^i + \delta v^j \partial_j v_0^i + v_0^j \partial_j \delta v^i = -\partial^i \left(\delta\Phi + \frac{\delta P}{\rho} \right)$$
$$\nabla^2 \delta\Phi = 4\pi G \delta\rho$$

Note that $\partial_i(\delta\rho v_0^i) \approx \delta\rho \partial_i H x^i = 3H\delta\rho$, where we ignore anything multiplies $v_0 \cdot \nabla$ (why?). Similarly, $\delta v^j \partial_j v_0^i = \delta v^j \partial_j H x^i = H \delta v^i$. We can replace $\delta = \delta\rho/\rho(t)$ such that $\partial_t \delta\rho = \partial_t(\delta\rho_0) = \rho_0 \partial_t \delta + \delta(-3H\rho_0)$. The equations are now

$$\rho_0 \partial_t \delta + \delta(-3H\rho_0) + 3H\delta\rho + \rho_0 \nabla \cdot \delta v = \rho_0(\partial_t \delta + \nabla \cdot \delta v) = 0$$
$$(\partial_t + H + v_0^j \partial_j) \delta v^i = -\partial^i \left(\delta\Phi + c_s^2 \delta \right)$$
$$\nabla^2 \delta\Phi = 4\pi G \delta\rho$$

Take ∂_i on both sides of Euler equation and replace $\nabla \cdot \delta v = -\dot{\delta}$, we have what we want:

$$\ddot{\delta} + 2H\dot{\delta} = (4\pi G \rho_0 + c_s^2 \nabla^2)\delta$$

The ∇^2 is still with respect to the physical coordinate. If we use comoving coordinate $x = r/a(t)$

$$\ddot{\delta}(t,x) + 2H\dot{\delta}(t,x) = \left(4\pi G \rho_0 + \frac{c_s^2}{a^2} \nabla^2 \right) \delta(t,x)$$

In Fourier space of comoving wave vector k:

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = \tilde{\delta} \left(4\pi G \rho_0 - \frac{v_s^2 k^2}{a^2} \right) \qquad \text{non-relativistic matter}$$

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = \tilde{\delta} \left(\frac{32\pi}{3} G \rho_0 - \frac{v_s^2 k^2}{a^2} \right) \qquad \text{relativistic radiation}$$

where

$$v_s = \begin{cases} c_s = 5 \text{ km/s} & \text{, for non-relativistic baryon} \\ \sigma & \text{, for cold dark matter} \\ c/\sqrt{3} & \text{, for relativistic radiation} \end{cases}$$

Perturbation growth

Here we assume the mode has length much larger than Jeans length. The density is the critical density

$$\rho_0 = \frac{3H^2}{8\pi G}$$

Matter-dominated	**Radiation-dominated**
$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = \frac{3}{2}H^2\tilde{\delta}$	$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = 4H^2\tilde{\delta}$

Assume $\tilde{\delta} \propto t^n$, the solutions are

$n = \begin{cases} -1 (\text{decay mode}) \\ \frac{2}{3} (\text{growth mode}) \end{cases}$	$n = -1(\text{decay mode}), +1(\text{growth mode})$
$\tilde{\delta} \propto t^{2/3} = a(t)$	$\tilde{\delta} \propto t = a^2(t)$
$M_\odot \sim 10^5 M_\odot$	$M_\odot \sim 10^{16} M_\odot$

Cold dark matter

CDM has a low σ that starts growing before the recombination. After recombination, the baryonic matter joins the potential well of CDM and starts from there. Otherwise, it doesn't have enough time to form large-scale structures.

Dark Energy

Cosmological constant

In Einstein's equation, we have a freedom to add an extra term $\Lambda g_{\mu\nu}$. The Friedmann equations with **cosmological constant** Λ are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}$$

with the third equation unchanged. This extra term can be incorporated in the $8\pi G\rho/3$ term, so it represents a certain energy density:

$$\epsilon_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

Evolution with dark energy

This Λ doesn't affect the early Universe when $a(t)$ is sufficiently small. However, it will dominant when $a(t)$ grows with time. Eventually, we will have

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) \approx \frac{\Lambda}{3} \quad \Rightarrow \quad a(t) \propto e^{\sqrt{\Lambda/3}t}$$

where Hubble parameter becomes a constant and Universe is expanding exponentially. Our event horizon size will also shrink exponentially with time. Light sources around us will get redshifted to infinity.

- Dimensionless parameters we can rewrite first Friedmann equation as

$$\frac{8\pi G\rho}{3H^2} + \frac{\Lambda}{3H^2} = \Omega_m + \Omega_\Lambda = 1 + \frac{kc^2}{a^2H^2}$$

So we would have $\Omega_m + \Omega_\Lambda > 1$ if the Universe is open, vice versa.

- Tests of big bang cosmology

  - Cosmological redshift

    Consider two wavefronts emitted from galaxy at comoving coordinate r, with wavelength $\lambda = c\Delta t$. Since light follows null geodesics:

$$c^2dt^2 - a^2(t)\frac{dr^2}{1-kr^2} = 0$$

    Integrate it up

$$\int_t^{t_0} \frac{dt}{a(t)} = \frac{1}{c}\int_r^0 \frac{-dr}{\sqrt{1-kr^2}} = \int_{t+\Delta t}^{t_0+\Delta t_0} \frac{dt}{a(t)} = \int_t^{t_0} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a(t)} - \int_t^{t+\Delta t} \frac{dt}{a(t)}$$

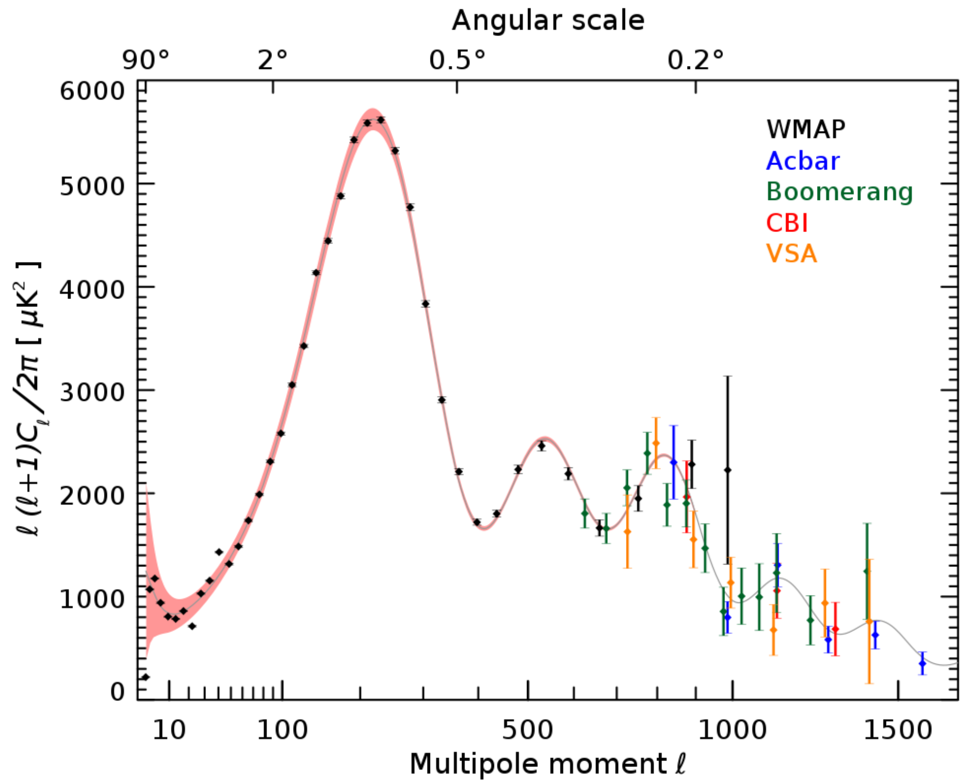
    Therefore,

$$\frac{\Delta t_0}{\Delta t} = \frac{a(t_0)}{a(t)} = \frac{\lambda(t_0)}{\lambda(t)} = 1 + z$$

    where z is the **cosmological redshift**. The present scale factor is $a_0 = a(t_0) = 1$. So the light will always be redshifted. without exception.

- Cosmic Microwave Background

  Since the density of Universe decreases with time, there must have been early time when the Universe is in a hot dense state and all atoms are ionized. So the main opacity is from electron scattering. As Universe cools, fewer photons are energetic enough to ionize hydrogen, so proton and electron recombines. Indeed, every line of sight does reach a thermal blackbody, aka **surface of last scattering**, but the temperature is low (2.7 K).



- Angular horizon

  The horizon scale at recombination is roughly

$$l_{horizon} \sim ct_{horizon} = ca_{horizon}^{3/2}$$

  assuming the matter-dominated universe. The angular distance is

$$d_A \sim \frac{1}{1+z}$$

  Therefore

$$\theta \sim \frac{l_{horizon}}{d_A} \sim a^{1/2} \sim \frac{1}{\sqrt{1+z}} \sim 1.7^{\circ}$$

- Anisotropy of CMB

  The anistropy is measured as $\delta T/T \sim 10^{-5}$, so the Universe is extremely isotropic.

  - Horizon problem

    At the time of recombination, the horizon size only subtends about 2deg on the sky today. So the CMB photons from opposite direction have never been in causal contact until now, yet they are almost exactly the same.

  - Flatness problem

    For the universe to be flat today, it has to be very flat at the beginning. Fridmann equation says:

$$H^2(a) = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} = H^2(a)(\Omega(a) - \frac{kc^2}{a^2H^2})$$

$$\Omega(a) - 1 = \frac{kc^2}{a^2H^2}$$

    For radiation dominated universe, $a \propto t^{1/2}$, so $\Omega(a) - 1 \propto t$. The error has to be extremely small for large current t.

- Uncertainty
  The uncertainty is large at low l because we only have one sky, so no averaging.

- Inflation

  - The inflation theory tries to explain these problems. During inflation at very early Universe, the vacuum energy with negative pressure caused an exponential expansion of a, which led causally connected regions to expand beyond the size of horizon. It also asks

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0$$

  such that the inverse term is small at small a. This leads to $\ddot{a} > 0$, and the cosmological constant can give this exponential expansion??

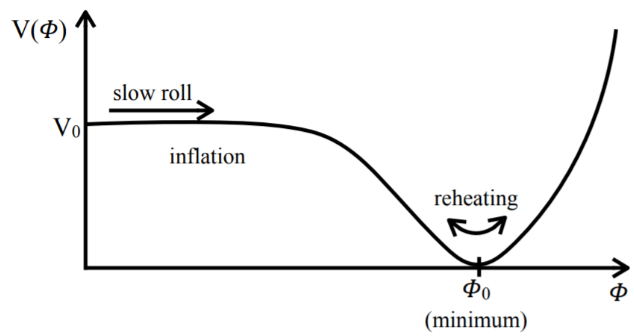
  - The cosmological could work, but it has two key problems:

    - acts too late
    - is constant, so it won't stop

  We need to invent another field ϕ with standard Lagrangian $\mathscr{L} = \partial_\mu \phi \partial^\mu \phi / 2 - V(\phi)$. The field evolution can be solved by relating $T_{\mu\nu}$ to Friedmann equations. Just remember the slow-roll approximation

$$\dot{\phi}^2 < V(\phi)$$

  The solution of slow-roll potential is shown to the right.



185 Questions

I *Solar Systems*

1. Discuss the observational status of searches for planets outside our own solar system. What are the most interesting things we've learned about exoplanets so far? Summarize the current state of the field of exoplanet observation and what it hopes to accomplish. Describe what's where in our solar system, including planets, dwarf planets, moons, asteroids and comets.

- Observational status: nearly 1000 exoplanets have been found. The first exoplanet found is called 51 Peg b. The host star is 51 Pegasi a in Pegasus constellation.
  - Doppler (radial-velocity) method - deduced from the periodic radial-velocity Doppler shifts in the spectral lines of the primary star, due to the motion of common center of mass. A Jupiter-sized planet causes very tiny Doppler shift: $\Delta\lambda/\lambda = v/c = 31/3e8 = \mathbf{10^{-7}}$. Our sensitive spectroscopic technique can detect as low as a few ppb.
  - Planetary transits - probability of observing a transit is $(\mathbf{r}_* + \mathbf{r}_\mathbf{p})/\mathbf{a}$, where a is distance between star(*) and planet(p).
  - Direct imaging - very challenging because the contrast between light reflected by planet and stellar light is very low, around 10 ppb. Or use coronagraph to block some of the stellar light to enhance contrast.
  - Microlensing - the planet near the lensing star causes perturbation of the magnification. This perturbation can be easily detected during a microlensing event.
  - Timing - some planets are discovered by their effects on the arrival times of signals from the system, such as radio pulses from a pulsar.
  - Astrometry - Gaia mission.
- Interesting things
  - Kepler mission

    It finds thousands of new exoplanets with orbital periods of less than 400 days and sizes larger than Mars. Still, we can use the Kepler discoveries to extrapolate the distribution of planets in our Galaxy. The data so far imply that planets like Earth are the most common type of planet, and that there may be 100 billion Earth-size planets around Sun-like stars in the Galaxy. About 2600 planetary systems have been discovered around other stars. In many of them, planets are arranged differently than in our solar system.
- Goals
  - Habitable exoplanets near solar system (20 pc).
  - Extraterrestrial life
  - Planet properties
  - Global statistics
- Our solar system
  - Planets
    - Terrestrial (rocky): 水金地火
    - Gas giants: 木土

        - Ice giants: 天海
    - Dwarf planets: Pluto, Ceres
    - Asteroids:
        - Minor planets in the asteroid belt. The largest one is Ceres. Total mass is roughly 4% of our moon.
        - They are planetesimals to form protoplanets. However, the **gravitational perturbation of Jupiter** causes them to collide more than to fuse together.
    - Comets: icy rocks that has coma when passing near Sun due to the stellar wind.

2. Describe, qualitatively, the standard model for the formation of the solar system, and discuss the observational evidence supporting this model.

- Old model - core accretion model
    - Approximately 4.6 billion years ago, the solar system was a cloud of dust and gas known as a **solar nebula**. Gravity collapsed the material in on itself as it began to spin, forming the sun in the center of the nebula. The remaining material began to clump together. Small particles drew together, bound by the force of gravity, into larger particles. The solar wind **swept away lighter elements** from the closer regions, leaving only rocky materials to create terrestrial worlds. But farther away, the solar winds had less impact on lighter elements, allowing them to coalesce into gas giants. In this way, asteroids, comets, planets and moons were created.
    - But the need for a rapid formation for the giant gas planets is one of the problems of core accretion. According to models, the process takes several million years, longer than the light gases were available in the early solar system.
    - At the same time, the core accretion model faces a **migration issue**, as the baby planets are likely to spiral into the sun in a short amount of time.
- New model including exoplanet statistics are being studied.

3. (19.) Explain the steps you would take to show that both the orbital period and the total energy of a Keplerian orbit depend only on the semimajor axis, and not on the orbital eccentricity.

- Orbit energy

The total energy is conserved, so at **perigee point**

$$
\begin{aligned}
E &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{Gm_1m_2}{r} \\
&= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \\
&= \frac{1}{2}\mu\dot{r}^2 + Gm_1m_2\left(\frac{a(1-e^2)}{2r^2} - \frac{1}{r}\right) \\
&= Gm_1m_2\left(\frac{a(1-e^2)}{2r_p^2} - \frac{1}{r_p}\right) \\
&= -\frac{Gm_1m_2}{2a}
\end{aligned}
$$

where $r_p = a(1-e)$. So E it's not dependent on e.

- Period

  The diffeq of dynamics is

$$
\dot{r} = \sqrt{\frac{2Gm_1m_2}{\mu}\left(-\frac{1}{2a} + \frac{1}{r} - \frac{a(1-e^2)}{2r^2}\right)} = \sqrt{\frac{2G(m_1+m_2)}{r^2}\left(-\frac{r^2}{2a} + r - \frac{a(1-e^2)}{2}\right)}
$$

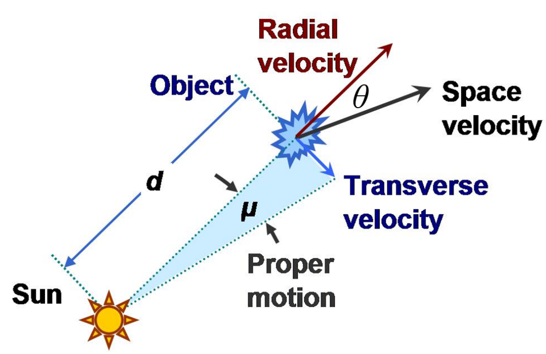
  Represent $r = a(1 - e\cos\psi)$

$$
d\psi(1 - e\cos\psi) = dt\sqrt{\frac{G(m_1+m_2)}{a^3}}
$$

  The eccentricity e term dies if you integrate for a full circle.

4. (15.) What is the moving cluster method of determining distances and how does it work?

- This is a method find distance from a remote cluster. From the perspective effect, all stars in the cluster, having (roughly) the same space velocity, look like they are heading to the same vanishing point. So you know angle θ between Earth and **vanishing point** (approximately the same as from star to the vanishing point, assuming the vanishing point is far far away).

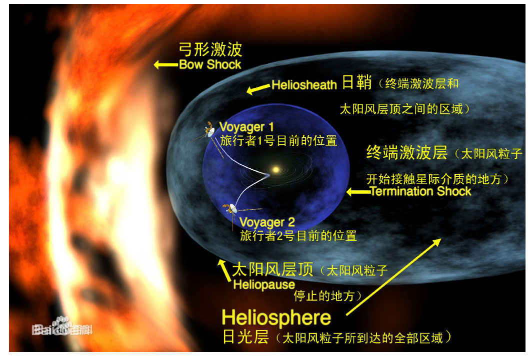


- Then by measuring radial velocity, you can find transverse velocity $v_r \tan\theta$. The proper motion μ in arcsec/yr can be measured, so

$$
d = \frac{v_r \tan\theta}{\mu}
$$

5. (25.) Calculate the approximate distance to the heliopause. Does the local interstellar medium begin at this boundary? Explain

- The Sun moves at around 200 km/s in the galaxy. Its stellar wind and dust interacts with ISM.
  - Heliosphere: the bubble continuously inflated by the sola wind.
  - Termination shock: the boundary where supersonic solar wind gets stopped by the interstellar medium and become subsonic.
  - Heliosheath: the transitional region between inner heliosphere and external atmosphere.
  - Heliopause: Outmost edge of the heliosphere. The two Voyagers spacecraft have reached the heliopause and now in the interstellar space.



- Distance to heliopause

  The heliopause is the equilibrium of internal solar wind pressure and external ISM pressure. The pressure inside is

$$P_{in} = \langle npv \rangle = \frac{1}{3} n_{in} m v_\infty^2$$

  where the density inside is determined by the **mass loss rate** of the Sun $n_{in} = \dot{M}_\odot / mv4\pi r^2$. The external pressure is

$$P_{out} = n_{out} k_B T$$

  The density and temperature outside is measured as 0.05 cm^{-3} and 10^4 K by Voyager, and mass loss rate of Sun is $10^{-13} M_\odot$/yr. Therefore, the distance is

$$r = \sqrt{\frac{\dot{M}_\odot v_\infty}{12\pi n k_B T}} = 143 \text{ AU}$$

  close to the real value of **123 AU**.

  The other way to estimate is to calculate the mean-free-path of particles, which is $1/(1\text{cm}^{-3}\pi(1\text{Å})^2) = 212$ AU. The actual cross section is larger because the solar wind plasma creates magnetic fields that interact with ISM.

- Yes, the interstellar medium starts from the boundary of heliosphere, the heliopause.

6. (21.) Describe the physics involved in the Earth-Moon interaction whereby the Earth's rotation rate is slowing and the orbital separation is increasing.

- As shown to the right, the bulge is little ahead of the moon, so it's feeling an extra little bit of force pulling it forward. So it's rotation is **accelerating**, and it moves to an orbit farther away (3 cm/yr).
- Conserved angular momentum → Earth is rotating slower (1 sec/50000yrs).
- This continues until the earth is tidally locked to the moon, aka synchronous rotation. So the bulge is permanently aligned with the moon.

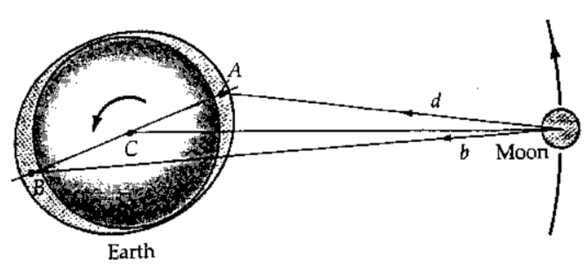


Figure 3–17

Tidal evolution. The Earth's tidal bulges (A and B) are driven by friction to be ahead of the Moon's orbital position. The friction slows the Earth's rotation, and the bulges accelerate the Moon in its orbit.

7. (23.) Describe Oort's theory of the origin of comets.

- Comets are believed to have two separate points of origin in the Solar System.
  - Short-period comets (those with orbits of up to 200 years) are generally accepted to have emerged from the **Kuiper belt** or scattered disc, two linked flat discs of icy debris beyond Neptune's orbit at 30 AU and jointly extending out beyond **100 AU** from the Sun.
  - Long-period comets, such as comet HaleBopp, whose orbits last for thousands of years, are thought to originate in the Oort cloud.
- The Oort cloud is a hypothesized spherical cloud of comets which may lie roughly **50,000 AU**, or nearly a light-year, from the Sun. This places the cloud at nearly a quarter of the distance to Proxima Centauri, the nearest star to the Sun.
- Astronomers believe that the matter composing the Oort cloud formed closer to the Sun and was scattered far out into space by the **gravitational effects** of the giant planets early in the Solar System's evolution. It is also speculated that the Oort cloud is, at least partly, the product of an exchange of materials between the Sun and its sister stars as they formed and drifted apart.

8. (24.) What is the 3:2 resonance involving Neptune and Pluto/Charon? And that involving the day and year on Mercury?

- In celestial mechanics, an orbital resonance occurs when two orbiting bodies exert a regular, periodic gravitational influence on each other, usually due to their orbital periods being related by a ratio of two small integers. The physics principle behind orbital resonance is similar in concept to pushing a child on a swing, where the orbit and the swing both have a natural frequency, and the other body doing the "pushing" will act in periodic repetition to have a cumulative effect on the motion. Orbital resonances greatly enhance the mutual gravitational influence of the bodies, i.e., their ability to alter or constrain each other's orbits. In most cases, this results in an unstable interaction, in which the bodies exchange momentum and shift orbits until the resonance no longer exists. Under some circumstances, a resonant system can be stable and self-correcting, so that the bodies remain in resonance. Examples are the 1:2:4 resonance of Jupiter's moons Ganymede, Europa and Io, and the 2:3 resonance between Pluto and Neptune. Unstable resonances with Saturn's inner moons give rise to gaps in the rings of Saturn. The special case of 1:1 resonance (between bodies with similar orbital radii) causes large Solar System bodies to eject most other bodies sharing their orbits.
- Radar observations in 1965 proved that the planet has a 3:2 spin-orbit resonance, rotating three times for every two revolutions around the Sun; the eccentricity of Mercurys orbit makes this resonance stable at perihelion, when the solar tide is strongest, the Sun is nearly still in Mercury's sky. Simulations indicate that the orbital eccentricity of Mercury varies **chaotically** from nearly zero (circular) to more than 0.45 over millions of years due to perturbations from the other planets. This is thought to explain Mercury's 3:2 spin-orbit resonance (rather than the more usual 1:1), since this state is more likely to arise during a period of high eccentricity.

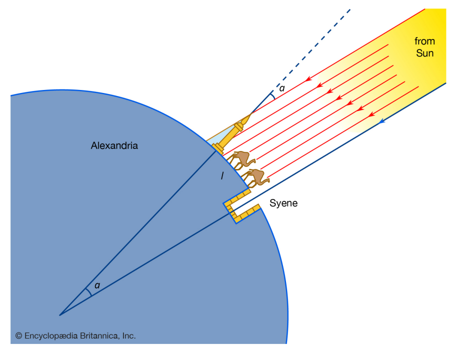
9. (13.) How is the astronomical unit measured in modern astronomy, and to what accuracy?

- AU is defined to be 149, 597, 870, 700m, which is approximately the distance between the sun and the earth.
- Modern measurement of AU:
    - Measured by **radar and telemetry from space probes.**
    - Nowadays, the AU is defined absolutely.

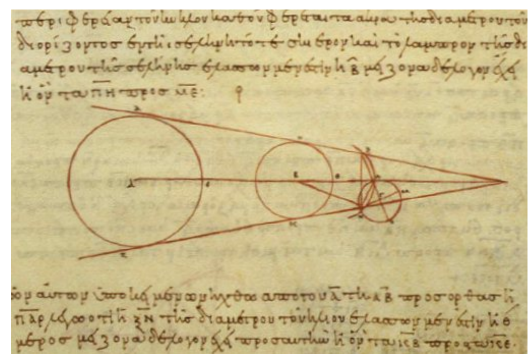
  This measurement yields 149, 597, 870, 700 ± 3m.

10. (1.) How did the Greeks determine the size of the Earth and the distance to the Moon?

- In 250 B.C., there was a Greek guy called Eratosthenes who came up with the idea of measuring the size of the Earth. He knows there was one day in summer, during which the sunlight shines straight into the bottom of the well in Egypt. He then measured the angle α shown in the right picture on the same day in Alexandria, and measured distance from his city to Egypt. With this data, he can calculate the circumference, thus the size, of spherical Earth.



- Two ways to measure distance to the moon:
    - Another Greek guy Aristarchus, at the same time (300 B.C.), used lunar eclipse to measure the distance. He knows Earth radius r and the period of moon around Earth $T = 30$ days. When the full moon eclipse happens, he counts how much time t the moon is under Earth's shadow. So we have $t/T = 2r/(2\pi d)$ where d is the distance we want.
    - The second way is to use lunar parallax. The ancient Greeks also know about this, but they don't have friends a few kilometers away to record the position of moon on the sky at the same time.



II *Stellar Structure & Evolution*

11. (33.) Describe the spectral classification scheme for stars: O, B, A, F, G, K, M. What are the characteristic effective temperatures for stars of each class? What are the characteristic luminosities for main-sequence stars of each class?

The modern classification is called Morgan-Keenan System. It has three levels

- Hot → cool: O, B, A, F, G, K, M
    - Subdivide hot → cool: 0, 1, ..., 9

0 or Ia	I	II	III	IV	V	VI	VII
hypergiants	supergiants	bright giants	regular giants	sub-giants	main-sequence	sub-dwarf	white dwarf

Class	Effective temperature[1][2]	Vega-relative chromaticity[3][4][a]	Chromaticity (D65)[5][6][3][b]	Main-sequence mass[1][7] (solar masses)	Main-sequence radius[1][7] (solar radii)	Main-sequence luminosity[1][7] (bolometric)	Hydrogen lines	Fraction of all main-sequence stars[8]
O	≥ 30,000 K	blue	blue	≥ 16 $M_\odot$	≥ 6.6 $R_\odot$	≥ 30,000 $L_\odot$	Weak	~0.00003%
B	10,000–30,000 K	blue white	deep blue white	2.1–16 $M_\odot$	1.8–6.6 $R_\odot$	25–30,000 $L_\odot$	Medium	0.13%
A	7,500–10,000 K	white	blue white	1.4–2.1 $M_\odot$	1.4–1.8 $R_\odot$	5–25 $L_\odot$	Strong	0.6%
F	6,000–7,500 K	yellow white	white	1.04–1.4 $M_\odot$	1.15–1.4 $R_\odot$	1.5–5 $L_\odot$	Medium	3%
G	5,200–6,000 K	yellow	yellowish white	0.8–1.04 $M_\odot$	0.96–1.15 $R_\odot$	0.6–1.5 $L_\odot$	Weak	7.6%
K	3,700–5,200 K	light orange	pale yellow orange	0.45–0.8 $M_\odot$	0.7–0.96 $R_\odot$	0.08–0.6 $L_\odot$	Very weak	12.1%
M	2,400–3,700 K	orange red	light orange red	0.08–0.45 $M_\odot$	≤ 0.7 $R_\odot$	≤ 0.08 $L_\odot$	Very weak	76.45%

🌞 Luminosity class Our Sun is G2V.

12. (41.) Make a dimensional analysis of the equation of hydrostatic equilibrium using a polytropic equation of state to find a general mass-radius relation for spherically-symmetric, self gravitating bodies. For which two polytropic indices is the configuration unstable?

- Motivation: a polytropic EoS is used for degenerate matter

  Remember the hydrostatic and mass continuity equation

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho \quad \text{and} \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho$$

  It can be solvable if we assume a polytropic equation of state

$$P = K\rho^{1+\frac{1}{n}}$$

  The scaling relation can be easily worked out

$$M \sim r^{\frac{n-3}{n-1}}$$

- The unstable indices are $n = 3$ and $n = 1$.

13. (43.) Write down the basic equations of the p-p chain that provides the Sun's nuclear power

- Hans Bethe proposed the pp-chain, the hydrogen burning in the Sun. A proton wanders for 10 billion years before reacting with another proton

$$p + p \rightarrow d + e^+ + \nu_e$$

  One of the proton undergoes $\beta+$ decay to a neutron, positron and a neutrino. The positron annihilates with an electron and releases 1.4 MeV (around twice of electron mass 0.5 MeV). Within 1 s,

$$p + d \rightarrow {}^3\text{He} + \gamma \text{ (5 MeV)}$$

  With 400 years, the ^{3}He reacts to ^{4}He via three branches

- p-p I (>80% of all branches)

$$^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + 2p + 13 \text{ MeV}$$

The total energy released is $13 + 2 \times (5 + 1) = 25$ MeV.

- p-p II (~10%)

$$\begin{aligned} &^3\text{He} + {}^4\text{He} \rightarrow {}^7\text{Be} + \gamma \\ &^7\text{Be} + e^- \rightarrow {}^7\text{Li} + \nu_e \\ &^7\text{Li} + p \rightarrow 2{}^4\text{He} \end{aligned}$$

This is also called lithium burning.

- p-p III (rarely happens)

$$\begin{aligned} &^3\text{He} + {}^4\text{He} \rightarrow {}^7\text{Be} + \gamma \\ &^7\text{Be} + p \rightarrow {}^8\text{B} + \gamma \\ &^8\text{B} \rightarrow {}^8\text{Be} + e^+ + \nu_e \\ &^8\text{Be} \rightarrow 2{}^4\text{He} \end{aligned}$$

Same $\beta+$ decay happens on ^{8}B atom.

- p-p IV (only theoretically possible. Never observed)

Compared with rest energy, the conversion coefficient is only 0.7 %. The CNO cycle happens for more massive stars > 1.3 $M_\odot$

14. (54.) Describe the types of stellar evolution that lead to type Ia, type Ib and type II supernovae. What are the observational differences among these?

- Supernova Classifications

  The types are classified on the absorption lines.

  - Type I - no hydrogen

    Type I supernovas have similar peak intensity and light curves.

    - Type Ia - has Si II line
      - Type Ia starts from a close binary system of a white dwarf and a companion, which becomes a red giant. The accretion from red giant to the white dwarf increases its mass, until the point where degenerate pressure can't hold gravity anymore.
      - Accretion triggers runaway nuclear reaction (positive feedback reaction)
      - Collision of two white dwarfs
    - Type Ib - no Si

      Core collapse like Type II, but no hydrogen shell (maybe lost from stellar winds).
    - Type Ic - no He

      Even the helium layer is ejected before core collapse in type Ic supernova.

    Type Ib and Ic are sometimes called stripped core-collapse supernova.

- Type II - has hydrogen

  Takes place in much more massive stars when their core collapsing are halted by forming a neutron star.

15. (86.) What is a Cepheid variable? What role do Cepheids play as distance indicators? Explain the underlying stellar physics involved.

- Cepheid variable is a type of star that pulsates radially with stable period and luminosity. It has a He core so it's already evolved off the main sequence. It's one of the two stars with very stable periodic luminosity (the other is RR Lyrae stars). Cepheids also have a stable period-luminosity relation. If you know the period, you can infer its absolute luminosity, which gives you distance if you compare it with apparent luminosity.

- Cepheid variables can be used as standard candles, objects with known intrinsic luminosity, to measure distances and Hubble constant. It has high luminosity so it's easily observed. Hubble Space Telescope can see Cepheids in Virgo clusters.

- Kappa-mechanism

  - The periodic expansion and contraction of the surface layers of the variable causes periodic pulsation. This means the star actually increases and decreases in size periodically.
  - Helium is the gas thought to be most active in the process. He III (doubly ionized) is more opaque than He II. The more helium is heated, the more ionized it becomes. At the dimmest part of a Cepheid's cycle, the ionized gas is opaque, heated by the star's radiation, and expand. As it expands, it cools, and so becomes less ionized and therefore more transparent. Then the expansion stops, and reverses due to the star's gravitational attraction. The process then repeats.

16. (34.) From a physics perspective, how does the quantity (B-V) help to determine a star's effective (surface) temperature?

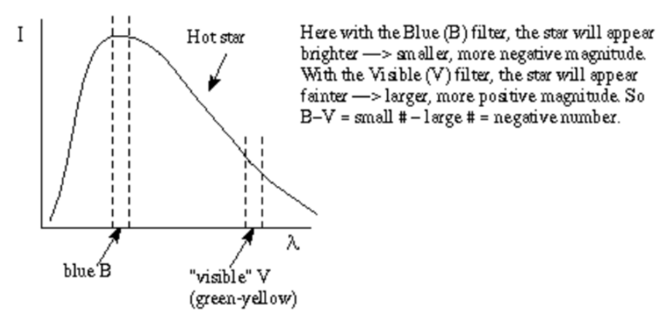
- The B-V value is defined as the flux ratio of Blue and Visible part of light.

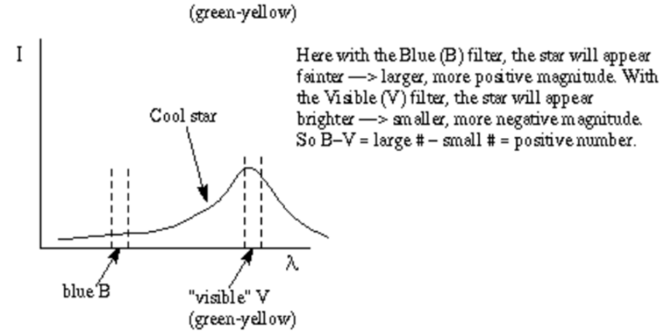
$$B - V = -2.5\left(\log\left(\frac{F_B}{F_{B,0}}\right) - \log\left(\frac{F_V}{F_{V,0}}\right)\right)$$
$$\propto 2.5\log\left(\frac{F_V}{F_B}\right)$$

  Remember Planck's law

$$I_\nu = \frac{4h}{c^2}\frac{\nu^3}{e^{h\nu/k_BT} - 1}$$

  You can tune the effective temperature T such that the flux ratio is equal to B-V value. Higher B-V ⇒ cooler stars.





17. (35.) Two stars are observed to have the same color and brightness. One of them is a giant at a greater distance than the other, which is a main sequence star. How could these be distinguished from spectroscopic measurements?

Because the atmospheres of **more luminous stars are less dense**, there are fewer collisions between atoms. Collisions can distort the energies of atomic orbitals, leading to broadening of the spectral lines. So in general, for stars of the same spectral type, narrower lines are usually produced by more luminous stars. Therefore, by looking at line width, we can distinguish the two.

18. (36.) If you are given an HR diagram, how would you go about constructing lines of constant stellar radius on the diagram?

Scaling relations

The luminosity has $L \propto r^2 T^4$. So for each constant r line, the luminosity goes $L \propto T^4$. The HR diagram reads $L \propto T^8$, so $L \propto M^2\sqrt{L} \Rightarrow L \propto M^4$, something between low-mass $\propto M^5$ and high-mass $\propto M^3$.

19. (41.) Make a dimensional analysis of the equation of radiative diffusion in stars to show that the luminosity of a star scales as its mass cubed, if the opacity is taken to be a constant.

Equation of radiative diffusion

$$\frac{dT}{dr} = -\frac{3L\kappa\rho}{4\pi r^2 16\sigma T^3}$$

For mass-dominanted EoS has

$$T \sim \frac{P}{\rho} \sim \frac{M}{r} \sim 1$$

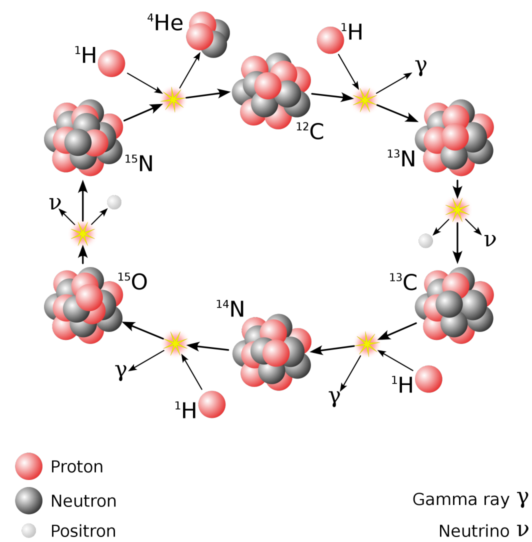
for main-sequence stars. So

$$L \sim \frac{T^4 r}{\kappa\rho} \sim \frac{M^3}{\kappa} \sim M^3$$

When the temperature is high enough, the hydrogens are fully ionized and opacity is just Thomson scattering cross section.

20. (45.) How does the CNO cycle work?

- Stars $> 1.3M_\odot$ have slightly higher core temperature to support this reaction. C, N and O act as catalysts in H-to-He burning. It's basically $^{12}C + H \rightarrow {}^{13}N \rightarrow {}^{13}C + H \rightarrow {}^{14}N + H \rightarrow {}^{15}O \rightarrow {}^{15}N + H \rightarrow {}^{12}C + {}^{4}He$
- p-p burning is more efficient than CNO in Sun's temperature. CNO releases more energy but requires higher temperature to operate.



21. (37.) Sketch the HR diagram for a typical globular cluster and open cluster. Identify the various observed populations and interpret them on the basis of stellar evolution theory.

- Globular cluster - tightly bound, 10^5 stars, mostly red

  It's a group of stars born at about the same time and same distance from us. Various regions of the H-R diagram are identified: Main Sequence (MS); Turn off (TO); Red Giant Branch (RGB); Helium flash occurs here at tip of RGB (Tip); Horizontal Branch (HB); Schwarzschild gap in the HB (Gap); Asymptotic Giant Branch (AGB); the final stellar remnants, White Dwarfs (WD), will lie off the bottom of the diagram.

- The turnoff point's B-V value says the mass of stars that runs out of hydrogen cores, which can be used to calculate the lifetime of the cluster. More massive stars burn out faster, so they've already at the RGB.
- Open cluster - open, 100 stars, mostly blue

22. (40.) Write down the four basic equations of stellar structure.

- Equation of hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

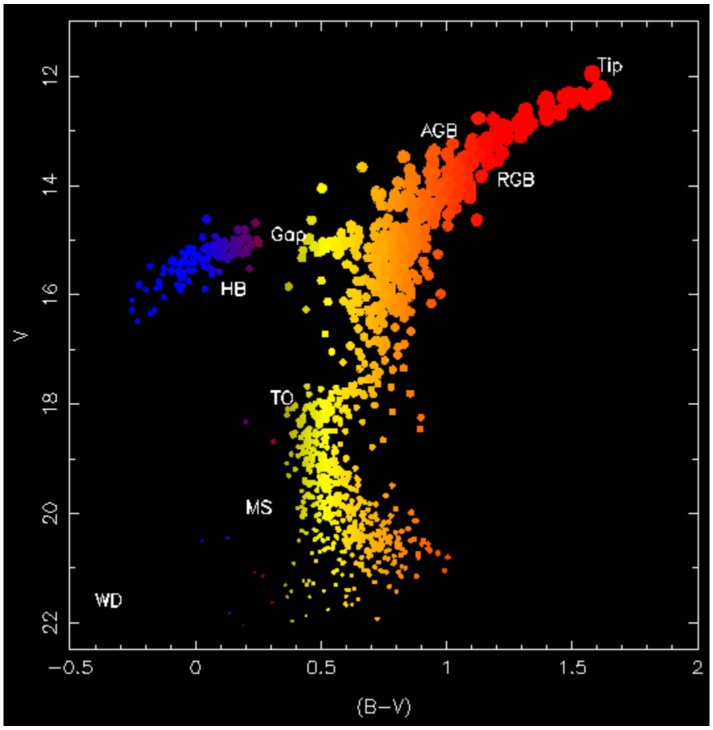
- Mass continuity equation

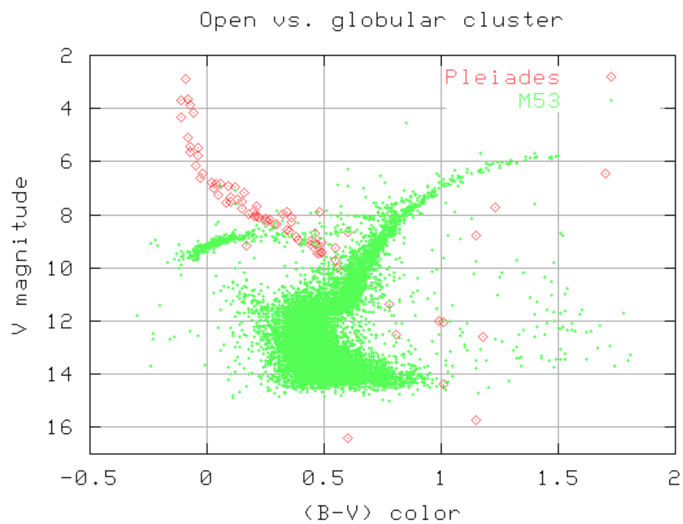
$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

- Energy continuity equation

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

- Equation of energy transport





- Radiative-dominant

$$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 16\sigma T^3(r)}$$

- Convective-dominant

$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma}\frac{T(r)}{P(r)}\frac{dP}{dr}$$

23. Describe the prominent neutrino producing reactions in the sun, and the experiments designed to detect them. What is the solar neutrino problem and how might they solve it?

- Nearly 90% of solar neutrino is from the start of p-p chain

$$p + p \rightarrow d + e^+ + \nu_e$$

- Solar neutrino problem

  Only 1/3 of the neutrinos detected are electron neutrino ν_e. The solution is neutrino oscillations.

  Some experiments:

  - Gallium to Germanium
  - Homestake experiment (2002 Nobel Prize) - a large tank of Cl is used to detect solar neutrino. $Cl + \nu_e \rightarrow Ar + e^-$
  - Kamiokande - pure water so the neutrino scatters with the electron and emit Cerenkov radiation.
  - IceCube - in South Pole looking for the high-energy neutrinos

  The neutrino has to be massive to oscillate because if it's massless particle, the time dilation is infinite and it won't have time to oscillate.

24. (52.) Using the known luminosity and mass of the Sun to estimate its nuclear lifetime.

- Nuclear burning efficiency: 0.7%. The lifetime is

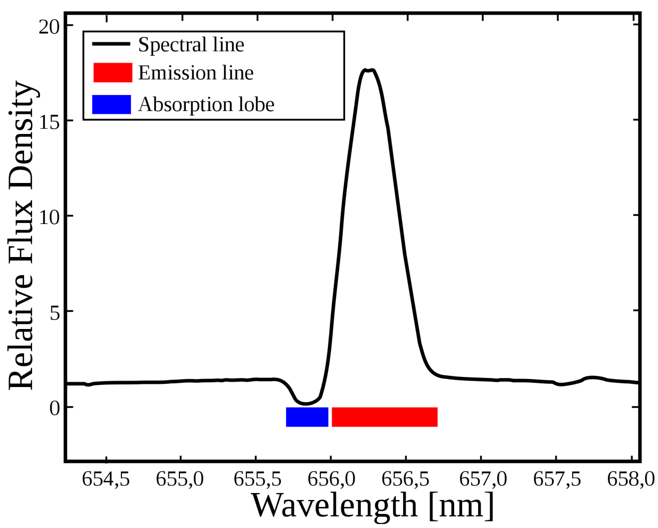
$$t = \frac{0.7\% M_\odot c^2}{L_\odot} = \frac{0.007(2e30)(3e8)^2}{3.9e26} = 10 \text{ billion yrs}$$

25. (87.) What are RR Lyrae stars, and how do they differ from Cepheids?

- It's a variable star with nice period-luminosity relation.
- Compared with Cepheid, it's more common, less luminous, shorter period. It is also old, metal-poor, low-mass star (Population II star). They can be used for smaller distance calibration.

26. What is a P Cygni line profile, and what does it signify?

- P Cygni is a variable star. The presence of both absorption and emission indicates the existence of a **gaseous envelope expanding away from the star**. The emission line arises from a dense stellar wind near to the star, while the blueshifted absorption lobe is created where the radiation passes through circumstellar material rapidly expanding in the direction of the observer. These profiles are useful in the study of stellar winds.



27. Describe the internal structure of the sun. How old is the sun?

- Hydrogen solid core
- Radiative gas envelope
- Convective gas envelope
- The Sun is 4.6 billion years old (half of the hydrogen fuel burned out).

28. What are "apparent magnitude," "absolute magnitude" and "bolometric magnitude"? What are U, B and V colors?

- Magnitude
  - The apparent magnitude is defined as (in visible band, 550 ± 45 nm):

$$m = -2.5 \log_{10} \left(\frac{F}{F_0} \right)$$

    whre F_0 is the flux of the star Vega, and $m_0 = 0$. So larger m, fainter star. The brightest star Sirius has $m = -1.4$, and the Sun has $m = -27$.
  - Absolute magnitude is defined as

$$M = m - 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right)$$

    Add some correction from distance. Sirius (2.6 pc away) has $M = 1.5$. Our Sun has $M = 4.8$.
  - Bolometric magnitude is the absolute magnitude including all wavelengths
- Colors
  - U - ultraviolet, 365 ± 35 nm
  - B - blue, 445 ± 45 nm
  - V - visible, 550 ± 45 nm

29. What are HH objects? T Tauri stars? Bipolar flows? OH masers? Where are they all found?

- Herbig–Haro (HH) object - a bright nebula ejected from a protostar

  It's formed by a shock on ISM from polar jets ejected from the protostar accertion disk.
- T Tauri stars - a very young star with the protoplanetary disk.
- Bipolar flow - jets ejected from a young star along the rotation axis. Its shock forms HH object.
- OH maser

  In the Milky Way, •OH masers are found in stellar masers (evolved stars), interstellar masers (regions of massive star formation), or in the interface between supernova remnants and ISM. Interstellar •OH masers are often observed from molecular material surrounding H II regions. But there are masers associated with very young stars that have yet to create UC H II regions. This class of •OH masers appears to form near the edges of very dense material, place where H_2O masers form. So, observations of •OH masers in these regions, can be an important way to probe the distribution of the H_2O in ISM.

30. (100.) What is a planetary nebula? What is our current understanding of the formation of planetary nebulae? What effects limit the lifetime of a planetary nebula?

- Planetary nebula is an ionized gas shell around a red giant star. It's a terrible misnomer because it has nothing to do with planets. Typically, it has 1 light year size, 0.1-1 solar mass. Famous nebulae include Ring Nebula (1 kpc away), Cassiopeia A (Cas A), and Crab Nebula (by SN1054, 2 kpc away).
- The planetary nebula is formed by stellar winds during giant phase. The outer shell is burning and the corona keeps expanding. The winds can also be driven by thermal radiation pressure (comparable to gravity for big star), or centrifugal force. An even more violent way is supernova explosion.
- The lifetime of planetary nebula phase after AGB phase is around 10000 years. The central star is the remnant of its AGB progenitor, an electron-degenerate carbon-oxygen core that has lost most of its hydrogen envelope due to mass loss on the AGB. As the gases expand, the central star undergoes a two stage evolution, first growing hotter as it continues to contract and hydrogen fusion reactions occur in the shell. In the second phase, it radiates away its energy and fusion reactions stop, as the central star is not heavy enough to generate the core temperatures required for carbon and oxygen to fuse. The star becomes a white dwarf, and the expanding gas cloud becomes invisible to us, ending the planetary nebula phase of evolution. For a typical planetary nebula, about 10,000 years passes between its formation and recombination of the star (from Doppler effect, the gas is measured to be moving away from the central stars at 10 - 30 km/s and characteristic length scales of nebula is 0.3 pc, giving the estimated age of 10,000 years).

31. (51.) Describe the various evolutionary phases of a low-mass (1 $M_\odot$) star and those of a high-mass (e.g. 12 $M_\odot$) star. Show the corresponding evolutionary tracks on an HR diagram.

- Hayashi track

  It can be shown that there is a minimum effective temperature (equivalently, a boundary on the right-hand side of the H-R diagram) cooler than which hydrostatic equilibrium cannot be maintained; this boundary corresponds to a temperature around 4000 K. Protostellar clouds cooler than this will contract and heat up until they reach the Hayashi boundary. Once at the boundary, a protostar will continue to contract on the Kelvin-Helmholtz timescale, but its effective temperature will no longer increase, as it will remain at the Hayashi boundary. Thus the Hayashi track is close to a vertical line on the H-R diagram. Stars at the Hayashi boundary are fully convective: this is because they are cool and highly opaque, so that radiative energy transport is not efficient, and consequently have large internal temperature gradients. Stars with masses < 0.5 Solar mass remain on the Hayashi track (i.e. are fully convective) throughout their premain sequence stage, joining the main sequence at the bottom of the Hayashi track. For stars with masses > 0.5 Solar mass the Hayashi track ends, and the Henyey track begins, when the internal temperature of the star rises high enough that its central opacity drops and radiative energy transport becomes more efficient than convective transport: the lowest luminosity on the Hayashi track for a star of a given mass is thus the lowest luminosity at which it is still fully convective.

- Helium flash

  For a star with a mass less than 2.25 solar masses, the core helium flash occurs when the core runs out of hydrogen, and the thermal pressure is no longer sufficient to counter the gravitational collapse. This causes the star to start contracting. During the contraction the core becomes hotter and hotter until it causes the outer layers to begin fusing hydrogen and expand outwards initiating the red giant stage. As the star continues contracting due to gravity, it eventually becomes compressed enough that it becomes degenerate matter. This degeneracy pressure is finally sufficient to stop further collapse of the most central material. As the rest of the core continues to contract and the temperature continues to rise, a temperature (1e8K) is reached at which the helium can start to fuse, and so helium ignition occurs. The explosive nature of the helium flash arises from its taking place in degenerate matter. Once the temperature reaches 100 million □ 200 million kelvins and helium fusion begins using the triple-alpha process, the temperature rapidly increases, further raising the helium fusion rate and, because degenerate matter is a good conductor of heat, widening the reaction region. However, since degeneracy pressure (which is purely a function of density) is dominating thermal pressure (proportional to the product of density and temperature), the total pressure is only weakly dependent on temperature. Thus, the dramatic increase in temperature only causes a slight increase in pressure, so there is no stabilizing cooling expansion of the core. This runaway reaction quickly climbs to about 100 billion times the star’ s normal energy production (for a few seconds) until the temperature increases to the point that thermal pressure again becomes dominant, eliminating the degeneracy. The core can then expand and cool down and a stable burning of helium will continue.

- Triple-alpha process

  The triple alpha process is a set of nuclear fusion reactions by which three helium-4 nuclei (alpha particles) are transformed into carbon. Older stars start to accumulate helium produced by the proton-proton chain reaction and the carbon-nitrogen-oxygen cycle in their cores. The products of further nuclear fusion reactions of helium with hydrogen or another helium nucleus produce lithium-5 and beryllium-8 respectively, both of which are highly unstable and decay almost instantly back into smaller nuclei. When the star starts to run out of hydrogen to fuse, the core of the star begins to collapse until the central temperature rises to 108 K (8.6 keV).

- Horizontal branch

  The horizontal branch (HB) is a stage of stellar evolution that immediately follows the red giant branch in stars whose masses are similar to the Sun’ s. The helium core flash that occurs to stars at the top of the red giant branch causes substantial changes in stellar structure, resulting in an overall reduction in luminosity, some contraction of the stellar envelope, and surfaces reaching higher temperatures. Horizontal branch stars are powered by helium fusion in the core (via the triple-alpha reaction) and by hydrogen fusion in a shell surrounding the core.

- Asymptotic giant branch

  The asymptotic giant branch is the region of the Hertzsprung-Russell diagram populated by evolving low to medium-mass stars. This is a period of stellar evolution undertaken by all low to intermediate mass stars (0.610 solar masses) late in their lives. Observationally, an asymptotic giant branch (AGB) star will appear as a red giant. Its interior structure is characterized by a central and inert core of carbon and oxygen, a shell where helium is undergoing fusion to form carbon (known as helium burning), another shell where hydrogen is undergoing fusion forming helium (known as hydrogen burning) and a very large envelope of material of composition similar to normal stars.

32. What is the Schwarzschild criterion for convective instability? What is the entropy distribution in a convective envelope?

- Schwarzschild stability condition

$$\left|\frac{dT}{dr}\right| < \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \left|\frac{dP}{dr}\right|$$

  If this is met, convection doesn’ t happen. If not, the convection equation is just the above with equal sign.

33. What are stellar populations? Give several examples of Pop. I and Pop. II objects in our galaxy.

- Pop. I:
  - Younger
  - Metal rich
  - In galactic disk

  Our Sun is Pop. I star.

- Pop. II:
  - Older
  - Metal poor
  - In bulge or globular cluster

34. What is the "mass-function" of a binary star system and how is it determined?

- The binary mass function or simply mass function is a function that constrains the mass of the unseen component (typically a star or exoplanet) in a single-lined spectroscopic binary star or in a planetary system. It's the lower limit of the mass of another body.

$$f_M = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{P_{\text{orb}} K^3}{2\pi G}$$

  where K is the measured radial velocity of the seen object M_1. Thus, the mass of unseen body $M_2 > f_M$ by definition. Essentially, this is caused by a degeneracy between mass and inclination.

35. What is meant by the "gravothermal collapse" of a globular cluster, and what can save the cluster from complete collapse?

- The cluster core has negative heat capacity (from virial theorem, $E_{tot} = E_{kinetic}$, so $C_V = dE/dT \sim 3/2Nk_B$). So as the core grows hotter, heat flows outward, which will drive the core to even higher temperature, and the system cannot reach thermal equilibrium.
- Conductive transfer of heat from the central region will raise the high central temperature faster than it raises the lower temperature of the outer parts. No equilibrium is possible; the center continues to contract (grows in density) and get hotter, sending out heat to the outer parts. If no other process intervenes, the core collapses to zero radius and infinite density in a finite time. Such a collapse is called gravothermal collapse.
- Its halted by formation of binaries that adds energy and cools the core.

36. Derive the equation for the radiative energy flux in a stellar envelope and show how the Rosseland mean opacity is introduced.

III *Compact Objects*

37. What are the most interesting things we've learned from stellar mass black hole observations? Masses? Spins? What are the most interesting things we've learned from supermassive black hole observations? Masses? Spins? Relation to host galaxies?

- Stellar black hole ($< 100 M_\odot$)
  - Interesting things
    - Event Horizon Telescope
    - X-ray
    - LIGO
- Supermassive black hole
  - Interesting things
    - Very Large Array

38. (57.) It is believed that most stars leave a collapsed remnant at the end of their evolution. What stars leave (i) white dwarfs? (ii) neutron stars? (iii) black holes?

- Chandrasekhar mass limit
  - EoS of ultrarelativistic quantum degenerate gas

$$P = \frac{1}{4}\left(\frac{3}{4\pi(2s+1)}\right)^{1/3} hcn_e^{4/3}$$

    The electron density is $n = (\rho/m_p)(Z/A) = (\rho/m_p)(1+X)/2$ where Z is atomic number, A is atomic mass. or X is the mass abundance of hydrogen atom.
  - Virial theorem

$$\bar{P}V = \frac{1}{4}\left(\frac{3}{4\pi(2s+1)}\right)^{1/3} hcn_e^{4/3}V = -\frac{1}{3}E_{gr} = \frac{GM^2}{3r}$$

$$M_{ch} \sim \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_p^2}$$

- White dwarf: support by electron degeneracy pressure

  The core has no hydrogen because they are all burned out, $X = 0$ or $Z/A \approx 1/2$. Then $M_{ch} = 1.4 M_\odot$. No white dwarf can exceed this limit.

- Neutron star: support by neutron degeneracy pressure

  Our poor understanding of neutron star EoS leads to theoretical calculation to be $M_{ch} = 3.2M_{\odot}$, while the actual mass limit is around $2M_{\odot}$.

- Black hole

  Mass larger than $2M_{\odot}$ will collapse to a black hole due to strong gravitation and nothing can support it.

39. (60.) What is a neutron star? What assumptions and inputs go into determining the upper mass limit for a neutron star? What is the approximate ratio of neutrons to protons (and electrons) in the interior of a neutron star?

- Upper mass limit

  - Tolman–Oppenheimer–Volkoff (TOV) limit

    12-km neutron star can be as massive as Sun. The upper mass limit of cold, non-rotating neutron star is called Tolman–Oppenheimer–Volkoff (TOV) limit, like Chandrasekhar limit for white dwarfs. It starts with Tolman-Oppenheimer-Volkoff equation (TOV equation), which is an equation for hydrostatic equilibrium that takes general relativity into account:

    $$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1}$$

    TOV used a very simple EoS for neutron interaction and got a wrong result of 0.8 $M_{\odot}$. Current models that include degeneracy pressure and strong force gives 1.5-3.0 $M_{\odot}$

  - Constraint from LIGO GW170817 - 2.16 $M_{\odot}$

    Recent calculation on the mass constraint is done by analyzing data from GW170817, a binary neutron star merger. The GW signal during the inspiral depends on the tidal deformability and stellar matter and weakly on EoS. So it's quasi-universal to all neutron stars. It's derived from mass-shedding limit and the result is 2.16 $M_{\odot}$.

- Neutron to proton ratio - 95% neutron, 5% proton

  - how to get it? From observation or theoretical calculation? No good answers for now. Section 2.5, excercise 2.22 of Shapiro's book.

  - Short-range correlations

    Even though proton is minority, it carries more energy than neutrons.

40. (61.) A double neutron star system in M31 emerges. What is the approximate energy emitted in gravitational radiation and what is the corresponding amplitude (strain) h observed here on Earth? Would LIGO be able to detect it?

- Binary neutron star with mass M_1 and M_2 separated by a.

🌞 Quadrupole approximation

$$h_{ij}^{TT} = \frac{2G}{c^4 r}\frac{d^2}{dt^2} I^{kl}\left(P_{li}P_{kj} - \frac{1}{2}P_{lk}P_{ij}\right)$$

The mass moment is

$$I^{kl} = \int d^3x T_{00} x^l x^k = M_1 x_1^l x_1^k + M_2 x_2^l x_2^k$$

where $x_1 = (r_1 \cos\Omega t, r_1 \sin\Omega t, 0)$ and $x_2 = (r_2 \cos(\Omega t + \pi), r_2 \sin(\Omega t + \pi), 0)$. The polarization direction is correct, so the projection tensor P_{ij} don't do anything. We have

$$h_{ij}^{TT}(t,r) = \frac{2G}{c^4 r}\ddot{I}_{ij}(t-r) = \frac{4Ga^2\Omega^2\mu}{c^4 r}\begin{bmatrix} -\cos 2\Omega(t-r) & -\sin 2\Omega(t-r) & \\ -\sin 2\Omega(t-r) & \cos 2\Omega(t-r) & \\ & & 0 \end{bmatrix}$$

🌞 Luminosity

$$L = \frac{G}{5c^5}\langle \dddot{I}_{ij}\dddot{I}^{ij}\rangle = \frac{G}{5c^5}\left(\frac{a^2\mu}{2}(2\Omega)^3\right)^2 (\sin^2 + (-\cos)^2 + (-\cos)^2 + (-\sin)^2) = \frac{32Ga^4\mu^2\Omega^6}{5} = \frac{32G^4M^3\mu^2}{5a^5}$$

where $M = M_1 + M_2$, total mass.

🌞 Time evolution

$$\frac{d}{dt}(E_k + E_p) = -L$$

where $E_k + E_p = -E_p/2 + E_p = E_p/2 = -GM\mu/2a$. So we have

$$\frac{da}{dt} = -\frac{64G^3M^2\mu}{5c^5a^3}$$

The energy emitted is simply

$$\int dt L = \frac{\Delta(E_p)}{2} = \frac{GM\mu}{2a_{merger}}$$

The merger distance is can be calculated by finding the slope of frequency at merger. For GW170817, the total energy released is $0.025M_\odot c^2$ with masses of 1.2 and 1.5 $M_\odot$. The merger distance a is around 50 km.

$$\frac{d\Omega}{dt} = -\frac{2\sqrt{GM}}{3a^{5/2}}\frac{da}{dt} = \frac{96}{5c^5}\left(GM_{chirp}\right)^{5/3}\Omega^{11/3}$$

where the **chirp mass** is $\mu^{3/5}M^{2/5}$.

🌞 Strain h_+

M31 is 770 kpc away from Earth. The strain at Earth is

$$h_+ = h_{11}^{TT}(r) = \frac{4Ga^2\Omega^2\mu}{c^4 r} = \frac{4G^2M_1M_2}{c^4 ra} = \frac{4G^2(1.5M_\odot)(1.2M_\odot)}{c^4(770\text{kpc})(50\text{km})} = 1\times 10^{-20}$$

It depends on the masses of stars, how far away, plus the total energy it released.

41. How does the orbital frequency of the innermost stable circular orbit around a black hole scale with its mass?

- The Schwartzchild metric is

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- For massive particle, we have $p^0 = (1 - r_S/r)^{-1}E$, $p^r = mdr/d\tau$, $p^\phi = g^{\phi\phi}p_\phi = L/r^2$. So $p_\mu p^\mu = -m^2$ gives

$$-\left(1 - \frac{r_S}{r}\right)^{-1} E^2 + \left(1 - \frac{r_S}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{L}{r^2}\right)^2 = -m^2$$

  For massive particles, the orbit equation is

$$m^2 \left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(\frac{L^2}{r^2} + m^2\right) \left(1 - \frac{r_S}{r}\right) = E^2 - V^2$$

  The circular orbit is found by taking $dV^2/dr = 0$. The solution is

$$r = \frac{L^2 \pm \sqrt{L^2(L^2 - 3m^2 r_S^2)}}{m^2 r_S}$$

  The stable innermost one is at $L^2 = 3m^2 r_S^2$ and $r = 3r_S$, and the orbital frequency is

$$\omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{p^\phi/m}{p^0/m} = \frac{L/r^2}{E/(1 - r_S/r)}$$

  At stable orbit (maybe not innermost),

$$L^2 = \frac{m^2 r_S r^2}{2r - 3r_S}$$

  and the $E = V$ is energy. The orbital frequency is reduced to

$$\omega = \sqrt{\frac{GM}{r^3}}$$

  Just like the Newtonian equation! At innermost radius, $r = 3r_S = 6GM/c^2$, so $\omega \propto 1/M$.

42. (70.) What is the Eddington limit and how is it manifested in (i) ordinary stars? (ii) accreting X-ray sources?

- Eddington luminosity is defined as

$$L_{Edd} = \frac{4\pi cGMm}{\sigma_T}$$

  It's a balance between radiation force and gravitational force. It doesn't have to be in an accreting disk

- For ordinary stars, the outer sphere could be hydrogen atom or helium, so the mass m could be m_p or something else. For mass larger than $50M_\odot$, the luminosity of main-sequence star starts to reach Eddington limit. It's difficult to form high-mass stars because the accreting process can be stopped from radiation pressure.

- Accreting X-ray sources: if the major star is neutron star (which pulsates), this sets a limit(Eddington limit) of the X-ray binary. Black holes are harder to tag in X-ray study. Super-Eddington luminosity is still under debate.

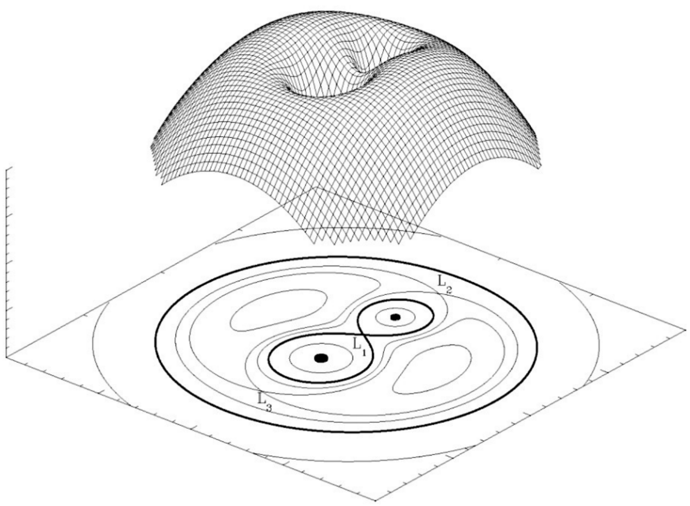
43. (81.) What is the Roche potential in a binary system? Describe carefully the assumptions that go into deriving it. Define the Roche limit.

- Roche potential

  In the corotating frame of a binary system M_1 and M_2, the Roche potential is

$$\psi(r) = -\frac{GM_1}{r} - \frac{GM_2}{r} - \frac{1}{2}\omega^2 r^2$$

  Note that the "kinetic energy" term has the minus sign due to this reference frame.



- Three assumptions
  - Point mass
  - Circular orbit, non-relativistic
  - Synchronized rotation with orbit.
- Roche limit

  The limit when the gravitational force and tidal force balance each other and destroys the secondary body. Consider the main mass to be M and secondary m with radius r, the separation d that balances two forces on the test particle u on the surface of m is

$$\frac{Gmu}{r^2} = GMu\left(\frac{1}{(d-r)^2} - \frac{1}{d^2}\right)$$

  So

$$d = \left(\frac{2m}{M}\right)^{1/3} r$$

  This is the Roche limit.

44. (82.) What is the Shakura-Sunyaev (alpha-disk) model for accretion disks? What are the assumptions that go into its derivation?

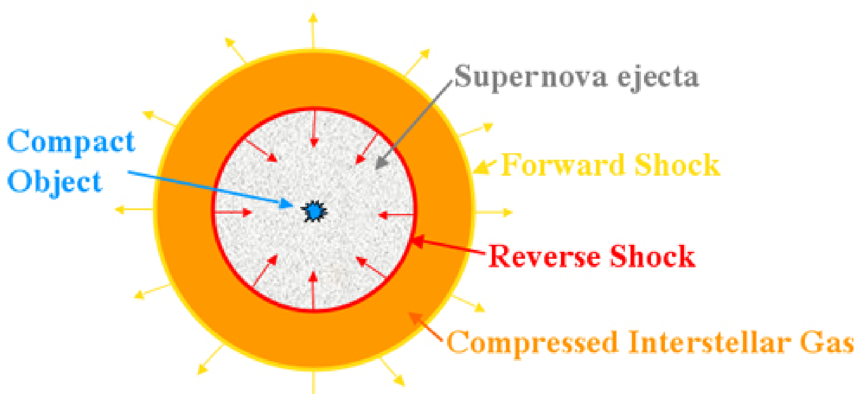
- α-disk model
  - Inward spiral
    As matter enters the accretion disc, it follows a trajectory called a tendex line, which describes an inward spiral. This is because particles rub and bounce against each other in a turbulent flow, causing frictional heating which radiates energy away, reducing the particles' angular momentum, allowing the particle to drift inwards, driving the inward spiral.
  - Angular momentum redistribution
    Turbulence-enhanced viscosity solves this problem of loss of angular momentum. The α model has the free parameter α:

$$\nu = \alpha c_s h$$

where ν is viscosity, c_s is sound speed, h is disk height. This viscosity gives enough friction to heat up gases to emit X-ray, and provides a loss channel of gravitational energy and angular momentum.

- Assumptions
  - the disk is axially symmetric
  - stationary disks
  - thin disks
  - the viscosity in the disk is small
  - Newtonian mechanics

45. (103.) Describe the various stages of evolution of a supernova remnant. What are the relevant physical processes during each phase? Explain why in the Sedov-Taylor phase of a supernova remnant, the radius expands at $t^{2/5}$.



- Free expansion phase
  The shock wave heats materials to millions of degrees Kelvin resulting in the emission of thermal X-rays. The shock wave also accelerates the ISM into an expanding shell which outputs synchrotron radiation due

to the acceleration of electrons in the presence of a magnetic field. This expanding shell surrounds an area of relatively low density, into which the supernova ejecta expands freely, typically with velocities of around 10^4 km/s. This free expansion phase lasts for around 100 years until the mass of the material swept up by the shock wave exceeds the mass of the ejected material. This coincides with the time when reverse shock has traversed all of the ejecta, significantly slowing the ejecta down, or stopping them altogether

- Sedov-Taylor phase

Rayleigh-Taylor instabilities arise once the mass of the swept up ISM approaches that of the ejected material. These instabilities mix the shocked ISM with the supernova ejecta and enhance the magnetic field inside the SNR shell. This phase lasts between 10,000 and 20,000 years.

The kinetic energy of bulk material is

$$E_k = \frac{3\pi}{4} R^3 \rho_0 \frac{v^2}{2} \propto \rho_0 R^3 (R/t)^2$$

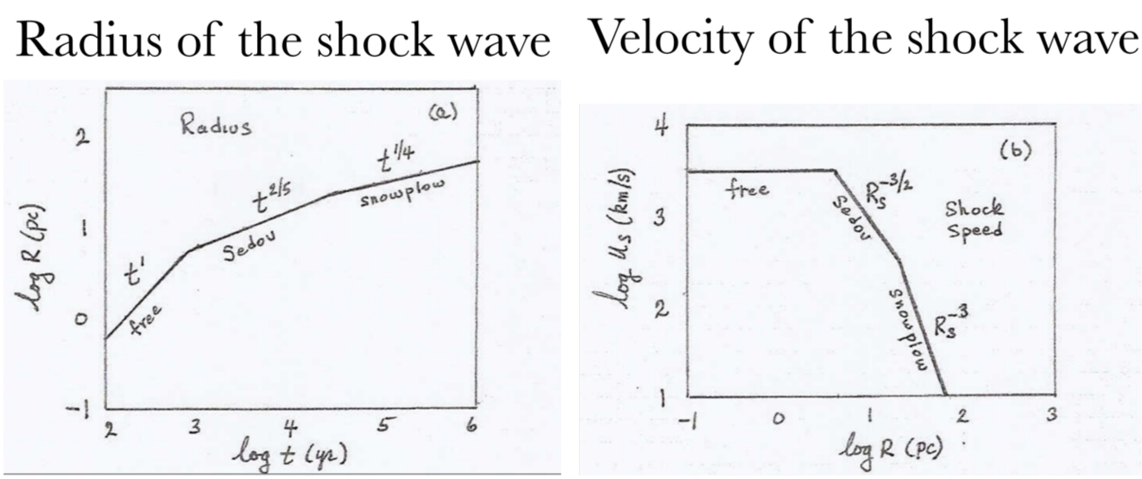
Some fixed portion of blast energy E_0 goes to the kinetic energy of the gas, so

$$R \propto \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5}$$

- Radiative cooling phase - snowplow

The shock wave continues to cool, and once temperatures drop below about 20,000 K, electrons start re-combining to form heavier elements, which radiates energy much more efficiently than the thermal X-rays and synchrotron emission produced thus far, further cooling the shock wave which ultimately disperses into the surrounding ISM.

Supernova remnants play a vital role in the evolution of galaxies. Apart from dispersing the heavy elements made in the supernova explosion into the ISM, they provide much of the energy that heats up the ISM and are believed to be responsible for the acceleration of galactic cosmic rays.



46. (58.) What is a white dwarf star? Why is the radius of a white dwarf a decreasing function of its mass? What is the basic physics that leads to the upper limit on the mass of a white dwarf (i.e. the Chandrasekhar limit)?

White dwarf is the end of stellar evolution of stars $< 1.4M_\odot$. It's supported by ultrarelativistic electron degenerate pressure with EoS

$$P \propto \rho^{4/3}$$

For nonrelativistic one, the EoS is $P \propto \rho^{5/3}$. Some dimensional analysis from hydrostatic and mass continuity equation

$$P \propto \frac{M\rho}{r} \sim \rho^{4/3} \quad \Rightarrow \quad r \sim M^{-1/3}$$

So larger mass, smaller radius. The Chandrasekhar limit states the max mass that can be supported by ultrarelativistic electron degenerate pressure:

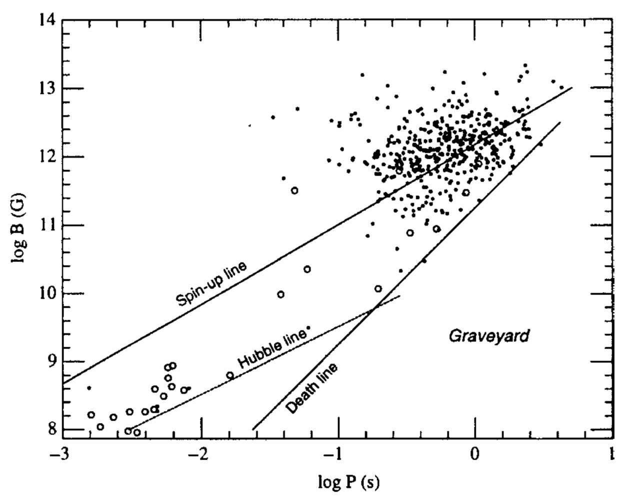
$$\bar{P}V \propto \frac{M^{4/3}}{r^4}r^3 \propto -E_{gr} \propto \frac{GM^2}{r}$$

The r part is canceled and

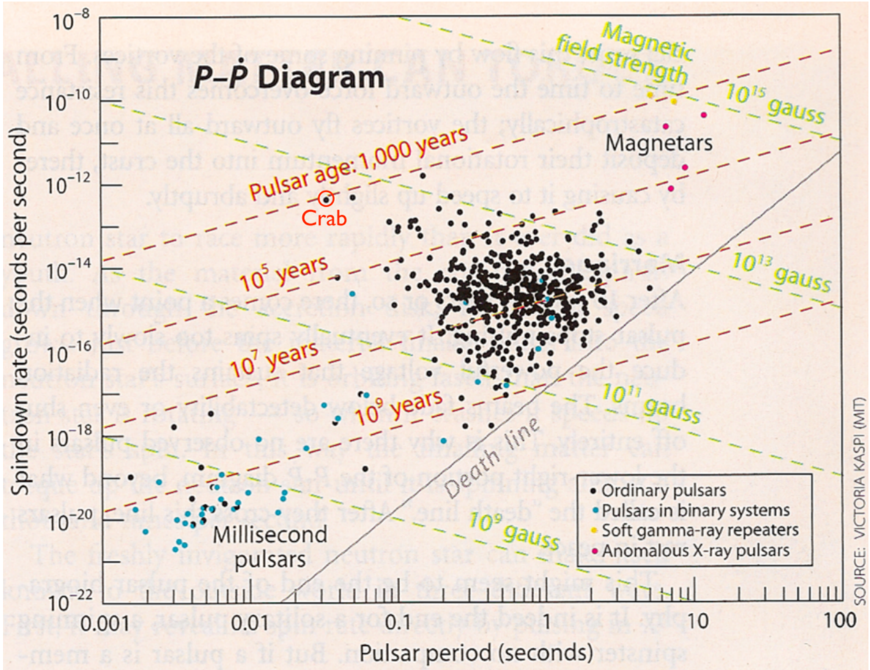
$$M_{ch} \propto \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_p^2}$$

47. (73.) What is a "millisecond pulsar"? What is the shortest spin period known for such an object? Estimate a lower bound to its mean density.

- Millisecond pulsars are old, rapidly rotating neutron stars that have been spun up or "recycled" through accretion of matter from a companion star in a close binary system. The transfer of angular momentum explains why the pulsar spins so fast.
- They are usually found in the binary systems (circled dots on the right plot). The low-mass X-ray binary sources are believed to be neutron stars accreting matter from inflated binary companions. Because of the orbital motion of the companion, the matter accreting onto the neutron star from its companion will carry a considerable amount of angular momentum. This is expected to increase the angular velocity of the accreting neutron star, leading to a decrease in rotation period.
- The accreted material on the neutron star covers up and buries the magnetic field so efficiently that very little magnetic field is present at the surface. That's why millisecond pulsars have low magnetic field.



48. (83.) Show from a simple dimensional analysis how the effective temperature of an accretion disk depends on accretion rate and distance from the central object.



- The luminosity of accretion disk is given by

$$L \propto \frac{GM\dot{M}}{r_{in}}$$

Assume it's a blackbody

$$L \propto r_{in}^2 T^4$$

So

$$T \propto \left(\frac{GM\dot{M}}{r_{in}^3}\right)^{1/4}$$

49. (84.) What evidence is there for "superluminal" jets in binary systems containing black holes, and how does one explain the superluminal motion?

- Superluminal - faster than light

  The apparent calculation of the speed of jets does not yield the actual speed of the object, as it fails to account for the fact that the speed of light is finite. The error in the above naive calculation comes from the fact that when an object has a component of velocity directed towards the Earth, as the object moves closer to the Earth that time delay becomes smaller. This means that the apparent speed as calculated above is greater than the actual speed. Correspondingly, if the object is moving away from the Earth, the above calculation underestimates the actual speed.

- Evidence: relativistic jet from M87 black hole.

50. Write down the fluid equations for conservation of mass and momentum that would describe a spherically symmetric, expanding supernova remnant. How would you derive the "jump conditions" for a strong

adiabatic shock.

- Recall fluid equations
  - Mass continuity:
$$\frac{d\rho}{dt} = -\rho\nabla \cdot v = -\rho\frac{1}{r^2}\partial_r(r^2 v)$$
  - Momentum continuity (Euler's equation)
$$\frac{dv^i}{dt} = g^i - \frac{\partial^i P}{\rho} = g^i - \frac{1}{\rho}\frac{\partial P}{\partial r}$$
- Adiabatic shock Also some continuity equations
$$\begin{cases} \rho_0 v_0 = \rho_1 v_1 \\ \rho_0 v_0^2 + P_0 = \rho_1 v_1^2 + P_1 \\ \frac{1}{2}v_0^2 + \frac{c_{s0}^2}{\gamma-1} = \frac{1}{2}v_1^2 + \frac{c_{s1}^2}{\gamma-1} \end{cases}$$

  The last equation is derived from the energy conservation
$$\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho} = \text{constant}$$
  with
$$\epsilon = \frac{P}{(\gamma - 1)\rho}$$

51. (72.) What is the significance of the Hulse-Taylor binary radio pulsar to physics?

- Hulse and Taylor found a pulsar with period changing in around 8 hours, because it's part of a binary system with another neutron star. After tens of years of observation, they calculated the change of period due to energy loss to GW and measured with the binary. The results agree beautifully. It's a precision test of GR and first indirect evidence of GW.

52. What minimum mass is required for a black hole powering a quasar of luminosity $10^{12}\ L_\odot$? What accretion rate is required?

- The luminosity from accretion disk has Eddington limit
$$L = \frac{4\pi G M m c}{\sigma_T} = 3 \times 10^4 L_\odot \frac{M}{M_\odot}$$
  with Thomson scattering cross section σ_T. So $M = 10^8 M_\odot$ is needed, which is supermassive black hole. The accretion rate is
$$\dot{M} = \frac{10^{12} L_\odot}{\eta c^2} = 2 \times 10^{-8} M_\odot/s$$

53. (55.) How much energy is typically released in a type II supernova? What fractions of that energy are in the forms of neutrinos, visible light, gas kinetic energy and gravitational radiation?

- Type II supernova - shows hydrogen, core collapse

  The gravitational binding energy released during core collapse is

$$E = \frac{3GM}{5}\left(\frac{1}{R_{in}^2} - \frac{1}{R_{out}^2}\right)$$

  Take $R_{in} = 10$ km and ignore the R_{out} part, and $M = 1.4M_\odot$, we have

$$E = 10^{47}\ \text{J}$$

  The energy release channels are

  - 1 %: luminosity
  - 0.1 %: kinetic energy
  - others: neutrino-antineutrino pairs

54. (85.) Describe a scenario whereby a neutron star can be formed in a binary system and have the system remain bound.

- Formation

  One of the binary evolves to a neutron star, so a 10 $M_\odot$ star ejects 8.6 $M_\odot$ and becomes a 1.4 $M_\odot$.

- Bound binary system

  For a binary system to remain bound, the total energy has to be negative. The transition (supernova explosion) is assumed to be instant. Before explosion, let M_1 be the supernova and M_2 be its companion in the circular orbit of a_i. Choose M_2's frame, the relative velocity of M_1 is

$$V_i = \omega a_i = \sqrt{\frac{GM}{a_i}}$$

  where $M = M_1 + M_2$ is total mass. During the explosion, the total energy changes of course. Right after explosion, the mass M_1 reduces to $M_1 - \Delta M = M_c$, and the relative velocity changes to V_f by a kick. The total energy right after explosion is

$$E_f = \frac{\mu_f V_f^2}{2} - \frac{GM_cM_2}{a_i}$$

  For bound orbit, $E_f < 0$.

  - If the explosion is spherically symmetric, there is no kick and $V_f = V_i$, so we have

$$\frac{M}{M - \Delta M} = 2 \qquad \Rightarrow \qquad \Delta M = \frac{M}{2}$$

    If the mass loss is more than half, the orbit would become unstable.

55. (59.) What is a "cataclysmic variable"? What are nova explosions, and what is the basic physics underlying these events?

- Cataclysmic variable is an accreting binary with white dwarf being the receiving star. It's a variable star, so its luminosity changes with time. The reason behind is the nova that happens on the white dwarf.

- Nova is a transient astronomical event that causes the sudden appearance of a bright, apparently "new" star, that slowly fades over several weeks or many months. Classical nova eruptions are the most common type. The accretion on the white dwarf creates a dense but shallow atmosphere. This atmosphere, mostly consisting of hydrogen, is thermally heated by the hot white dwarf and eventually reaches a critical temperature causing ignition of rapid runaway fusion. The nova remnant then cools down so the luminosity decreases.

56. (65.) What is the gravitational redshift from the surface of a neutron star?

- Recall the Schwartzchild metric

$$ds^2 = -(1 - r_S/r)dt^2 + (1 + r_S/r)^{-1}dr^2 + r^2 d\Omega^2$$

For a photon far away with some 4-momentum p, the energy measured by different observers are

$$E = -p^\mu u_\mu$$

The static observer in a metric is $u = (u(r), 0, 0, 0)$ and $g_{\mu\nu}u^\mu u^\nu = -1$. So

$$u(r) = \sqrt{1 - \frac{r}{r_S}}$$

The energy measured is

$$E = -p_0 u(r)$$

So the energy ratio is

$$\frac{E_\infty}{E(r)} = \sqrt{1 - \frac{r_S}{r}}$$

Convert to redshift

$$\frac{\lambda_\infty}{\lambda(r)} = \frac{1}{\sqrt{1 - \frac{r_S}{r}}}$$

57. (68.) How much rotational kinetic energy can be stored in a neutron star? How does this help resolve the "energy budget" for the Crab nebula?

- A millisecond pulsar can store a lot of rotational energy.

- The ionosphere of Earth gives the lower frequency of 10 MHz. The Crab nebula has the luminosity of 10^{38} erg/s of non-thermal radiation. It's something like a power law, in all band. The spinning magnetic dipole radiation has 2x frequency of the pulsar rotation, and it accounts for 99% of the energy loss of the pulsar, but the frequency is too low to be detected on ground-based observatory. The spin-down of pulsar solves the energy budget by providing enough energy for nebula luminosity.

58. (66.) Explain the basic physics underlying type I X-ray bursts on neutron stars. Explain why very little mass is ejected during such a burst. Compare this phenomenon with recurrent novae.

- Type I X-ray bursts are thermonuclear explosions that occur in the envelopes of accreting neutron stars. It is analogue to the classical nova that happens on the white dwarf. But the X-ray burst releases much less energy than a classical nova explosion (about 10^{-5} as much). The reasons are:
  - since the gravity on the surface of a neutron star is much greater (by about 105) than that on a white dwarf, a much thinner layer is required to reach the temperature and density to ignite the explosion.
  - the surface area of the neutron star is much smaller than that of the white dwarf. Like classical novae, X-ray bursts repeat, but the interval between repetitions is much shorter – a few hours, instead of decades.
- Classical novae radiate most of their energy at optical and ultraviolet wavelengths, whereas X-ray bursts radiate the energy as X-rays. In classical novae, the layer is ejected from the star by the explosion. In X-ray bursters, the gravity of the neutron star is so great that the explosion can lift the material only a few hundred meters, and then it falls down again. We think that X-ray bursters do not pulse because the magnetic fields on the neutron stars in bursters are relatively weak. Therefore, the gas that falls onto the neutron star is deposited on the entire surface, not channeled to the magnetic poles as in a pulsar.

59. (67.) Explain why you might expect most of the emission of an accreting neutron star to be in the form of X-rays. How does an X-ray pulsar pulse?

- Given the high luminosity (accreting rate) and small radius (10 km), the effective temperature is extremely high, at 10^7 K. So the emission peaks at X-ray.
- An X-ray pulsar consists of a magnetized neutron star in orbit with a normal stellar companion and is a type of binary star system. The misalignment of rotation and magnetic axis causes the hot spot (at magnetic pole) to rotate in and out of view.

60. (75.) Derive a plausible relation between the luminosity of an X-ray pulsar and its spin-up rate.

- Unlike radio pulsars which are all spinning down due to energy losses in the form of relativistic particles and magnetic dipole radiation, some X-ray pulsars have been found to be increasing their rate of spin (spinning up).

- Over 99% of radio pulsars are **single** objects that radiate away their rotational energy in the form of relativistic particles and magnetic dipole radiation. In contrast, X-ray pulsars are members of binary star systems and accrete matter from either stellar winds or accretion disks. The accreted matter transfers angular momentum to (or from) the neutron star causing the spin rate to increase or decrease at rates that are often hundreds of times faster than the typical spin down rate in radio pulsars.
- The spin-up rate can be calculated as

$$\frac{d}{dt}(I\Omega) = \dot{M}\frac{J}{m}$$

where

$$\frac{J}{m} = \frac{mvR}{m} = \omega R^2 = \sqrt{GMR}$$

is the angular momentum per unit mass near the radius of neutron star. So

$$I\dot{\Omega} = \dot{M}\sqrt{GMR} - \dot{I}\Omega \approx \dot{M}\sqrt{GMR}$$

because the accreting particle spins way faster than the self spin of neutron star itself.

$$\dot{\Omega} = \frac{\dot{M}\sqrt{GMR}}{I} = \frac{5}{2}\frac{L}{\sqrt{GM^3R}}$$

61. (80.) Why are more X-ray binaries and radio pulsars found (per unit mass) in globular clusters than in other parts of the galaxy?

- Low-Mass X-ray Binaries (LMXBs) is orders-of-magnitude more numerous (per unit mass) in Globular clusters than Galactic disk. This overabundance is due to the production of compact binary systems containing primordially-produced neutron stars via stellar interactions within the high-density cluster cores.

62. (77.) What are "magnetars", and what evidence do we have that they exist?

- Magnetars are differentiated from other neutron stars by having even stronger magnetic fields, and by rotating more slowly in comparison. It's the most magnetic thing known.
- Starquakes triggered on the surface of the magnetar disturb the magnetic field which encompasses it, often leading to extremely powerful gamma-ray flare emissions. In 1979, a soft gamma-ray burst happens and various spacecrafts in solar system detects it. This is the first evidence of magnetars.

63. (144.) Describe the basic features of the "fireball" model of gamma-ray burst afterglows. What is the emission mechanism? What inputs are required?

- Gamma-ray burst (GRB) is one of the most ill-understood stuff in astro world. One popular theory is that they come from neutron star merger events. It's one of the most energetic events in the whole Universe (highest one reaches 9000 SN).

- The burst is produced by internal shocks. The "inner engine" that produces the relativistic energy flow is hidden from direct observations. This model requires a compact internal "engine" that produces a wind – a long energy flow (long compared to the size of the "engine" itself) –rather than an explosive "engine" that produces a fireball whose size is comparable to the size of the "engine". Not all the energy of the relativistic shell can be converted to radiation (or even to thermal energy) by internal shocks. The remaining kinetic energy will most likely dissipate via external shocks that will produce an "afterglow" in different wavelength. This afterglow was recently discovered, confirming the fireball picture. Wtf does it mean exactly?

64. (62.) Approximately how many binary systems in the galaxy are thought to contain a black hole? What is the evidence for this?

- So far, about 20 X-ray binaries have been identified to be probable black hole binaries. Except for three persistent sources, the others are all transient sources. From the current statistics of the transients, the total number of black-hole binaries in our Galaxy is estimated to be as many as 1000 or even more.

- X-ray telescopes (XRTs) are the best hunting ground for new stellar-mass BHs with 17 cases currently known and estimated masses between 4 - 14 $M_\odot$. These are only the tip of the iceberg of an estimated dormant population of $\sim 10^3$ BH binaries and $\sim 10^9$ steller-mass BHs in the Galaxy.

65. (104.) X-ray emission from a nearby supernova remnant is observed to peak at 0.5 keV. Estimate the velocity with which the blast wave is propagating through the interstellar medium

- Wien's law:

$$\lambda T = 2.9 \times 10^{-3} mT$$

So $T = 10^6$ K. Then use EoS to find velocity

$$\frac{\rho k_B T}{\mu m_p} = P = \rho v^2$$

66. (69.) How are the magnetic fields of neutron stars estimated in (i) X-ray binaries? (ii) radio pulsars

- X-ray binaries

The general environment of the hot plasma at the magnetic poles of an accreting neutron star emitting an X-ray continuum spectrum with cyclotron features. The continuum photons produced inside the hot region by thermal bremsstrahlung and comptonization trying to escape through the surface have a finite probability of being resonantly scattered in the outer layers, thereby producing an apparent absorption line (Cyclotron Resonance Scattering Features; CRSF). The resonance occurs because of quantized energy states (Landau levels) of electrons with respect to their motion in circular orbits transverse to the magnetic field direction. The fundamental energy where the feature appears (corresponding to the energy difference between the ground state and the first excited state) is related to the magnetic field strength.

Radio pulsar

$$\frac{dE_{tot}}{dt} = I\Omega\dot{\Omega} = \frac{2\pi B^2 R^6 \Omega^4 \sin^2\alpha}{3\mu_0 c^3}$$

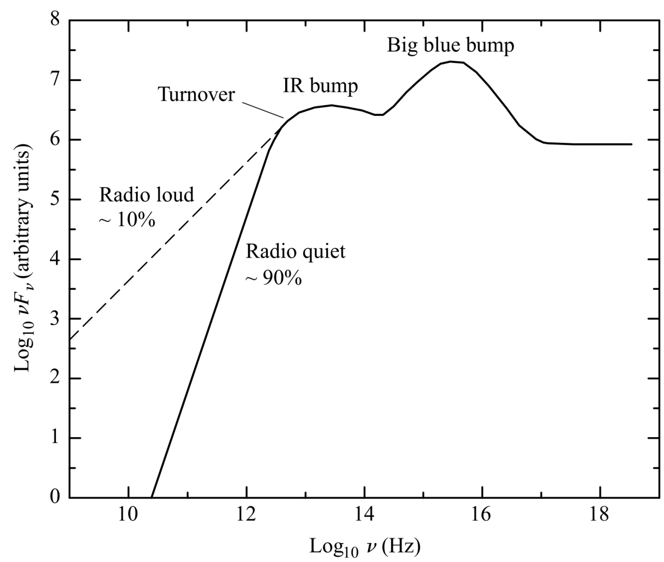
Measure change of period to find magnetic field.

67. (138.) Sketch the frequency distribution of the emission of a typical QSO from radio to X-ray frequencies.

The most notable feature of this spectral energy distribution is its persistence over some 10 order of magnitude in frequency. Power law,

$$F_\nu \propto \nu^{-1}$$

A pure power law spectrum is the signature of synchrotron radiation. The big blue bump is generally believed to be due to an optically thick accretion disk. A thermal infrared bump is probably due to warm dust grains. The typical spectrum of a quasar shifted to rest wavelength. Quasars emit very strongly in UV.



68. (71.) How would you calculate the magnetospheric radius of an accreting neutron star?

This radius (Alfven radius) is where the magnetic energy density $B^2/2\mu_0$ and kinetic energy density $\rho v^2/2$ become comparable. Given the mass loss

$$\dot{M} = 4\pi r^2 \rho v$$

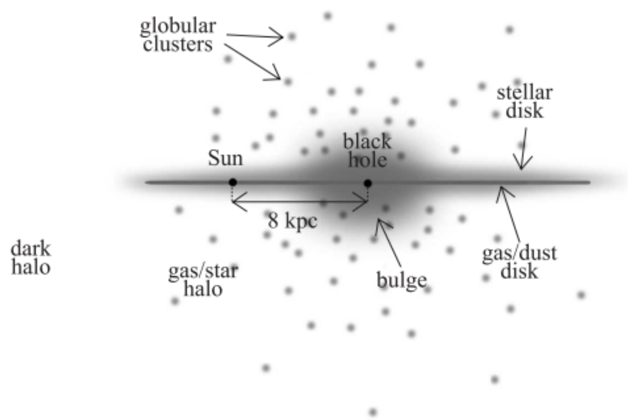
and $1/r^3$ magnetic field fall-off from a dipole

$$B(r) = B_s(R/r)^3$$

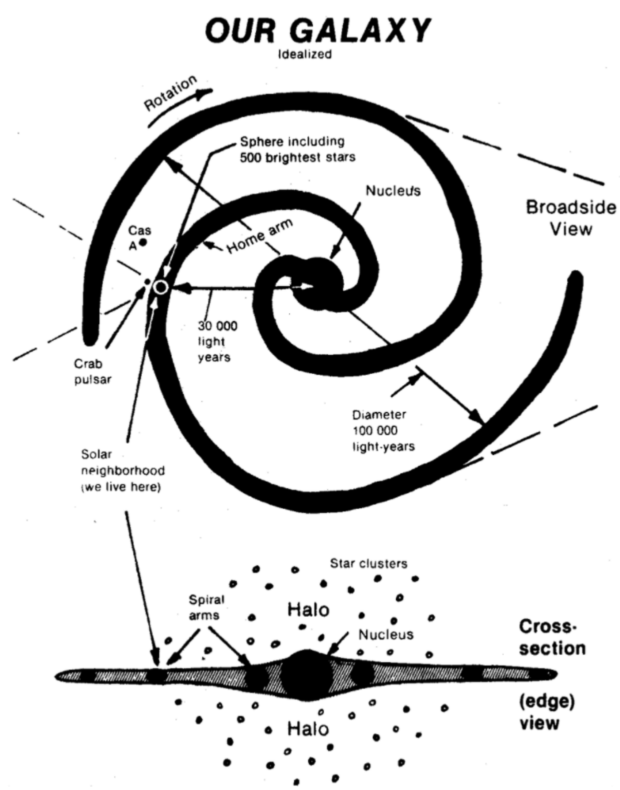
You can express the radius in terms of surface magnetic field strength and mass loss rate.

IV *Galaxy*

69. (109.) Make a sketch of the Galaxy to scale (top and side views), and indicate its various features and properties



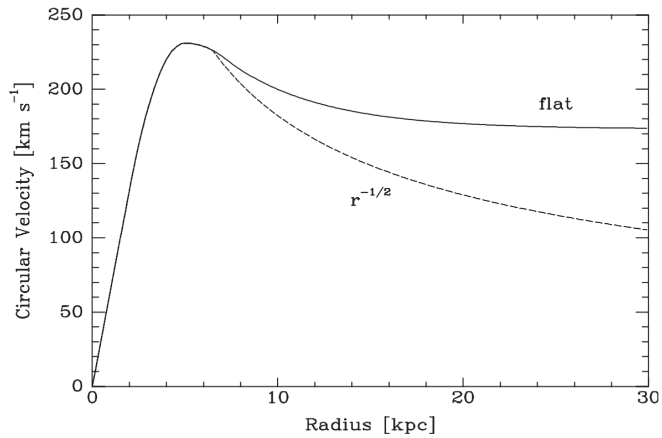
- Features
  - The spheroid
    - Supermassive black hole - millions of $M_\odot$ with small luminosity.
    - Bulge - a bunch of old stars because no metals in atmospheres (there were no giants). Contains 10 million stars per pc^3.
    - Gas/dust disk - components for star formation.
    - Stellar disk
    - Globular clusters - 0.1 million stars
    - Dark matter halo - hot neutrino or cold dark matter.



70. (114.) What evidence do galaxy rotation curves provide for dark matter? How are galaxy rotation curves measured?

- Rotation curve - circular velocity as a function of radius $v(r)$
  The rotation is flat beyond the region emitting visible light, meaning $M \sim v^2 r \sim r$. These invisible massive stuff is thus dark matter. The asymptotic circular velocity depends on the mass, or luminosity of the galaxy. The Tully-Fisher relation is

$$v_c \approx 220 \left(\frac{L}{L_*}\right)^{0.22} \quad \text{km/s}$$



- Measurement
  - Measuring the line-of-sight velocity from the Doppler shift of a known transition:
    - 21 cm line for H I region
    - Balmer Hα emission line for H II region
    - Radio lines of CO for molecular clouds

71. (136.) What do we know about the masses of galaxies and clusters of galaxies and how do we know it?

- Galaxy mass
  - Value
    - we know our Sun lies $R = 8$ kpc away from the galaxy center by measuring distances to globular clusters and determining the centroid of their distribution.
    - we know the velocity of the Sun $v = 240$ km/s by measuring the velocity of our neighborhood.
    - The mass of galaxy can be calculated by Kepler's third law

$$M_{\text{galaxy}} = \frac{\omega^2 a^3}{G} = \frac{v^2 R}{G} = 10^{11} \ M_\odot$$

      This is only the mass of visible matters in galaxy.
  - Composition
    - The flat rotation curve tells us the actual mass of galaxy extends beyond the visible matters (3 kpc), possibly to 30 kpc. So the total mass is around $10^{12} M_\odot$, 90% of which are dark.
- Galaxy cluster mass
  - A cluster of galaxies is usually a gravitationally bound system and does not expand with the expansion of the Universe. So it is the clusters of galaxies rather than galaxies themselves which are moving away from each other with the expansion of the Universe.
  - Galaxies typically move with $v = 1000$ km/s in the cluster with typical size 1 Mpc. So the total mass of galaxy cluster is

$$M_{\text{cluster}} = \frac{v^2}{RG} = 10^{14} \ M_\odot$$

    So a cluster can hold up to 100 galaxies.

- The total luminosity of galaxy is around $10^{12} L_{\odot}$, meaning that there are much more matter that are not luminous. At most 30% of the dark matter in a galaxy cluster may be attached to galaxies.

72. (112.) How are gas and dust distributed in our Galaxy? Why are they distributed differently from the stars?

- The interstellar dust particle scatters blue light more than the red light. So the observed stellar light becomes redder. The dust is confined in a 150-pc layer at the mid-plane of galaxy. This is the **zone of avoidance**. You won't see much if you look in the dust disk direction due to extinction and reddening.

$$\rho(r, z) = \rho_0 e^{-r/r_d} e^{-|z|/z_d}$$

  where $z_d = 150$ pc.

- The ISM inside a galaxy is usually found to be distributed rather nonuniformly. In parts of the spiral arms, the ISM seems to form a succession of clumps like beads on a string. Parker found that a uniform distribution of the interstellar medium would be unstable. This Parker instability is related to magnetic buoyancy and is presumably the cause behind the ISM fragmenting into clumps. 99% of the ISM is gas, and 1% is dust.

- The dust particles are mostly metal and far smaller than the dust on my table. It's more like "interstellar smoke". Stars are formed by clumps of matter and move in orbits, but gas and dust are more freely moving. what's more?

73. (113.) Sketch the rotation curve of our Galaxy, with approximate scales on the axes. How can information about the rotation curve be determined from 21 cm observations?

- The sketch is in Question 70. Remember that the horizontal scale is in kpc, and vertical scale is in km/s. The velocity peaks at 6 kpc with 230 km/s and flat at 200 km/s.

- If we know the rotation curve for the galaxy and assume that the gas is in circular orbit around the galactic center, we can use 21-cm line profiles to map the spiral arms.

74. (119.) What is the density profile of a self-gravitating isothermal gas sphere? What is the corresponding phase-space distribution function?

- Isothermal sphere in hydrostatic equilibrium

  Using hydrostatic equation:

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

  and ideal gas EoS

$$P = nk_BT = (\rho/m_p)k_BT$$

we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -\frac{Gm_p}{k_B T}4\pi r^2 \rho$$

The singular solution assumes $\rho = Cr^b$ and $b = -2$. For non-singular solution, needs numerical integration to do.

- The phase-space density is kinda complicated. why do we need that?

75. What is two-body relaxation? For a self-gravitating cluster of N objects, each of mass m, with a velocity dispersion s, what is the relaxation time? How long does it take for a massive object (M » m) to sink to the bottom of a cluster potential well?

- Two-body relaxation: a star's orbit is altered due to the gravitational interaction with another star.

  The perturbation of perpendicular speed δv is

$$\delta v = \frac{2Gm}{bv}$$

  where b is the impact parameter, m is the stationary mass, and v is the speed of another mass flying towards it.

- Relaxation time

  The velocity dispersion can be calculated from Virial Theorem

$$s^2 = \frac{GM}{R}$$

  where R is the size of galaxy 10 kpc. The total velocity perturbation is

$$\begin{aligned}\delta v^2 &= \sum_i \delta v_i^2 \\ &= \int_{b_{min}}^{b_{max}} dN(\delta v(b))^2\end{aligned}$$

  The number of bodies in the impact parameter $(b, b+db)$ is

$$dN = n(s\Delta t)(2\pi b db)$$

  Integrate it up, we get

$$\delta v^2 = \frac{8\pi G^2 m^2 n}{v}\Delta t \ln\left(\frac{b_{max}}{b_{min}}\right)$$

  When the system is relaxed, the final perturbated velocity would be just the virial velocity. So we get the relaxation time

$$t_{relax} = \frac{s^3}{8\pi G^2 m^2 n \ln(b_{max}/b_{min})}$$

Mass segregation

Mass segregation is the process by which heavier members of a gravitationally bound system, such as a star cluster or cluster of galaxies, tend to move toward the center, while lighter members tend to move farther away from the center. The kinetic energy is thus equipartitioned. The most massive stars in a cluster can segregate more rapidly than the less massive stars. The segregation time is intuitively

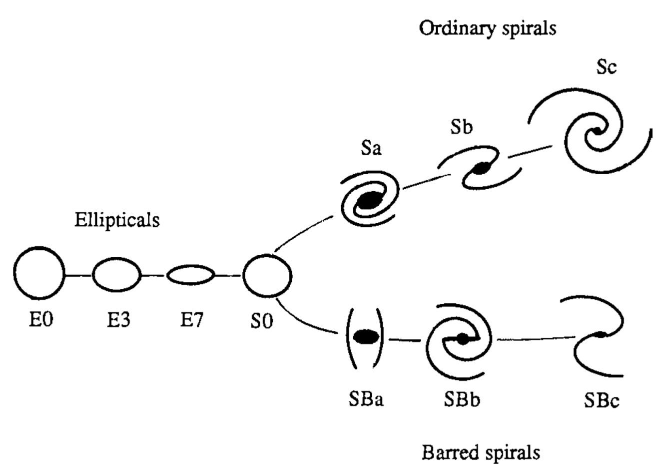
$$t(M) = \frac{m}{M} t_{relax}$$

76. (129.) What is the "Local Group"?

- It's the galaxy group that contains Milky Way. Some numbers to remember
  - Size: 3 Mpc
  - Mass: $2 \times 10^{12} M_\odot$
  - Two large member: Milky Way and Andromeda Galaxy (M31) ($\sim 10^{12} M_\odot$), and Triangulum Galaxy (M32) ($\sim 10^{10} M_\odot$)

77. (131.) What fraction of all galaxies are spirals? Ellipticals? SOs? How are these ratios different in loose and dense clusters of galaxies? Why? Give several possible explanations

- Hubble tuning fork
  - Spiral galaxy: more luminous, star-borning
  - Elliptical galaxy: ellipsoid-shaped. Has old stars, like our center bulge region. Lack of gas and dust. It's formed by galaxy collisions.



- The typical fractions of spirals and ellipticals in a population of galaxies depend on the environment (Dressler, 1980). In the central regions of rich clusters of galaxies, only about 10% of the galaxies may be spirals. In contrast, the spirals may constitute nearly 80% of the bright galaxies in the low-density regions of the Universe. Higher density → more collisions → more elliptical galaxies.

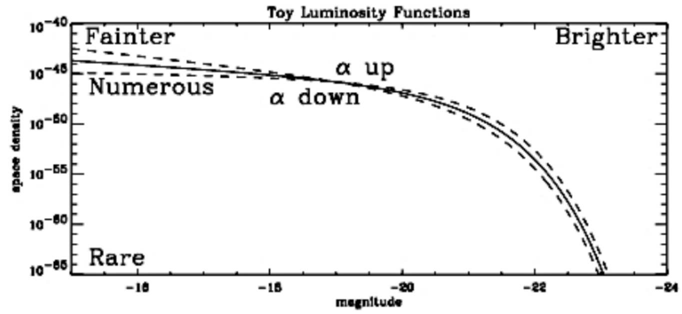
78. (133.) What is the "Schechter luminosity function"? What is the luminosity of a typical bright galaxy?

- Schechter luminosity function

$$n(L)\,\mathrm{d}L = \phi^* \left(\frac{L}{L^*}\right)^\alpha \mathrm{e}^{-L/L^*} \frac{\mathrm{d}L}{L^*}$$

  Three parameters to tune: magnitude ϕ^*, decreasing speed α, cutoff L_*.

- The typical luminousity of bright galaxy is $L_* = 1$ billion $L_\odot$



79. (120.) What is the radial Jeans equation in spherical coordinates for a spherically symmetric system, keeping terms associated with an anisotropic velocity dispersion tensor? How is it used to measure masses for galaxies and clusters?

- The radial Jeans equation can be derived from Liouville's theorem

$$\frac{df}{dt} = 0$$

  and plug in all spherical coordinates and velocities. Assume the spherical symmetry and steady-state, we have

$$\frac{1}{n}\frac{\partial}{\partial r}(n\sigma^2) = -\frac{\partial \Phi}{\partial r} = -\frac{GM(r)}{r^2}$$

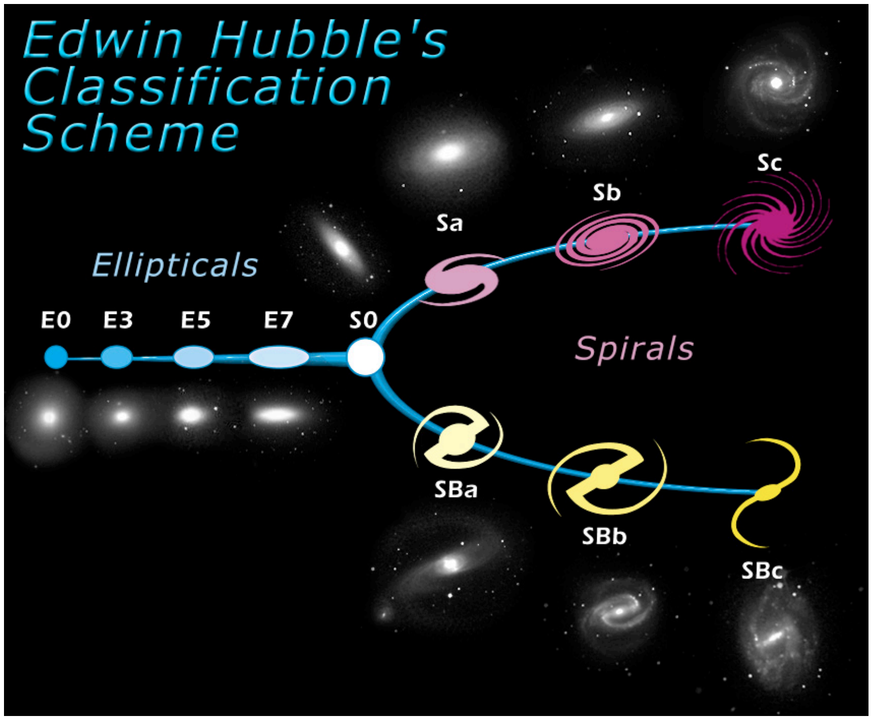
  By measuring $\partial\sigma^2/\partial r$, etc, we can estimate the mass of galaxy

$$M(r) \propto \frac{r\sigma^2}{G}$$

  For galaxy, $\sigma = 300$ km/s, $r = 10$ kpc. For cluster, $\sigma = 1000$ km/s, $r = 1$ Mpc.

80. Describe Hubble's classification scheme for galaxies and explain why it is useful.

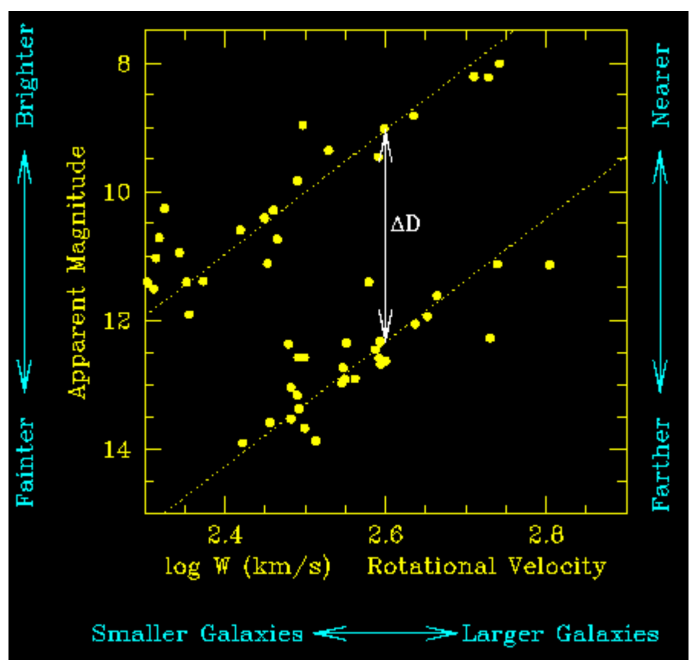
- The Hubble sequence shown again



- It had a WRONG physical meaning of early → old galaxies from left → right of the diagram. what else useful?

81. What is the "Tully-Fisher method"? Can it be used to determine H_0? How about q_0?

- Tully-Fisher relation
  Tully-Fisher relation is a correlation for spiral galaxies between their luminosity and how fast they are rotating. The bigger the galaxy is, the faster it is rotating. That means that if you know the rotation velocity of the spiral galaxy, you can tell by using this Tully-Fisher relation its intrinsic brightness, then you can calculate its distance.
  For a cluster of galaxies where you can observe tens of galaxies, you can measure the distance to each galaxy via Tully-Fisher relation and then take the average to calculate the distance to that cluster. The further the cluster of galaxies is located, more the Tully-Fisher relation is shifted downward on this diagram.



- Knowing the distance to the clusters and the radial velocity, you can find $H_0 = v/d$.
- what is q_0?

82. (132.) What are cD galaxies and where are they found? How might they be formed?

- Type-cD galaxy is a galaxy morphology classification, a subtype of type-D giant elliptical galaxy. Characterized by a large halo of stars. They are also known as supergiant elliptical galaxy.

- They can be found near the centres of some rich galaxy clusters.

- cD galaxies are believed to grow via mergers of galaxies that spiral in to the center of a galaxy cluster. This "cannibalistic" mode of growth leads to the large diameter and luminosity of the cDs. Remains of "eaten" galaxies sometimes appear as a diffuse halo of gas and dust, or tidal streams, or undigested off-center nuclei in the cD galaxy.

- Dynamical friction

  Dynamical friction is believed to play an important role in the formation of cD galaxies at the centres of galaxy clusters. This process begins when the motion of a large galaxy in a cluster attracts smaller galaxies and dark matter into a wake behind it. This over-density follows behind the larger galaxy and exerts a constant gravitational force on it, causing it to slow down. As it loses kinetic energy, the large galaxy gradually spirals toward the centre of the cluster. Once there, the stars, gas, dust and dark matter of the large galaxy and its trailing galaxies will join with those of other galaxies who preceded them in the same fate. A giant or supergiant diffuse or elliptical galaxy will result from this accumulation. The centers of merged or merging galaxies can remain recognizable for long times, appearing as multiple "nuclei" of the cD galaxy.

83. (122.) What is the "fundamental plane" for elliptical galaxies?

- The fundamental plane is NOT a physical plane. It's a function or correlation among three most important variables of the elliptical galaxy: luminosity L, radius R, and velocity dispersion σ.

  The luminosity-velocity relation of elliptical galaxy is called Faber-Jackson relation. It's analogous to the Tully-Fisher relation for spiral galaxy. It's a power law.

- Will they ask us for the specific fundamental plane? we won't remember this function.

84. (124.) What is a violent relaxation? How does the phase space distribution function it produces differ from that of an isothermal gas?

- Since the collisional relaxation time in a galaxy is so enormous $t_{two-body}$, one may think that the stellar velocity distribution in a galaxy would be completely unrelaxed and would have the signature of some initial primordial velocity distribution. This is NOT true. If a galaxy forms by contracting from a larger volume, then the gravitational field at a point inside the galaxy will keep changing drastically during the contraction time. It can be shown that a time-changing potential has some effects analogous to the effects of collision. This is called violent relaxation.

- Phase mixing is the simplest mechanism that causes relaxation in gravitational N-body systems. Although phase-mixing is a relaxation mechanism, in that it drives the system towards a state in which the phase-space density is more and more uniform, it does not cause any loss of information: the system preserves all knowledge of the initial conditions.

- Violent relaxation requires both a time-varying potential and mixing to occur simultaneous.

$$t_{vr} = \left\langle \frac{(dE/dt)^2}{E^2} \right\rangle^{-1/2}$$

  It's very fast, so it's called violent.

- what to say on its phase space distribution?

85. (117.) Derive an equation whose solution gives the vertical density profile for an isothermal, self-gravitating (galactic) disk.

- From Question 74: for isothermal sphere: $P = (\rho/m_p)k_B T$. Assume uniform gravitational field g, we have

$$\frac{d}{dz}\left(\frac{\rho}{m_p}k_B T\right) = -g\rho$$

  The solution of density profile is exponential

$$\rho = \rho_0 \exp\left(-\frac{g m_p}{k_B T} z\right)$$

86. (118.) Why do some galaxies have prominent spiral structure and others do not? Briefly describe the density wave theory of spiral structure.

- Winding problem:

  Originally, astronomers thought the arms of a spiral galaxy were material. However, if this were the case, then the arms would become more and more tightly wound, since the matter nearer to the center of the galaxy rotates faster than the matter at the edge of the galaxy. The arms would become indistinguishable from the rest of the galaxy after only a few orbits.

- Lin-Shu density wave theory:

  The arms were not material but instead made up of areas of greater density, similar to a traffic jam on a highway. The cars move through the traffic jam: the density of cars increases in the middle of it. The traffic jam itself, however, moves more slowly. In the galaxy, stars, gas, dust, and other components move through the density waves, are compressed, and then move out of them.

87. (111.) How many globular clusters does our Galaxy contain? How are they distributed in space? What fraction of the total mass of the Galaxy do they contain?

- The Milky way contains more than 150 of globular clusters.
- Of the globular clusters within the Milky Way, the majority are found in a halo around the galactic core, and the large majority are located in the celestial sky centered on the core.
- $150 * 100000/1e12 = 0.1\%$

88. (116.) What is the Oort limit and how is it determined? How much dark matter in the solar neighborhood is inferred from the Oort limit?

- Oort cloud
  - Two types of comets:
    - Short-period comet: they are in the same plane of solar system. 10 AU away.
    - Long-period comet: they are from Oort cloud, 2000 AU away.
- Oort limit
  Oort also measured velocity dispersion $\langle v_z \rangle^2$ and calculated gravitational field strength

$$\frac{d}{dz}(n\langle v_z \rangle^2) = gn$$

  Then he measured density from g. He found that this number is 2 times larger than the density from visible stars, dark matter! It's also a limit of amount of ISM around the solar neighborhood.

89. What is the difference between a "tube" orbit and a "box" orbit, and how are the latter helpful in building galaxies?

- The intrinsic shape of elliptical galaxies is not clear yet, and could be either a spheroid (i.e., the body of revolution obtained by rotating an ellipse around one of its axes) or a triaxial ellipsoid, of the form $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.
- The " tube" and " box" orbits are orbits in a 2D non-axisymmetric potential, where no component of the angular momentum is conserved.

90. (125.) A globular cluster at a distance of 10 kpc, containing 10^6 stars, subtends an angle of 1/3 arcminute. Estimate the velocity dispersion among its stars.

A simple problem on Virial theorem. The radius of cluster is R = 10 kpc ×1/3 arcmin/2 = 0.5 pc. The total kinetic energy is

$$T = \frac{Nm}{2}u^2$$

and potential energy is

$$U = \sum_{i,j} \frac{-Gm_i m_j}{r_{ij}} \sim -\frac{N(N-1)}{2}\frac{Gm}{R}$$

Virial theorem $2T + U = 0$ gives

$$u^2 = \frac{GNm}{2R} = \frac{GM}{2R}$$

91. (115.) What are the Oort A and B coefficients and what basic information about the Galaxy can be determined from them?

The Oort constants (discovered by Jan Oort) A and B are empirically derived parameters that characterize the local rotational properties of our galaxy, the Milky Way, in the following manner

$$A = \frac{1}{2}\left(\frac{V_0}{R_0} - \left.\frac{dV}{dR}\right|_{R_0}\right)$$

$$B = \frac{1}{2}\left(-\frac{V_0}{R_0} - \left.\frac{dV}{dR}\right|_{R_0}\right)$$

where V_0 and R_0 are the rotational velocity and distance to the Galactic center, respectively, measured at the position of the Sun. They depend only on the motions and positions of stars in the solar neighborhood. The values of these constants are A = 14.82 ± 0.84 km/s/kpc and B = 12.37 ± 0.64 km/s/kpc. From the Oort constants, it is possible to determine the orbital properties of the Sun, such as the orbital velocity and period, and infer local properties of the Galactic disk, such as the mass density and how the rotational velocity changes as a function of radius from the Galactic center.

92. (121.) Define the following simple "laws" and "profiles" for stellar systems: Hubble, King, de Vaucouleurs, exponential.

Hubble's Law: $v = H_0 d$

King's stellar dynamics model:

$$f(x,v) = \begin{cases} A(e^{-\beta E(x,v)} - 1)\ , \text{ if } E(x,v) < 0 \\ 0\ , \text{ if } E(x,v) > 0 \end{cases}$$

where $f(x,v)d^3x d^3v$ is the number of self-gravitating particles within volume d^3x having the ends of their velocity vectors lying within the volume d^3v in the velocity space.

- de Vaucouleurs law (the surface brightness for elliptical galaxies)

$$I(r) = I_e \exp\left[7.7\left(\left(\frac{r}{r_e}\right)^{1/4} - 1\right)\right]$$

where r_e is called the effective radius within which half of the luminosity is contained (if the image of the galaxy happens to be circular), and $I(r_e) = I_e$.

- Exponential law (the surface brightness for spiral galaxies):

$$I(r) = I_0 e^{-r/r_d}$$

where r_d is the distance at which the intensity falls to $e^{-1}I_0$ and gives a measure of the size of the disk.

93. (134.) What is the ”Faber-Jackson law”? Give a plausible derivation.

- The velocity dispersion σ of an elliptical galaxy is related to its intrinsic luminosity by Faber-Jackson relation:

$$\sigma \sim 220 \left(\frac{L}{L_*}\right)^{1/4} \text{ km/s}$$

To derive this, we start from the scaling law of stars

$$L \sim M^2 \sim (\sigma^2 R)$$

So $\sigma \sim L^{1/4}$

V *Cosmology*

94. Summarize the state of the experimental field of 21cm cosmology and what it hopes to accomplish.

- 21-cm cosmology

  The 21-cm line occurs in neutral hydrogen, due to differences in energy between the spin triplet and spin singlet states of the electron and proton.

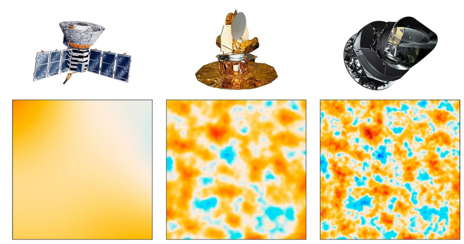
  - Goals
    - To probe the Dark Age of the Universe from recombination to reionization. It can provide a picture of the matter power spectrum in the period after recombination by mapping the intensity of redshifted 21-cm radiation
    - It also can tell us how the universe was reionized, as neutral hydrogen which has been ionized by radiation from stars or quasars will appear as holes in the 21 cm background.
  - Experiments

    The observation of 21-cm line is difficult because of interference from television transmitters and the ionosphere, plus noise from synchrotron emission and bremsstrahlung from galaxy.

- Experiment to Detect the Global Epoch of Re-ionization Signature (EDGES)
  A radio receiver in Australia to find 21-cm line. It measures the absorption profile of the cosmic radio-frequency spectrum, which is expected to show a strong 21-cm transition line absorption signal from Lyman-α radiation from some of the earliest stars
- Precision Array for Probing the Epoch of Reionization (PAPER) these things need more details
- Low Frequency Array (LOFAR)
- Murchison Widefield Array (MWA)
- Giant Metrewave Radio Telescope (GMRT)
- Dark Ages Radio Explorer (DARE)
- Large-Aperture Experiment to Detect the Dark Ages (LEDA)

95. Summarize the state of the experimental CMB field and what it hopes to accomplish.

- Projects and experiments on CMB
  - Observation



      - Cosmic Background Explorer (COBE, 1989) satellite detected and quantified the large scale anisotropies at the limit of its detection capabilities.
      - Wilkinson Microwave Anisotropy Probe (WMAP, 2001) made more precise measurements of the large scale anisotropies over the full sky.
      - Planck Surveyor (2009) satellite made even more precise measurement.
      - South Pole Telescope (2012). The extreme cold keeps the amount of water vapor in the air low, which causes noise.
- Goals
  - Provide evidence of theories that explain what happened before recombination. Cosmic string theory or inflation theory.
    - Polarization: Break polarization vectors into divergent (“E-mode”) and circulating (“B-mode”). “B-modes” of the polarization of the background radiation could provide evidence of the primordial GW produced by inflation.
    - The CMB doesn’t have polarization to the zero-th order because of homogeneity. The Thomson scattering of CMB photons gives it polarization.

96. (101.) Write down the Jeans equation for a disturbance propagating in a self-gravitating medium. From this show how to find the critical wavenumber for propagating modes. What is the Jeans mass?

Jeans equation

Liouville theorem

This theorem basically states that the phase space density $f(x, v)$ behaves like an incompressible fluid.

$$\frac{d}{dt} f(x, v) = 0$$

where d/dt is material derivative along a trajectory. It is true because the phase space volume remains constant through time for Hamiltonian dynamics.

$$dq' \cdot dp' = (dq + \dot{q}\delta t) \cdot (dp + \dot{p}\delta t) = dq \cdot dp \left[1 + \left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right) \delta t\right] = dq \cdot dp$$

because of Hamilton equation

$$\frac{\partial \dot{q}}{\partial q} = \frac{\partial}{\partial q}\frac{\partial \mathscr{H}}{\partial p} \quad \text{and} \quad \frac{\partial \dot{p}}{\partial p} = \frac{\partial}{\partial p}\frac{-\partial \mathscr{H}}{\partial q}$$

Jeans equation

Jeans equation is derived from Liouville theorem using a Hamiltonian with gravitational potential. The actual form are

$$\frac{\partial n}{\partial t} + \sum_i \frac{\partial (n\langle v_i \rangle)}{\partial x_i} = 0$$

$$\frac{\partial (n\langle v_j \rangle)}{\partial t} + n\frac{\partial \Phi}{\partial x_j} + \sum_i \frac{\partial (n\langle v_i v_j \rangle)}{\partial x_i} = 0 \qquad (j = 1, 2, 3.)$$

You won't remember this.

Critical wavenumber

When the gas is at static equilibrium, its property ρ_0, P_0, and v_0^i have zero time derivative. The two continuity equations are trivial, with equation of motion be

$$\nabla P_0 = \rho_0 g_0 = -\rho_0 \nabla \Phi_0$$

The first order perturbations are ρ_1, P_1, v_1, and Φ_1. Solve energy continuity equation $\partial_t(P/\rho^\gamma) = 0$, we have

$$\frac{P_0 + P_1}{P_0} = \left(\frac{\rho_0 + \rho_1}{\rho_0}\right)^\gamma \quad \Rightarrow \quad P_1 \approx P_0 \gamma \frac{\rho_1}{\rho_0} = c_s^2 \rho_1$$

Plug it in the other two equations:

$$\partial_t \rho_1 + \rho_0 \partial_i (v_1^i) = 0$$

$$\partial_t v_1^i = (g_0 + g_1)^i - \frac{\partial^i (P_0 + P_1)}{(\rho_0 + \rho_1)} = -\partial^i (\Phi_1 + P_1/\rho_0)$$

with another one $\partial_i \partial^i \Phi_1 = 4\pi G \rho_1$. Assume negligible enhanced gravity $\Phi_1 \approx 0$ to solve. These two equations above can combine to

$$\partial_t^2 \rho_1 - \partial_i \partial^i (\Phi_1 \rho_0 + c_s^2 \rho_1) \approx (\partial_t^2 - c_s^2 \partial_i \partial^i) \rho_1 = 0$$

Clearly a wave equation (the sound/acoustic wave). The general solution is the integration of all Fourier modes. Pick one mode $e^{i(kx-\omega t)}$ and plug in without ignoring Φ_1

$$(-i\omega)^2 - (4\pi G \rho_0 + c_s^2 (ik)^2) = -\omega^2 - (4\pi G \rho_0 - c_s^2 |k|^2) = -\omega^2 + c_s^2 (|k|^2 - |k_J|^2) = 0$$

Apparently, the frequency would be imaginary if we have $|k| < |k_J| = \sqrt{4\pi G \rho_0}/c_s$ (the critical wavenumber).

Jeans mass

The corresponding length scale is **Jeans length**

$$\lambda_J = \frac{2\pi}{k_J} = \frac{2\pi c_s}{\sqrt{4\pi G\rho_0}} = \sqrt{\frac{\pi\gamma k_B T}{G\rho_0 \bar{m}}}$$

If things happen at scales larger than λ_J, the enhanced gravitation can dominate over acoustic waves. So the corresponding **Jeans mass** is

$$M_J = \frac{4\pi}{3}\rho_0\lambda_J^3$$

For example, $n_0 = 10^2$ /cm^{-3} and $T = 100$ K, the Jeans mass is $10^5 M_\odot$. A more complex yet realistic theory is done by Spitzer.

97. How does a density fluctuation grow over cosmic history and how does the answer depend on its comoving wavelength?

Solving the perturbation fluid equation using the cosmic expansion background, we have

Matter-dominated

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = \tilde{\delta}\left(4\pi G\rho_0 - \frac{v_s^2 k^2}{a^2}\right)$$

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = \frac{3}{2}H^2\tilde{\delta}$$

Radiation-dominated

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = \tilde{\delta}\left(\frac{32\pi}{3}G\rho_0 - \frac{v_s^2 k^2}{a^2}\right)$$

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} = 4H^2\tilde{\delta}$$

Assume $\tilde{\delta} \propto t^n$, the solutions are

Matter-dominated:

$$n = -1(\text{decay mode}), \frac{2}{3}(\text{growth mode})$$

$$\tilde{\delta} \propto t^{2/3} = a(t)$$

$$M_\odot \sim 10^5 M_\odot$$

Radiation-dominated:

$$n = -1(\text{decay mode}), +1(\text{growth mode})$$

$$\tilde{\delta} \propto t = a^2(t)$$

$$M_\odot \sim 10^{16} M_\odot$$

98. (149.) Summarize the observational evidence in favor of the standard "Big Bang" model of the universe.

Four pillars of Big Bang theory

- Hubble's law and the expansion of space

  That space is undergoing metric expansion is shown by direct observational evidence of the cosmological principle and the Copernican principle.

- Cosmic microwave background radiation

  The CMB radiation was found to be approximately consistent with a blackbody spectrum in all directions; this spectrum has been redshifted by the expansion of the universe, and today corresponds to approximately 2.725 K.
- Big Bang nucleosynthesis

  The abundances of primordial elements (^{3}H, etc.) depend on a single parameter, the ratio of photons to baryons. This value can be calculated independently from the detailed structure of CMB fluctuations.
- Galactic evolution

  Observational differences about galaxies supports the Big Bang because we have observed differences in older galaxies farther away from Earth than for younger ones closer to Earth.

99. (155.) What is the Hubble constant? Explain at least three independent methods to measure it. How does it relate to the age of the universe?

- Measure H_0
  - Current conflicting values of H_0 from standard candle measurements ($H_0 = 73$ km/s/Mpc from Cepheid variable, type Ia supernova) and CMB measurement ($H_0 = 67$ km/s/Mpc).
  - H_0 Lenses in COSmological MOnitoring of GRAvItational Lenses (COSMOGRAIL)' s Wellspring (H0LiCOW, procounced Holy-Cow)

    The team measured H_0 by measuring time delay of the very distant quasar signal lensed by a galaxy. As time goes, the object recedes each other and causes change of path, thus changes time delay.
- The age of Universe can be derived from the expansion coefficient a. $a \propto t^{2/3}$ for matter-dominated and $a \propto t^{1/2}$ for radiation-dominated universe. The Hubble constant is thus

$$H(t_{now}) = H_0 = \frac{\dot{a}}{a} = \frac{2}{3t_{now}}$$

  So t_{now} is roughly H_0

100. Compute the age of our universe using the Friedmann equation.

- Re-write Friedmann equation as

$$H^2(t) = H_0^2(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda)$$

  For flat universe, $k = 0$, $\Omega_r \approx 0$. Replace $H = \frac{\dot{a}}{a}$ and integrate both sides to get the age of universe. $t_0 \approx 1/H_0$

101. (156.) How can the age of the universe be estimated empirically? Do the current estimates agree with that obtained from the Hubble constant?

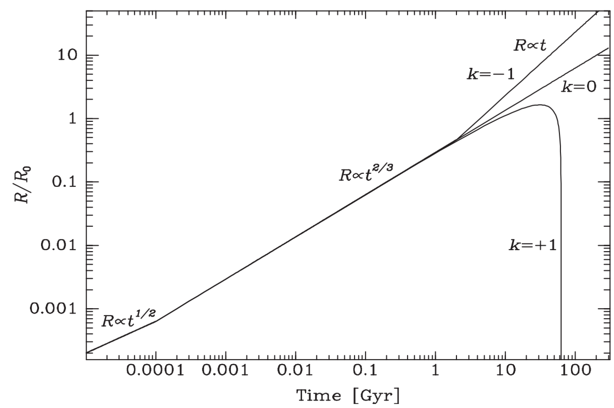
- Turn-off points from the globular clusters.
- Yes.

102. What is the Robertson-Walker metric? What are the Friedmann equations? How does the universe expand if it is (i) radiation dominated? (ii) matter dominated with $\Omega \ll 1$, $\Omega = 1$, $\Omega \gg 1$? (iii) dominated by a cosmological constant?

- The expansion is summarized in the figure to the right.
- For the dark-energy dominated:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}$$

$$a(t) \propto e^{\sqrt{\Lambda/3}t}$$



103. (162.) What was the temperature of the universe at recombination, and why did recombination happen then?

- Two ways to solve for the temperature at recombination
  - The redshift of recombination is known $z = 1100$. So the recombination photon is redshifted today, and the whole blackbody spectrum is redshifted in the same way.

$$T_{rec} = (1 + z)(2.7\text{ K}) = 3000\text{ K}$$

  - Use the Saha equation

$$\frac{n_{\mathrm{p}} n_{\mathrm{e}}}{n_{\mathrm{H}}} = \left(\frac{m_{\mathrm{e}} k_{\mathrm{B}} T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \exp\left(-\frac{E_{\mathrm{I}}}{k_{\mathrm{B}} T}\right)$$

  Assume the ionization percent is 50%, and $n_p = 1.6(1 + z)^3$, $T = 2.7(1 + z)$, the z can be solved and thus the temperature. It's pretty much the same as the first one.

104. (165.) When (or at what redshift) did the universe become (i) matter dominated? (ii) optically thin to electron scattering? What temperature was the CMB at these times? What significance did these events have for the CMB?

- The redshift of the transition is

$$\frac{\rho_r}{\rho_m} = \frac{1.7\rho_{r,0}a_0^4}{a^4}\frac{a^3}{\rho_{m,0}c^2a_0^3} = 1 \quad \Rightarrow \quad \frac{a_0}{a} \approx 3500$$

- Recombination $z = 1100$. 3000 K. That was the last scattering surface of CMB, when everything is freezed.

105. (150.) What fractions of each of the following is composed of "dark matter": (i) The disk of the Milky Way, (ii) The Milky Way, (iii) The Coma cluster of galaxies, (iv) The universe. How are each of these estimates arrived at?

- Disk of Milky Way - 50%. with a big error bar. Oort limit.
- Milky Way - 84%. Rotation curve
- Coma galaxy cluster - 90%. It is found that the galaxies were moving too fast for the cluster to be bound together by the visible matter of its galaxies. Uses Virial Theorem.
- Universe - 27%. This is from CMB measurement by Planck mission. The stucture formation also implies dark matter.

106. What evidence is there for dark energy and what might it be?

- Evidence
  - Supernova
    Supernove is a standard candle. Adam Riess et al. found that "the distances of the high-redshift SNe Ia were, on average, 10% to 15% farther than expected in a low mass density ΩM = 0.2 universe without a cosmological constant". This means that the measured high-redshift distances were too large, compared to nearby ones, for a decelerating universe.
  - CMB
    For the shape of the universe to be flat, the mass–energy density of the universe must be equal to the critical density. The total amount of matter in the universe (including baryons and dark matter), as measured from the CMB spectrum, accounts for only about 30% of the critical density.
  - Large-scale structure
    The theory of large-scale structure, which governs the formation of structures in the universe (stars, quasars, galaxies and galaxy groups and clusters), also suggests that the density of matter in the universe is only 30% of the critical density.

107. Is most of the hydrogen in the universe neutral or ionized? What is the Ly alpha forest? Why are cosmologists interested in it?

- Most of hydrogen in the universe is ionized. When hydrogen is dispersed at low density in the intergalactic medium, the gas is vulnerable to photoionization, and once ionized, the time for recombination exceeds the Hubble time. If hydrogen clouds are confined to sufficient density that they are self-shielding to the ionizing background, they are vulnerable to instability, collapse and star formation, which over time, locks the hydrogen into long lived stars.
- The light from all distant quasars is seen to be partially absorbed by numerous clouds of gas along the line of sight. A small fraction ($\sim 10^{-4}$) of the hydrogen in these clouds is neutral, and is manifest as a "forest" of redshifted absorption lines (mostly Lyman-α) in the spectrum of each quasar. Each absorption line is at the wavelength of Lyman-α redshifted according to the distance of the particular absorbing cloud.
- The Lyman-alpha forest is an important probe of the intergalactic medium and can be used to determine the frequency and density of clouds containing neutral hydrogen, as well as their temperature. Searching for lines from other elements like helium, carbon and silicon (matching in redshift), the abundance of heavier elements in the clouds can also be studied. The Lyman-alpha forest observations can be used to constrain cosmological models.

108. (168.) What are the anisotropies in the CMB? How do their amplitudes depend on angular scale?

- The anisotropy comes mainly from three effects:
  - Gravitational redshift of photons coming out of potential wells
  - Peculiar velocity - if matter that photon last scatters off is moving away, the photon gets Doppler shift
  - Denser - if photon is deep in a well, time runs slower, so " universe is younger" in the well and it' s hotter in the well

109. What are acoustic peaks in the CMB power spectrum? What do their locations and amplitudes tell us?

- These peaks are the modes of acoustic oscillation of baryon-photon fluid. Some wavelengths are preferred when it oscillates just a full cycle of recombination time, etc.
- Application
  - Location of 1st peak (0.7°): angular size of CMB tells us if the Universe is curved or not.
  - Amplitude of 1st peak: Ω_m can be measured from the height of the first peak of the power spectrum. For larger Ω_m, universe is younger for a given value of a, and thus less time for structure to form, resulting in lower amplitude of 1st peak.
  - Ratio of 1st and 2nd peak: Ω_b. The first peak corresponds to the first round of compression to the maximum density, while the second peak corresponds to 1 compression and 1 rarefaction. Because photons and baryons are coupled, they experience damped harmonic oscillation from the viscosity. Baryons pull photons into compression peaks, and therefore enhances compression and weakens rarefaction. Thus, the ratio of the even and odd peak increases with increasing Ω_b.

110. What are the "flatness problem" and the "isotropy problem" in standard Big-Bang cosmology? How does the inflationary model resolve these problems?

- Flatness problem

  The flatness problem is a cosmological fine-tuning problem within the Big Bang model of the universe. Such problems arise from the observation that some of the initial conditions of the universe appear to be fine-tuned to very ’ special’ values, and that a small deviation from these values would have had massive effects on the nature of the universe at the current time. To see this, note that for a matter dominated case. For our universe to be flat today, it had to be very flat at high redshift.

- Horizon (isotropy) problem

  Why the two sides of the sky has such a high isotropy? They should not be causally connected. Inflation explains it by arguing that the universe is causally connected during a exponentially expanding phase.

- Monopole problem

  The magnetic monopole problem says that if the early universe were very hot, a large number of very heavy, stable magnetic monopoles would be produced. Monopoles are expected to be copiously produced in Grand Unified Theories at high temperature, and they should have persisted to the present day, to such an extent that they would become the primary constituent of the universe. Not only is that not the case, but all searches for them have so far turned out fruitless, placing stringent limits on the density of relic magnetic monopoles in the universe. A period of inflation that occurs below the temperature where magnetic monopoles can be produced would offer a possible resolution of this problem: monopoles would be separated from each other as the universe around them expands, potentially lowering their observed density by many orders of magnitude.

111. (148.) Specify (at least) six stages in the standard method of determining the extragalactic distance scale.

- Cepheids

  Use the period-luminosity relation for Cepheids

- Type Ia supernovae

  Take advantage of the similarity of Type Ia supernovae light curves, from which we get the peak magnitude

- Tully-Fisher Relation

  Take advantage of the relation between luminosity of a spiral galaxy and its maximum rotation velocity.

- $D - \sigma$ Relation

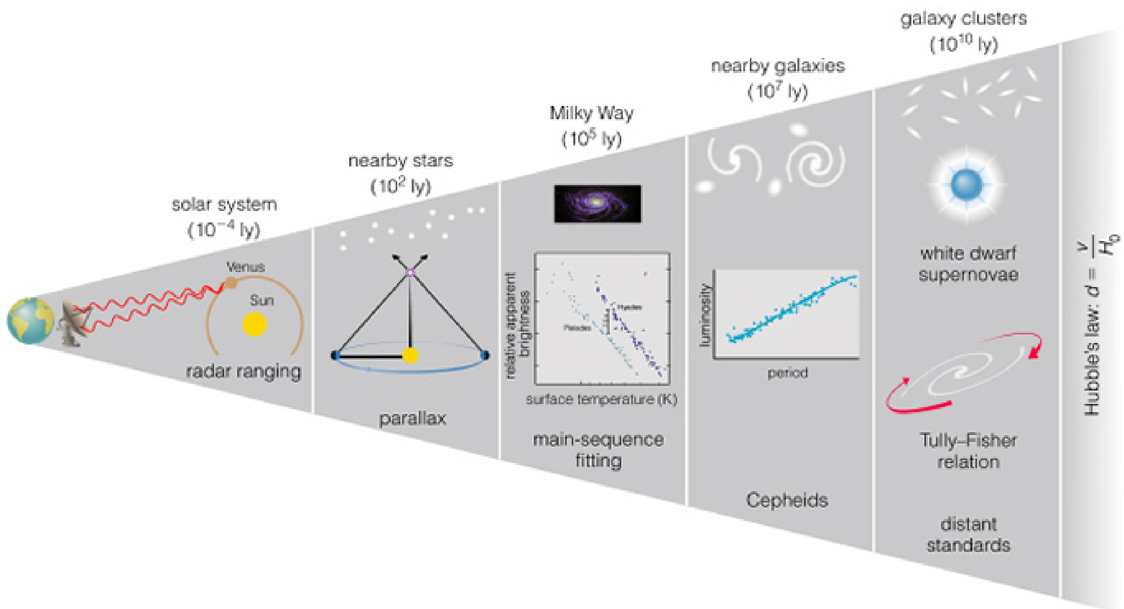
  Take advantage of the relation between the velocity dispersion σ to the diameter D of an elliptical galaxy.

- The Globular Cluster Luminosity Function

  The globular cluster luminosity function (number of globular clusters as a function of brightness) can be well described by a Gaussian, which has a well-defined peak at a known turnover magnitude.

- Parallax

- Moving cluster method



- Main sequence fitting

112. (151.) What is the baryonic contribution to the cosmological mass density and how is it determined?

- From power spectrum of the CMB measurement, we can find Ω_b by ratio of the first and the second peak height. The first peak corresponds to the first round of compression to the maximum density, while the second peak corresponds to 1 compression and 1 rarefaction. Because photons and baryons are coupled, they experience damped harmonic oscillation from the viscosity. Baryons pull photons into compression peaks, and therefore enhances compression and weakens rarefaction. Thus, the ratio of the even and odd peak increases with increasing Ω_b.
- Study of nucleosynthesis in the Big Bang also produces an upper bound on the amount of baryonic matter in the universe6. The theory of BBN gives a qualitative prediction of the production of the light " elements" d, ^{3}He, ^{4}He, and ^{7}Li. And since this prediction depends on baryon to photon ratio, we can deduce the baryon density.

113. (163.) How did the particle content of the universe evolve? What is the most abundant particle in the universe today?

- From the big bang nucleosynthesis, we can get d, ^{3}He, ^{4}He, and ^{7}Li. The mass fraction ratio for ^{4}He is about 25%, and others are quite small. After the BBN, except for the decay of radio-active materials, the amounts don' t change much. Heavier atoms were created by stars. The abundances of primeval deuterium can be measured using the Lyman-alpha forest of absorption lines in high-z molecular clouds observed in front of quasars.
- The most abundant massive particle is neutrino.

114. (167.) Explain how the apparent magnitudes of distant supernovae can be used to distinguish between cosmological models. What is the apparent magnitude difference for a supernova at z 1 between a model with $\Omega_m = 1, \Omega_\Lambda = 0$ and one with $\Omega_m = 0.3, \Omega_\Lambda = 0.7$? Is $\Omega_\Lambda \approx 0.7$ a "natural" value from the particle physics point of view?

- Type Ia supernovae produces consistent peak luminosity (2e44 J) because of the uniform mass of white dwarfs that explode via the accretion mechanism. The stability of this value allows these explosions to be used as standard candles to measure the distance to their host galaxies because the visual magnitude of the supernovae depends primarily on the distance.

- With known luminosity, we can measure the luminosity distance, which can be computed as

$$D_L = \frac{c}{H_0}(1+z)\int_0^z \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda}}$$

  We can constrain Ω stuff from D_L.

115. (173.) What is "Press-Schechter theory" and why is it wrong?

- The Press-Schechter formalism is a mathematical model for predicting the number of objects (such as galaxies or galaxy clusters) of a certain mass within a given volume of the Universe. Assume probability distribution of mass/density fluctuations δ_m is Gaussian with variance σ (which is a function of mass and the slope of $P(k)$, n):

$$P(m) = \frac{1}{\sqrt{2\pi\sigma_m^2}} e^{-\delta_m^2/2\sigma_m^2}$$

  Integrate it from critical density to infinity, the total probability, or the fraction of mass collapsed is

$$F(>m) = \int_{\delta_c}^{\infty} P(m) d\delta_m = \frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{2\sigma_m}\right)$$

  There is a missing factor of 2 if we take limit $\lim_{m\to 0}\sigma_m = \infty$ and erfc(0) = 1:

$$F(>0) = 1/2$$

  We add a fudge factor of 2 to solve it.

116. (160.) Describe the synthesis of light elements in the Big Bang. Why can their abundances constrain Ω_b? Does the microwave background play a role in Big Bang nucleosynthesis?

- Big Bang nucleosynthesis refers to the production of nuclei other than those of the lightest isotope of hydrogen during the early phases of the universe. Primordial nucleosynthesis is believed to be responsible for the formation of a heavier isotope of hydrogen known as deuterium (H-2 or D), the helium isotopes He-3 and He-4, and the lithium isotopes Li-6 and Li-7. The key parameter which allows one to calculate the effects of BBN is the number of photons per baryon (baryon-to-photon ratio η). This parameter corresponds to the temperature and density of the early universe and allows one to determine the conditions under which nuclear fusion occurs. From this we can derive elemental abundances.

- BBN will result in mass abundances of about 75% of H-1, about 25% helium-4, about 0.01% of deuterium, trace amounts (on the order of 10^{-10}) of lithium and beryllium, and no other heavy elements. That the observed abundances in the universe are generally consistent with these abundance numbers is considered strong evidence for the Big Bang theory. For a long time, to test BBN theory against observations one had to ask: can all of the light element observations be explained with a single value of the baryon-to-photon ratio? Or more precisely, allowing for the finite precision of both the predictions and the observations, one asks: is there some range of baryon-to-photon values which can account for all of the observations? More recently, the question has changed: Precision observations of the CMB give an independent value for the baryon-to-photon ratio. Using this value, we ask if the BBN predictions for the abundances of light elements are in agreement with the observations. The present measurement of helium-4 indicates good agreement, and yet better agreement for helium-3. But for lithium-7, there is a significant discrepancy between BBN and WMAP, and the abundance derived from Population II stars.

- Before nucleosynthesis began, the temperature was high enough for many photons to have energy greater than the binding energy of deuterium; therefore any deuterium that is formed was immediately destroyed (a situation known as the deuterium bottleneck). Hence, the formation of helium-4 is delayed until the universe became cool enough to form deuterium (at about $T = 0.1$ MeV), and for it to remain; after which there was a sudden burst of element formation. However, very shortly thereafter, at twenty minutes after the Big Bang, the universe became too cool for any further nuclear fusion and nucleosynthesis to occur. At this point, the elemental abundances were nearly fixed, and only change was the result of the radioactive decay of some products of BBN (such as tritium).

117. (172.) What is the galaxy correlation function? How do the correlation functions of galaxies and clusters of galaxies differ? What are "Lyman break galaxies"? Are they clustered?

- A correlation function $\xi(r) = \langle\delta(x)\delta(x + r)\rangle$ describes the distribution of galaxies in the universe. By default, correlation function refers to the two-point autocorrelation function. For a given distance, the two-point autocorrelation function is a function of one variable (distance) which describes the probability that two galaxies are separated by this particular distance. It can be thought of as a lumpiness factor - the higher the value for some distance scale, the more lumpy the universe is at that distance scale. Clusters of galaxies have 20 times larger scale than galaxies.

- Lyman-break galaxies are star-forming galaxies at high redshift that are selected using the differing appearance of the galaxy in several imaging filters due to the position of the Lyman limit. The Lyman-break galaxy selection technique relies on the fact that radiation at higher energies than the Lyman limit at 91.2 nm is almost completely absorbed by neutral gas around star-forming regions of galaxies. In the rest frame of the emitting galaxy, the emitted spectrum is bright at wavelengths longer than 91.2 nm, but very dim or imperceptible at shorter wavelengths - this is known as a " dropout" , or " break" , and can be used to find the position of the Lyman limit. Light with a wavelength shorter than 91.2 nm is in the far-ultraviolet range and is blocked by the Earth' s atmosphere, but for very distant galaxies the wavelengths of light are stretched considerably because of the expansion of the Universe. For a galaxy at redshift $z = 3$, the Lyman break will appear to be at wavelengths of about 360 nm, which is long enough to be detected by ground- or space-based telescopes.

- Yes. The strong clustering and the large bias of the LBGs are consistent with biased galaxy formation CDM theories and provide additional evidence that these systems are associated with massive dark matter halos. The results of the clustering of LBGs at $z = 3$ emphasize that apparent evolution in the clustering properties of galaxies may be due as much to variations in effective light-to-mass bias parameter among different galaxy samples as to evolution in the mass distribution through gravitational instability.

118. (177.) What are the thermal and kinetic Sunyaev-Zel'dovich effects?

- The Sunyaev-Zel' dovich effect (often abbreviated as the SZ effect) is the result of high energy electrons distorting the cosmic microwave background radiation (CMB) through inverse Compton scattering, in which the low energy CMB photons receive an average energy boost during collision with the high energy cluster electrons. Observed distortions of the cosmic microwave background spectrum are used to detect the density perturbations of the universe. Using the Sunyaev-Zel' dovich effect, dense clusters of galaxies have been observed.

- Since the Sunyaev-Zel' dovich effect is a scattering effect, its magnitude is independent of redshift. So clusters at high redshift can be detected just as easily as those at low redshift. Another factor which facilitates high-redshift cluster detection is the angular scale versus redshift relation: it changes little between redshifts of 0.3 and 2, meaning that clusters between these redshifts have similar sizes on the sky.

  The Sunyaev-Zel' dovich effect can be divided into:

  - Thermal effects, where the CMB photons interact with electrons that have high energies due to their temperature.
  - Kinematic effects, a second-order effect where the CMB photons interact with electrons that have high energies due to their bulk motion (Hubble flow).

119. (169.) What range of physical scales are being probed by current and future CMB experiments? Are the anisotropies related to galaxy formation? What is the current observational status?

- WMAP: WMAP was designed to determine the geometry, content, and evolution of the universe via a 13 arcminute FWHM resolution full sky map of the temperature anisotropy of the cosmic microwave background radiation.

- Planck: The basic scientific goal of the Planck mission is to measure CMB anisotropies at all angular scales larger than 5 to 10 arcminutes over the entire sky with a precision of 2 parts per million.

120. (171.) What are the main sources of foreground emission that CMB experiments have to contend with?

- The most prominent of the foreground effects is the dipole anisotropy caused by the Doppler shift of the CMB due to the translational motions of Sun and Earth relative to the local-to-Earth comoving frame that participates in the (mean) expansion of the universe. The dipole anisotropy and others due to Earth' s annual motion relative to the Sun and numerous microwave sources in the galactic plane and elsewhere must be subtracted out to reveal the extremely tiny variations characterizing the fine-scale structure of the CMB background.

- In particular, these foregrounds are dominated by galactic emissions such as Bremsstrahlung, synchrotron, and dust that emit in the microwave band.

121. (175.) What is the ”spherical top hat” model for the formation of galaxies and clusters? What is the significance of the number $18\pi^2$?

- The spherical top-hat model of collapse is the simplest analytic model of the nonlinear evolution of a discrete perturbation, where we assume that spherical overdensity evolves as “mini-closed universe” (by Birkhoff' s theorem) while the background evolves according to the usual Friedman equation. Combining with the conservation of energy for a spherical gravitational potential:

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = -\frac{GM}{r_{max}}$$

You can have a parametric solution for the evolution of overdensity. By substituting $r/r_{max} = \sin^2(\eta/2)$, we have

$$\begin{cases} t(\eta) & = \frac{1}{2}\sqrt{\frac{3}{8\pi G\rho_{max}}}(\eta - \sin\eta) \\ r(\eta) & = r_{max}\frac{1-\cos\eta}{2} \end{cases}$$

This is the evolution of the perturbed density. At maximum r_{max} $(\eta = \pi)$, the time is

$$t_p = \frac{\pi}{2}\sqrt{\frac{3}{8\pi G\rho_{p,max}}}$$

During this time, the background unperturbed density evolves as

$$\rho_u = \frac{3H^2}{8\pi G} = \frac{3}{8G}\left(\frac{2}{3t_p}\right)^2 = \frac{16}{9\pi^2}\rho_{p,max} \quad \Rightarrow \quad \frac{\rho_p}{\rho_u} = \frac{9\pi^2}{16}$$

Instead of collapsing to a single point, the halo virilizes to equilibrium. Virial theorem says

$$2T_{eq} + U_{eq} = 0$$

and

$$T_{eq} + U_{eq} = \frac{U_{eq}}{2} = U_{max} \quad \Rightarrow \quad r_{eq} = r_{max}/2, \ \eta = 2\pi$$

This leads to

$$\rho_{p,eq} = \rho_{p,max}\left(\frac{r_{eq}}{r_{max}}\right)^2 = \frac{\rho_{p,max}}{8} \quad \text{and} \quad t_{eq} = 2t_p$$

and

$$\rho_{u,eq} = \frac{16}{9\pi^2}\left(\frac{1}{2}\right)^2\left(\frac{1}{8\rho_{p,eq}}\right) = \frac{1}{18\pi^2}\rho_{p,eq}$$

In summary,

$$\begin{cases} \delta < \dfrac{9\pi^2}{16} & \text{, field} \\ \delta > \dfrac{9\pi^2}{16} & \text{, collapsing object} \\ \delta > 18\pi^2 & \text{, collapsed object} \end{cases}$$

122. (154.) What is the "Hubble Deep Field"? What have we learned from it?

- The Hubble Deep Field (HDF) is an image of a small region in the constellation Ursa Major, constructed from a series of observations by the Hubble Space Telescope. It covers an area 2.5 arcminutes across. The field is so small that only a few foreground stars in the Milky Way lie within it; thus, almost all of the 3,000 objects in the image are galaxies, some of which are among the youngest and most distant known. By revealing such large numbers of very young galaxies, the HDF has become a landmark image in the study of the early universe.

- While quasars with high redshifts were known, very few galaxies with redshifts greater than one were known before the HDF images were produced. The HDF, however, contained many galaxies with redshifts as high as 6, corresponding to distances of about 12 billion light-years.

- The HDF galaxies contained a considerably larger proportion of disturbed and irregular galaxies than the local universe; galaxy collisions and mergers were more common in the young universe as it was much smaller than today. It is believed that giant elliptical galaxies form when spirals and irregular galaxies collide.

- Another important result from the HDF was the very small number of foreground stars present. For years astronomers had been puzzling over the nature of dark matter, mass which seems to be undetectable but which observations implied made up about 90% of the mass of the universe. One theory was that dark matter might consist of Massive Astrophysical Compact Halo Objects (MACHOs) - faint but massive objects such as red dwarfs and planets in the outer regions of galaxies. The HDF showed, however, that there were not significant numbers of red dwarfs in the outer parts of our galaxy.

123. (153.) What is the "Lyman limit", and how does it relate to observations of high-redshift galaxies?

- The Lyman limit is the short-wavelength end of the hydrogen Lyman series, at 91.2 nm. It corresponds to the ionization energy of hydrogen, 13.6 eV. The Lyman-break galaxy selection technique relies on the fact that radiation at higher energies than the Lyman limit at 91.2 nm is almost completely absorbed by neutral gas around star-forming regions of galaxies. In the rest frame of the emitting galaxy, the emitted spectrum is bright at wavelengths longer than 91.2 nm, but very dim or imperceptible at shorter wavelengths, aka " break" , and can be used to find the position of the Lyman limit. Light with a wavelength shorter than 91.2 nm is in the far-ultraviolet range and is blocked by the Earth' s atmosphere, but for very distant galaxies the wavelengths of light are stretched considerably because of the expansion of the Universe. For a galaxy at redshift z = 3, the Lyman break will appear to be at wavelengths of about 360 nm, which is long enough to be detected by ground- or space-based telescopes.

124. (176.) Why are voids spherical?

- Voids are believed to have been formed by baryon acoustic oscillations in the Big Bang by collapses of mass followed by implosions of the compressed baryonic matter.

- Consider a single wave originating from this overdense region in the center of the plasma. This region contains dark matter, baryons and photons. The pressure results in a spherical sound wave of both baryons and photons moving with a speed slightly over half the speed of light outwards from the overdensity. The dark matter only interacts gravitationally and so it stays at the center of the sound wave, the origin of the overdensity. Before decoupling, the photons and baryons move outwards together. After decoupling the photons are no longer interacting with the baryonic matter so they diffuse away. This relieves the pressure on the system, leaving a shell of baryonic matter at a fixed radius. This radius is often referred to as the sound horizon. Without the photo-baryon pressure driving the system outwards, the only remaining force on the baryons is gravitational. Therefore, the baryons and dark matter (still at the center of the perturbation) form a configuration which includes overdensities of matter both at the original site of the anisotropy and in a shell at the sound horizon. The ripples in the density of space continue to attract matter and eventually galaxies formed in a similar pattern, therefore one would expect to see a greater number of galaxies separated by the sound horizon than by nearby length scales.

125. (159.) What is the parametric solution (in terms of an "eccentric anomaly") for the evolution of the radius of curvature and age in a dust-only universe?

- The governing equation is simply the evolution of radius in a spherical potential

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = -\frac{GM}{r_{max}}$$

By substituting $r/r_{max} = \sin^2(\eta/2)$, we have

$$\begin{cases} t(\eta) &= \frac{1}{2}\sqrt{\frac{3}{8\pi G\rho_{max}}}(\eta - \sin\eta) \\ r(\eta) &= r_{max}\frac{1-\cos\eta}{2} \end{cases}$$

where $M = 4\pi\rho_{max}r_{max}^3/3$.

126. (164.) What mass should a neutrino have to close the universe? Compare it with current upper bounds (or detection).

- The number density of the relativistic fermion is

$$dN = \frac{g}{\exp\left[\frac{E-\mu(T)}{k_B T}\right]+1}\frac{d^3p d^3x}{h^3}$$

At non-degenerate relativistic limit, $E = pc \ll \mu(T)$, so we have

$$n = g\int \frac{d^3p}{h^3}\frac{1}{\exp\left[\frac{pc}{k_B T}\right]+1} = 300 \text{ cm}^{-3}$$

at 2.7 K. We know the mass density of neutrinos:

$$\rho = \Omega_{DM}\rho_c = 0.25\times(8.7\text{e-30 g/cm}^3) = 2.2\text{e-30 g/cm}^3$$

The energy for each one is just $\rho c^2/n = 3.6$ eV.

- From the SN 1987A measurement, the upper limit on the electron neutrino is 16 eV, consistent with the results of laboratory experiments that place the upper limit at 2.2 eV. The strongest upper limit on the masses of neutrinos comes from cosmology: the Big Bang model predicts that there is a fixed ratio between the number of neutrinos and the number of photons in the cosmic microwave background. If the total energy of all three types of neutrinos exceeded an average of 50 eV per neutrino, there would be so much mass in the universe that it would collapse. This limit can be circumvented by assuming that the neutrino is unstable; however, there are limits within the Standard Model that make this dicult. A much more stringent constraint comes from a careful analysis of cosmological data, such as the cosmic microwave background radiation, galaxy surveys, and the Lyman-alpha forest. These indicate that the summed masses of the three neutrino varieties must be less than 0.3 eV.

127. (161.) How does the average density of luminous matter in the universe compare with that inferred from gravity? With the amount inferred from primordial nucleosynthesis? Is this enough to close the universe?

- From the CMB measurements, Ω_b is 4 - 5% and Ω_c (dark matter density) is 22 - 25% ($\Omega_m = \Omega_b + \Omega_c$). Ω_m can be measured from the height of the first peak of the CMB spectrum.

- From BBN, we get that Ω_b is 5%.

- Since $\Omega_m < 1$, it's not enough to close the universe.

128. What is the meaning and purpose of a log N –log S curve? Explain the current interpretation of this curve for gamma-ray bursts.

- Fluence, S, is defined as the total energy received per unit area of detector surface during the burst (the energy flux integrated over the duration of the burst). Let E be the energy of a gamma-ray burst, located at distance r from the Earth. Then we have $r(S) = \sqrt{E/4\pi S}$ for spherical symmetry. Given a constant volume density of sources n, we have

$$N = \frac{4\pi}{3}r^3 n \propto S^{-3/2} \quad \Rightarrow \quad \log N \sim -\frac{3}{2}\log S$$

- The Compton results show that this proportionality is violated when S is small enough to include the more distant, fainter sources. This implied that there is an edge to the distribution; the burst sources do not extend outward without limit. This limit could be the edge of the observable universe. what does this mean?

129. (174.) What is "biased galaxy formation"?

- Biased galaxy formation refers to phenomenological models of galaxy formation in simulations (or analytic theories) that lack sufficient resolution or physical content to allow galaxies to form directly. Crudely speaking, " bias" as used in this context refers to the difference between the galaxy distribution and that of all matter: $\delta_{galaxy} = b\delta_{CDM}$, where b is the bias factor.
- Bias was originally invoked to explain the stronger correlations of galaxy clusters compared with galaxies themselves. It is shown that the regions of high density (plausibly those regions that preferentially form galaxies) in a Gaussian random field are more strongly correlated than the overall field itself. The CDM model couldn't explain the clustering properties of galaxies on small scales if galaxies simply traced the mass distribution in the N-body simulations. A bias is introduced where galaxies only form where Universe was particularly dense. In other words, we assume a strongly non-linear relations between distributions of dark and luminous material.

VI *Interstellar Medium*

130. (93.) Discuss the various "phases" of gas in the interstellar medium. Are these phases in pressure equilibrium? The phases of gas in interstellar medium are

- H I cloud

  Use 21-cm line to analyze its velocity, temperature.

  - Cold intercloud medium - narrow absorption line and broad emission line
  - Warm intercloud medium - broad absorption line and narrow emission line
- H II cloud
  - Stromgren radius

    A radius around a hot star in which photoionization rate (aka photon emission rate) and recombination rate is balanced.

    $$Q = \frac{4}{3}\pi r_{strom}^3 n_p n_e \langle \sigma_{rec} v \rangle$$

    where Q is the number of ionizing photons rate and σ_{rec} is the cross section of recombination.
- H_2 molecular cloud

  The molecule doesn't have any radio lines, but it can be inferred from absorption lines in the UV spectra of background sources. The molecule CO is most extensively studies because it has convenient radio lines from transitions of rotational levels.

- Hot corona gas

  From supernova explostion. Very hot (1 million K) and emit X-ray due to bremsstrahlung.

- Yes, they are roughly at equilibrium.

131. (99.) What is an HII region? Estimate how the Stromgren radius scales with the luminosity of the ionizing source and with the ambient density.

- Emission lines from H II region (Ionized hydrogen by UV light (< 91.2 nm) formed around young O and B stars):
  - Pretty much all transitions lines from hydrogen.
  - Bremsstrahlung
  - Excited C, N, O atoms de-excite by emitting a photon. In lab, they de-excite by collisions because photon emission is too slow.

- Stromgren radius

  A radius around a hot star in which photoionization rate (aka photon emission rate) and recombination rate is balanced.

  $$Q = \frac{4}{3}\pi r_{strom}^3 n_p n_e \langle \sigma_{rec} v \rangle$$

  where Q is the number of ionizing photons rate and σ_{rec} is the cross section of recombination.

132. (105.) Explain what bremsstrahlung radiation is. From what kinds of astrophysical objects is such radiation observed?

- The bremsstrahlung is the free-free emission of ionized gas at millions of degrees. The charged particle in plasma radiates when accelerated or decelerated due to mutual Coulomb interactions among themselves. Its usually X-ray.

- bremsstrahlung can be observed at
  - H II region
  - Galaxy clusters
  - Hot corona gas from supernova explosion

133. (107.) What is synchrotron radiation? From what kind of astrophysical objects is such radiation observed?

- Physics

- Cyclotron: low-speed, narrow-line spectrum
- Synchrotron: relativistic, continuum spectrum

  The power of synchrotron emission is

$$P = \frac{c}{6\pi\epsilon_0}\left(\frac{q}{a}\frac{\beta^2}{1-\beta^2}\right)^2$$

  where $\beta = v/c$. The radiation points along the velocity direction for relativistic speed.

- Astro objects
  - Pulsar
  - ISM
  - Jet

134. (49.) What is meant by "free-free" absorption? How is this different from electron scattering?

- Free-free absorption is the reverse process of bremsstrahlung.
- The electron scattering releases another photon for either Compton scattering (inelastic) or Thomson (elastic). But the free-free absorption is absorbing a photon completely. All of the momentum/energy of photon is used to accelerate the charged particle.

135. (38.) What is the Saha equation and how is it used in stellar structure calculations?

- The Saha equation tells us what fraction of a gas is ionized, as a function of the temperature, density, and ionization energies of the atoms.

$$\frac{n_{i+1}n_e}{n_i} = \frac{(2\pi mkT)^{3/2}}{h^3}\frac{2g_{i+1}}{g_i}e^{-(\epsilon_{i+1}-\epsilon_i)/kT}$$

  where n_i is the density of atoms in the i-th state of ionization, that is with i electrons removed, and ϵ_i is the energy required to remove i electrons from a neutral atom, creating an i-level ion.

- It is used to explain the spectral classification of stars. From this equation (combined with Boltzmann equation), we see that strength of spectral lines depends not only on the composition of stellar atmosphere, but strongly on temperature. Therefore, using spectral lines, we can classify stars with their e□ective temperature.

136. (141.) The spectra of non-thermal radio sources frequently show a "break", i.e. a change in the slope of the flux-density vs. frequency curve. The frequency at which the break occurs (never well defined) is used to estimate the "age" of the source. Describe how this is done.

- The flux density is usually a power law of the frequency $F = \nu^n$. Two different sources
  - Thermal radio source: dependent solely on the temperature of the emitter. Black-body radiation, bremsstrahlung.
  - Non-thermal radio source: depends on the population of excited state, etc. Synchrotron radiation, maser radiation.

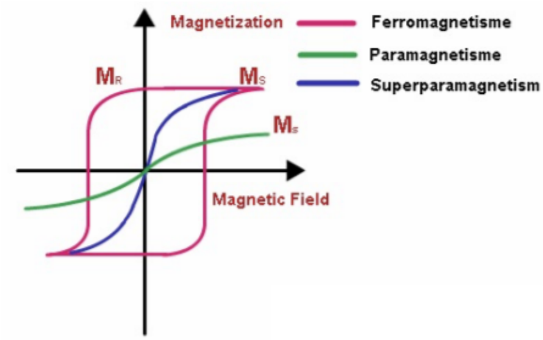
As time goes, more high-energy electrons are depleted by synchrotron radiation so there is a break on the spectrum. That's how it tells the age.

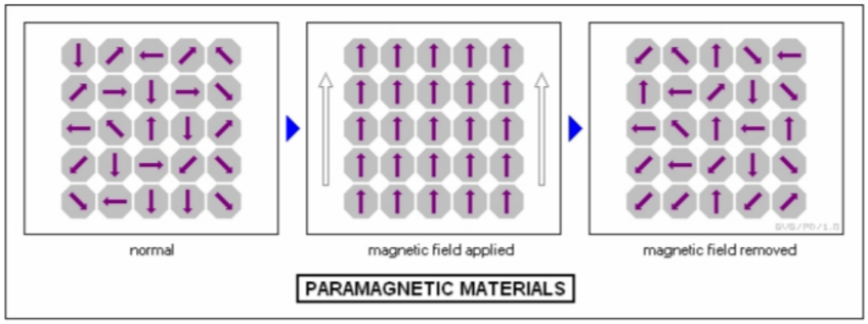
137. Explain, from a statistical mechanics point of view, why the Balmer lines are most prominent in A stars with an effective temperature of $\approx 10^4$K

- Balmer lines are produced due to atomic transitions to the n=2 atomic state from higher states. If the stellar surface temperature is too high, then hydrogen is completely ionized and such atomic transitions do not take place. On the other hand, a low surface temperature would imply that all hydrogen atoms are mostly in the ground state n=1, with very few atoms occupying the states n=3, 4, ... Only for intermediate stellar surface temperature , the levels n=3,4, ... are well populated and appropriate atomic transitions take place to produce the Balmer lines.

138. (90.) Explain how interstellar dust grains can result in linear polarization of transmitted starlight. How is the direction of polarization related to the average direction of the interstellar magnetic field (as projected on the plane of the sky)?

- The polarization of starlight is attributed to directional extinction arising when dust grains along the line of sight are aligned in the presence of a magnetic field. Only the larger grains in the size range responsible for extinction are efficient polarizers. Small grains are much less well aligned and/or much less anisotropic compared with large grains. The degree of alignment is highly sensitive to ambient physical conditions as well as to particle size. The magnetic field runs along the spiral arm of galaxy.
- The general principles of magnetic alignment: a spinning grain tends to become orientated with its longest axis perpendicular to the angular momentum vector, and **paramagnetic or superparamagnetic relaxation** imposes alignment of the angular momentum with respect to the galactic magnetic field. The mean direction of the electric vector in the transmitted beam is thus parallel to the mean field direction, i.e. the observed polarization traces the magnetic field on the sky.
- Paramagnetic relaxation





Magnetic Property	Direction of Polarization (I) Relative to External Field	Relative Magnetic Susceptibility (χ) in ppm	Typical Materials
Diamagnetism	Opposite	−10	Water, fat, calcium, most biologic tissues
Paramagnetism	Same	+1	Molecular O_2, simple salts and chelates of metals (Gd, Fe, Mn, Cu), organic free radicals
Superparamagnetism	Same	+5000	Ferritin, hemosiderin, SPIO contrast agents
Ferromagnetism	Same	> 10,000	Iron, steel

139. (95.) Explain the physics of 21 cm radio emissions from neutral hydrogen atoms.

The interstellar hydrogen gas would emit radiation at the radio wavelength of 21 cm. The proton and the electron in the hydrogen atom can have their spins either parallel or antiparallel. The state with parallel spins has slightly higher energy than the state with antiparallel spins. When transition from the higher state to the lower state takes place, radiation with wavelength 21 cm is expected to be emitted. This is, however, a ’ forbidden’ atomic line and it is not easy to see this line in laboratory experiments. Since interstellar space has huge amount of hydrogen with very low density such that an atom in the higher state is unlikely to de-excite due to collisions, it should be possible to receive emission from interstellar hydrogen at this spectral line.

140. (142.) What is the ”equipartition energy” of a synchrotron source? Why is this energy interesting?

For synchrotron source, astronomers often assume that the energy density of relativistic particles (electrons and ions) are equal to the energy density of magnetic field, for several reasons:

- It is physically plausible - systems with interacting components often tend toward equipartition.

- Large and luminous extragalactic radio sources such as Cyg A have enormous energy requirements even near equipartition; the problem of explaining the large energy is even worse otherwise.
- It eliminates an unknown parameter and permits estimates of the relativistic particle energies and the magnetic field strengths of radio sources.

- Given synchrotron radiation, we can use equipartition energy (which corresponds to minimum total energy) to estimate particle energy and magnetic field strength.

141. (145.) What kinds of sources are detected at TeV energies? How far from the Milky Way can they be observed, and why?

- The Crab Nebula is the strongest steady TeV source in the sky.
- Extragalactic sources (up to $z = 0.5$, 2000 Mpc)

  Predominant sources are BL Lac object, which is a type of active galaxy with an active galactic nucleus (AGN). BL Lac objects have spectra dominated by a featureless non-thermal continuum. The observed nuclear phenomenology of BL Lacs is interpreted as being due to the effects of the relativistic jet that is closely aligned to the line of sight of the observer, aka relativistic beaming.
- Supernovae remnants (up to 10 kpc)

  A supernova remnant (SNR) is the structure resulting from the explosion of a star in a supernova. The supernova remnant is bounded by an expanding shock wave, and consists of ejected material expanding from the explosion, and the interstellar material it sweeps up and shocks along the way.
- Pulsar wind nebulae (up to 10 kpc)

  Pulsar winds are composed of charged particles accelerated to relativistic speed by the rapidly rotating, superstrong magnetic field of the spinning pulsar. The pulsar wind streams into the interstellar medium, creating a standing shock wave, where it is decelerated to sub-relativistic speed.

142. (50.) What is the dominant absorption mechanism in the Sun's atmosphere that leads to the production of the Fraunhofer spectrum?

- The Fraunhofer lines are typical spectral absorption lines. These dark lines are produced whenever a cold gas is between a broad spectrum photon source and the detector. In this case a decrease in the intensity of light in the frequency of the incident photon is seen as the photons are absorbed, then re-emitted in random directions, which are mostly in directions different from the original one. This results in an absorption line, since the narrow frequency band of light initially traveling toward the detector, has been effectively scattered in other directions. For type G star (the Sun), Ca II, Fe I and other neutral metal lines are the major lines.

143. (92.) Why is the gas in the interstellar medium largely transparent at visible wavelengths?

The ability of a particle to scatter a beam of light depends on both the size of the particle and the wavelength of the radiation involved. As a rule of thumb, only particles having diameters comparable to or larger than the wavelength can significantly influence the beam, and the amount of scattering produced by particles of a given size increases with decreasing wavelength. Consequently, dusty regions (100-nm particle size) of interstellar space are transparent to long-wavelength radio and infrared radiation but opaque to shorter-wavelength optical, ultraviolet, and X-ray radiation. This dimming of starlight by interstellar matter is called extinction.

144. Sketch a typical cooling function $\Lambda(T)$ for diffuse interstellar gas and identify its prominent features. Overplot a hypothetical heating curve and show how to identify points of thermal equilibrium and their stability

Define the cooling function Λ as the rate at which energy is lost from unit volume in unit time. Main cooling channels:

- Fine structure cooling:

  The process of fine structure cooling is dominant in most regions of the ISM, except regions of hot gas and regions deep in molecular clouds. It occurs most efficiently with abundant atoms having fine structure levels. Collisions will excite atoms to higher levels, and they will eventually de-excite through photon emission, which will carry the energy out of the region.

- Cooling by permitted lines:

  At higher temperatures, more levels than fine structure levels can be populated via collisions. For example, collisional excitation of the $n = 2$ level of hydrogen will release a Lyα photon upon de-excitation. In molecular clouds, excitation of rotational lines of CO is important. Once a molecule is excited, it eventually returns to a lower energy state, emitting a photon which can leave the region, cooling the cloud.

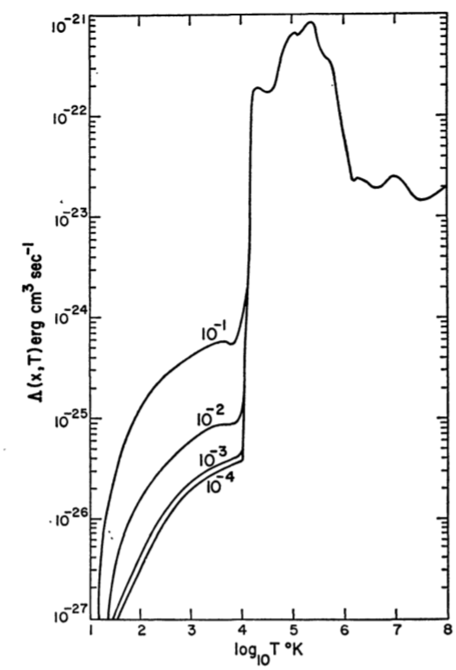


FIGURE 2. The interstellar cooling function $\Lambda(x, T)$ for various values of the fractional ionization x. The labels refer to the values of x.

- The hypothetical heating function γ gives the energy loss function $\mathscr{L} = \Lambda - \Gamma$. To have stable equilibrium, we need
$$\mathscr{L} = 0 \quad \text{and} \quad \frac{\partial \mathscr{L}}{\partial T} > 0$$
such that a decrease of the temperature decreases the energy loss rate, and vice versa. Thus the system can be in stable equilibrium corresponding to the temperatures at A and B, with the intermediate temperatures ruled out. It is proposed that the H I clouds and the warm intercloud medium correspond to the two distinct thermal equilibrium states of the neutral gas.

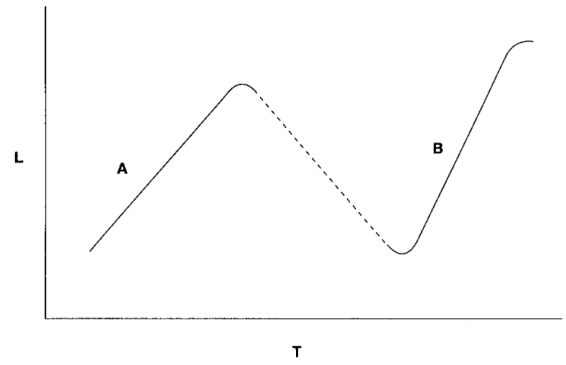


Fig. 6.14 A schematic sketch of $\mathcal{L}$ as a function of T for a system which has two possible stable equilibrium configurations.

145. (18.) What is "brightness temperature"? What are typical brightness temperatures of:

- Brightness temperature is the temperature a black body in thermal equilibrium with its surroundings would have to be to duplicate the observed intensity of a grey body object at a frequency ν.
$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}} - 1}$$
- Some examples: (what's the observed intensity?)
  - H II region at 5 GHz
  - A radio pulsar at 1 GHz
  - A compact extragalactic radio source at 5 GHz
  - The Sun at 100 MHz
  - A Citizen's Band radio

146. (147.) What are current explanations for the X-ray background? Discuss the 3 keV, the 1 MeV and the GeV portions of the spectrum quantitatively.

- 3 keV:
  - Below 1 keV the Galaxy (and probably a local hot bubble) dominates the X-ray intensity, and the extragalactic component is shielded by the interstellar medium.
  - Normal active galactic nuclei (AGNs) explain the cosmic X-ray background (CXB) below several hundreds keV. Its spectrum was well measured by the HEAO-1 mission and fits remarkably well a thermal bremsstrahlung model at a temperature of 30 keV.
  - About half of these sources are galaxies with modest X-ray luminosity produced by stellar-size black holes in binary star systems, hot gas within the galaxy, remnants of supernova explosions, or a combination of the above. The other half of the X-ray sources seen in the Deep Fields are active galaxies and their more powerful cousins, quasars. These extremely luminous galaxies have supermassive black holes in their centers that are sucking in huge quantities of gas.

- MeV: unknown
- GeV: The leading candidates are unresolved active galactic nuclei (AGN), especially blazars.

147. (106.) Make a simple classical argument to show that the spectrum of radiation from monoenergetic electrons with a speed v impinging on ions at an impact parameter b would be roughly flat up to a frequency $\sim v/b$.

- The accelearation of the charged particle is

$$a(t) = \frac{F(t)}{m} = \frac{kZe^2}{m(b^2 + v^2t^2)}$$

where $k = 1/4\pi\epsilon_0$. Using the Larmor formula to find the radiation power

$$P(t) = \frac{2}{3}\frac{ke^2a^2}{c^3} = \frac{2}{3}\frac{Z^2k^3e^6}{c^3}\frac{1}{m^2(b^2 + v^2t^2)^2}$$

Take Fourier transform to get frequency spectrum

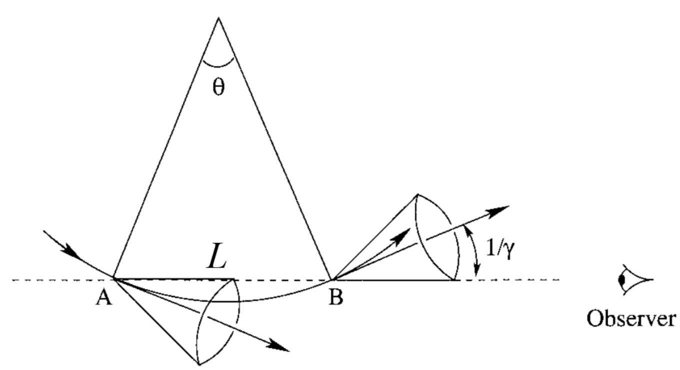
$$P(\omega) = \int dte^{-i\omega t}P(t)$$

At low frequency, the integration turns to a constant, so it's flat. The only characteristic time scale is b/v, so the cutoff starts at v/b

148. (108.) What is the spectrum of radiation from a single ultra-relativistic electron in a magnetic field? What is the spectrum of an ensemble of such electrons with energy spectrum $\propto E^p$ for $E_1 < E < E_2$?

- Single electron relativistic beaming
The beaming effect makes observer only see radiation from $1/\gamma$ cones. So the signal happens from point A to B, with distance L. The relativistic gyration frequency is

$$\omega_r = \frac{\omega_{nr}}{\gamma} = \frac{eB}{m\gamma}$$



The frequency we observe is peaked at the time difference of signals from A and B:

$$\Delta t = \frac{L}{v} - \frac{L}{c}$$

where v is the rotation speed. The approximation for small angle $1/\gamma$ gives peak frequency

$$\nu = 1/\Delta t = \gamma^2\frac{eB}{m}$$

- For a collection of electrons with distribution

$$N(E)dE \propto E^{-p}dE$$

Each electron with energy E contributes frequency $\nu \sim \gamma^2 \sim E^2$, so $E \sim \sqrt{\nu}$ and $dE \sim 1/\sqrt{\nu}d\nu$. So we have

$$f(\nu)d\nu = \nu N(E)dE = \nu\nu^{-p/2}d\nu/\sqrt{\nu} = \nu^{(1-p)/2}d\nu$$

149. (146.) Explain at least two different ways of determining the cosmic ray energy density in an external galaxy.

- Synchrotron radio emission:

  The equipartition assumption enables deduction of the cosmic ray proton energy density in star forming galaxies. There thus appears to be a remarkable equipartition of energy amongst the gas, the magnetic field and the cosmic rays.

- Supernova rates:

  The energy density can be estimated from the observed rate of core-collapse SN and the deduced residency time of cosmic ray proton in the insterstellar medium - once a fraction of SN kinetic energy that is channeled into particle acceleration has been assumed.

150. (91.) If a typical interstellar dust grain is 0.2 microns in size, and starlight suffers an extinction of 1 magnitude per kpc, estimate the space density of dust grains

- The percentage of light being scattered is $n\sigma d$, where $\sigma = \pi(d/2)^2$.

151. (96.) What are the Einstein A and B coefficients for a spectral line, and what are the relationships among them?

For a bounded atomic gas with two energy levels $E_1 < E_2$ and density n_1 at E_1 and n_2 at E_2, there are three Einstein coefficients: the coefficient of

- spontaneous emission A
- stimulated emission B
- absorption B'

These can be calculated from perturbative quantum field theory, but we can still infer something. The change of n_2 due to emission and absorption is

$$dn_2 = -An_2 - BI(\omega)n_2 + B'I(\omega)n_1$$

where $\omega = (E_2 - E_1)/\hbar$ is the photon with intensity $I(\omega)$ and right frequency to trigger stimulated emission and absorption. The spontaneous emission doesn't depend on the photons in the environment. Conservation of numbers give

$$dn_1 + dn_2 = 0$$

Assume gas at equilibrium ($dn_1 = dn_2 = 0$) and follow Boltzmann distribution ($n \propto e^{-\beta E}$), we have

$$I(\omega) = \frac{An_2}{B'n_1 - Bn_2} = \frac{A}{B'e^{\beta\hbar\omega} - B}$$

I know the Planck's law (two polarizations)

$$I(\omega) = 2 \times \frac{c}{4\pi}u(\omega) = \frac{\hbar\omega^3}{\pi^2c^2}\frac{1}{e^{\beta\hbar\omega} - 1} = \frac{A}{B'}\frac{1}{e^{\beta\hbar\omega} - B/B'}$$

So we have

$$B = B' \qquad \text{and} \qquad \frac{A}{B} = \frac{\hbar\omega^3}{\pi^2c^2}$$

The stimulated emission coefficient equals to the absorption.

152. (97.) Explain quantitatively why stimulated emission is important and spontaneous emission is usually ignored in the radio domain, whereas the reverse is true in the optical domain. Given a thermal spectrum at some temperature T, at what frequency would the two emission rates be equal?

- The ratio of stimulated emission over spontaneous emission is

$$\frac{BI(\omega)}{A} = \frac{1}{e^{\hbar\omega/kT} - 1}$$

Remember B/A is just a constant. In radio domain, $\hbar\omega/kT \ll 1$ and $I(\omega)$ is huge. So the stimulated emission dominates. Vice versa for optical domain. For equal rates, just make $e^{\hbar\omega/kT} = 2$.

153. Name five molecules found in the interstellar medium and comment on how they are detected.

- H_2

  H_2 is the most abundant molecule. Unfortunately, H_2 is very difficult to observe directly because the molecule does not have any emission or absorption lines in the visible or radio portions of the spectrum at the cool temperatures typical of the ISM. 21-cm line is one way.

- CO

  It's necessary to use other molecules, such as CO, as tracers of H_2 by making the assumption that their abundances are proportional to the abundance of H_2. CO emits radiation by changing either their rotational or vibrational states. A change in the rotational state of the CO molecule results in a photon emitted at millimeter wavelengths.

- H_2O

  The formation of molecular hydrogen and water occurs largely on dust grains. The comet tail is just water vapor.

- NH_3, CH_4

  These molecules are formed by chemical reactions.

VII *Plasma Physics*

154. (31.) What is "Faraday rotation"? How is it used in astronomy?

- Definition: It's the rotation of polarization plane of the low-frequency pulsar signals caused by interstellar plasma. It's the angular dispersion of magnetic field.

- Physics: The propagation of EM wave parallel to the magnetic field direction is different from vacuum. There are extra terms from plasma oscillations in the wave equation. As a result, the left and right circularly polarized waves propagate in different speeds and phases.

$$k_L = \frac{\omega}{c}\sqrt{1 - \frac{\omega_P^2}{\omega(\omega + \omega_c)}} \quad \text{and} \quad k_R = \frac{\omega}{c}\sqrt{1 - \frac{\omega_P^2}{\omega(\omega - \omega_c)}}$$

  where ω_P is the plasma frequency and ω_c is the cyclotron frequency. Therefore, the polarization angle of the light changes as it propagates.

$$\theta = \frac{k_L - k_R}{2} z$$

- Usage: The time dispersion also happens to the low-frequency waves. The magnitude of magnetic field in ISM can be derived by measuring angular and time dispersions. Specifically, it measures

$$\theta \propto \int_0^L n_e B_{||} dl$$

155. (27.) What is the MHD approximation? What are the MHD equations for continuity, momentum conservation and energy conservation?

- MHD (Magneto-Hydro-Dynamics) equations

  - Relate E to B.

$$\nabla \times B = \mu_0 j + \frac{1}{c^2}\frac{\partial E}{\partial t} \approx \mu_0 j = \mu_0 \sigma (E + v \times B)$$

    where σ is the electrical conductivity. The E can be solved.

- Dynamics of B (Induction equation)
  Plug E into the dynamic part of B in Maxwell's equation

$$\partial_t B = -\nabla \times E = \nabla \times (v \times B) + \frac{\nabla^2 B}{\mu_0 \sigma}$$

- Equation of motion
  we have an extra magnetic force

$$\frac{\delta F_{mag}}{\delta V} = \rho_c v \times B = j \times B = \frac{1}{\mu_0}(\nabla \times B) \times B = \frac{1}{\mu_0}(B \cdot \nabla)B - \frac{1}{\mu_0} B \cdot \nabla B \, (\text{"}\nabla |B|^2/2\text{"})$$

  Plug it in equation of motion:

$$\frac{dv^i}{dt} = g^i - \frac{\partial^i}{\rho}\left(P + \frac{|B|^2}{2\mu_0}\right) + \frac{1}{\mu_0 \rho} B^j \partial_j B^i$$

  So magnetic field introduces an additional pressure ($|B|^2/2\mu_0$) and a tension force along magnetic field lines ($B^j \partial_j B^i / \mu_0 \rho$).

- Mass continuity

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v)$$

- Energy continuity

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0$$

- MHD Approximation

  - Ignore displacement current - $\nabla \times B = \mu_0 j + \frac{1}{c^2}\frac{\partial E}{\partial t} \approx \mu_0 j$
  - Assume very high conductivity - $\mu_0 j = \mu_0 \sigma (E + v \times B) \approx \mu_0 \sigma E$
    and the induction equation becomes

$$\partial_t B = \nabla \times (v \times B)$$

  - Assume perfect fluid with adiabatic condition - zero viscosity

156. (29.) What is meant by "temperature"? What is the "pressure tensor" for a plasma?

- Temperature

  - Temperature is a measure of the stochastic motion of a many-body system. For solid, it measures the jittering motion of bound molecules as if they are harmonic oscillators. For plasma, it measures the kinetic speed of charged particles. The temperature of an isotropic fluid is diagonal and all the same.
  - For plasma, the motion might not be isotropic, for example, all of the particles are spinning around a magnetic field bundle. The temperature in the direction of magnetic field is zero because no particle is moving in that direction.

- Pressure tensor

  From Jeans equation, we can organize velocity dispersions to a pressure tensor

$$P_{ij} = nm\langle\ (v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)\ \rangle = nm(\langle v_i v_j\rangle - \langle v_i\rangle\langle v_j\rangle)$$

  and Jeans equation looks like

$$nm\left(\frac{\partial}{\partial t} + \langle v_i\rangle\frac{\partial}{\partial i}\right)\langle v_j\rangle = nF_j - \frac{\partial P_{ij}}{\partial i}$$

  The density is constant if you follow a volume element δV along the derivative.

157. (29.) What is meant by the term "temperature anisotropy"?

The plasma under galactic magnetic field has different temperatures $T_{||}$ and $T_{\perp}$ anistropies.

158. What is the physical significance of the plasma frequency? What is the dispersion relation for electro-magnetic waves in a plasma?

- The Maxwell equation is modified by having a current vector J from the plasma. The plasma frequency thus depends on the charge density, charge-to-mass ratio:

$$\omega_p = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$$

  The dispersion relation is

$$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

  If the plasma frequency is too large, the k can be imaginary, so the EM waves are reflected by the plasma boundary like a metal surface.

159. What are the three types of MHD waves in a magnetized plasma? Are magnetic fields important in the propagation of waves in the interstellar medium? In a star?

Name	Type	Propagation
Sound wave	longitudinal	k || B
Alfven wave	longitudinal	k || B
Magnetosonic wave	transverse	k ⊥ B

They are important.

160. What is the "ideal Ohm's law" for a plasma? What does "frozen-in" mean?

- Ohm's law:

$$J = \sigma(E + v \times B)$$

  Ideal Ohm's law:

$$E + v \times B = 0$$

- When magnetic Reynolds number is large, which happens when length scale is big (so most astronomical events are in this case) or resistivity is small, magnetic flux is frozen. In other words, the magnetic flux moves with the plasma; if the plasma column is bent the magnetic field lines are also bent, and if one end of the plasma column is twisted, then the magnetic field lines are also twisted. The magnetic field in an astrophysical system can almost be regarded as a plastic material which can be bent, twisted or distorted by making the plasma move appropriately.

VIII *Gravitational Physics*

161. Summarize the state of the experimental gravitational wave detection and what it hopes to accomplish.

- Current GW detectors
  - LIGO
    - DARM sensitivity: $\sim 10^{-20}$ m/$\sqrt{\text{Hz}}$, strain sensitivity: $\sim 10^{-23}$ m/$\sqrt{\text{Hz}}$.
    - Angle-averaged binary NS range: 134 Mpc.
    - Event rate: nearly once per week in O3.

    Current upgrade A+ plans to install frequency-dependent squeezing and new coating.
  - Virgo
  - KAGRA
- Future GW detectors
  - Cosmic explorer
  - LIGO-Voyager
  - LISA

162. (178.) How does a gravitational lens work? Explain what needs to be measured to use a gravitational lens system to measure the Hubble constant. What is the current status of these determinations of H_0?

- Gravitational lensing happens when light trajectory is bent towards the star due to its gravitation. The bending angle in weak-field is

$$\alpha \propto \frac{M}{b} = \frac{4GM}{bc^2} = \frac{2r_s}{b}$$

  where M is the mass, b is the impact parameter, and r_s is the Schwartzchild radius. This is the same as the change of velocity in two-body relaxation $/v = 2r_S/b$.

- Measure H_0
  - Current conflicting values of H_0 from standard candle measurements ($H_0 = 73$ km/s/Mpc from Cepheid variable, type Ia supernova) and CMB measurement ($H_0 = 67$ km/s/Mpc).
  - H_0 Lenses in COSmological MOnitoring of GRAvItational Lenses (COSMOGRAIL)' s Wellspring (H0LiCOW, procounced Holy-Cow)

    The team measured H_0 by measuring time delay of the very distant quasar signal lensed by a galaxy. As time goes, the object recedes each other and causes change of path, thus changes time delay.

163. (74.) What is the "Shapiro time delay"?

- Literature
  - It's the time delay of the passage of light over finite coordinate distance according to a Schwarzschild metric.
  - Shapiro proposed a test: bounce radar beams off the surface of Venus and Mercury and measure the round-trip travel time. When the Earth, Sun, and Venus are most favorably aligned, Shapiro showed that the expected time delay, due to the presence of the Sun, of a radar signal traveling from the Earth to Venus and back, would be about 200 microseconds.
  - It's the "fourth test" of GR in addition to three tests proposed by Einstein: gravitational redshift, bending of light, and precession of orbit.
- Physics

  Calculate from Schwartzchild metric. The result is

$$\Delta t = -\frac{2GM}{c^3} \ln(1 - \hat{r} \cdot \hat{x})$$

  where $\hat{r}$ is the unit vector to the source, and $\hat{x}$ is the unit vector pointing to the mass M.

164. What is gravitational microlensing, what do we learn from it, and how are various experiments studying this phenomenon?

- Unlike strong or weak lensing, microlensing from a low-mass lens (a planet) is too weak such that the images can't be resolved separately. But it still causes magnitude to change slightly and detectable. You have to monitor the light curve (intensity vs. time) to tell if there's a microlensing. The Einstein angle is around microarcsec, so it's called microlensing.
- The only parameter from the light curve is the timescale of the whole process. The Einstein angle is

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{d_s - d_l}{d_s d_l}}$$

where d_s is distance to source and d_l is distance to lens. Let $u = \Delta\theta/\theta_E$ with $\Delta\theta$ being the angular difference, the magnification is

$$A(u) = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}$$

So $A(u) > 1$ and $A(u) \to \infty$ at $u = 0$ (ideal point source but not physical). A single event only tells you the timescale, and you need other parameters to determine the lens mass, proper velocity, etc.

- First successful microlensing: Optical Gravitational Lensing Experiment (OGLE) 1992

165. What is the "weak lensing effect"? Explain how it can be used to measure the masses of clusters of galaxies. What other ways are there to estimate the masses of clusters? Compare the different methods.

- Weak lensing effect:

  The distortions of background sources are much smaller and can only be detected by analyzing large numbers of sources to find coherent distortions of only a few percent. The lensing shows up statistically as a preferred stretching of the background objects perpendicular to the direction to the center of the lens.

- By measuring the shapes and orientations of large numbers of distant galaxies, their orientations can be averaged to measure the shear of the lensing field in any region. This, in turn, can be used to reconstruct the mass distribution in the area: in particular, the background distribution of dark matter can be reconstructed. Since galaxies are intrinsically elliptical and the weak gravitational lensing signal is small, a very large number of galaxies must be used in these surveys. The results of these surveys are important for cosmological parameter estimation, to better understand and improve upon the Lambda-CDM model, and to provide a consistency check on other cosmological observations. They may also provide an important future constraint on dark energy.

- Other methods to estimate cluster masses

  - Stellar light:
    Counting the number of luminous stars in galaxies and count galaxies in a cluster.
  - Galaxy velocity dispersion:
    Treat the cluster as a virialized gravitational potential well. Just do Kepler's law from measured velocity to find the total mass.
  - Sunyaev-Zel'dovich effect:
    As a CMB photon goes through a cluster, it will interact with some of the high energy electrons in the cluster's hot gas (the same ones responsible for the X-ray emission). The result is that the CMB photon gains a little bit of energy, causing the CMB to appear hotter in the direction of the cluster than it would be in the cluster's absence. Through a conspiracy of math and physics, the change in CMB intensity is essentially due to the cluster's mass alone - and the SZ signal is basically independent of the cluster's redshift. With known temperature, the mass can be found.

166. What is Lense-Thirring precession, how might we observe it, and what evidence there is for it.

- Lense-Thirring precession is a relativistic correction to the precession of a gyroscope near a large rotating mass such as the Earth. It is a gravitomagnetic frame-dragging effect.
- It's hard to describe. Watch https://www.youtube.com/watch?v=dNswvk3oP$_k$

IX *Observational Astronomy*

167. (11.) Explain how a radio interferometer manages to produce sub-arcsecond images when none of the constituent dishes has an angular resolution of better than 1 arcmin.

- Parabolic antenna

  Aka dish antenna. The cross section of the dish has a parabolic shape such that an incident plane wave will have the same distance reflected to the sensor at focus. A spherical source emitted from the focus will also be reflected to plane wave by the parabolic dish.

- Radio interferometer
  - A large array of radio antenna. It's basically a telescope with a very large and incomplete-filled apertures.
  - Aperture synthesis imaging

    Each pair of antenna forms a unique distance (baseline) towards the source. For n antenna, you can get $n(n-1)/2$ baselines and these much points of the sky. With Earth rotating, the baselines are changing with time to get more points for the whole image.
  - Rayleigh criterion

    The angular resolution of the telescope is determined by the diffraction limit

    $$\theta = 1.22\frac{\lambda}{D}$$

    The aperture of radio telescopes is determined by the max baseline length, which is larger than the diameter of a single dish.

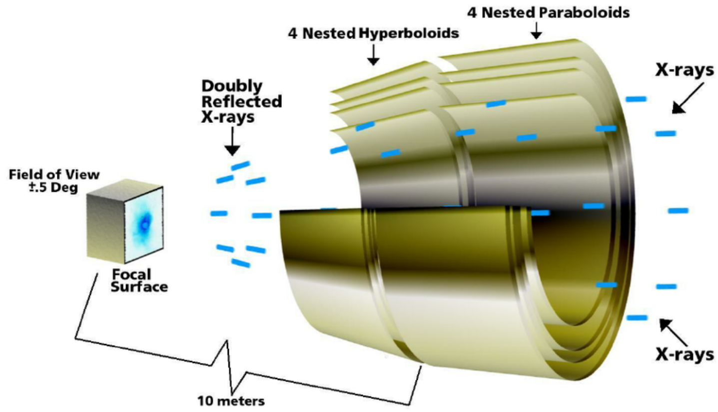
168. (12.) How does an X-ray telescope image? Why is it necessary to have the reflection take place at grazing angles of incidence?

- Grazing angles of incidence
  - High energy X-ray photons will pass through most objects.
  - The grazing angle of incidence will only deflect the photons at tiny angle, which can be guided to the focus. The physics behind is Bragg's law

    $$2d\sin\theta = N\lambda$$

    where *theta* is angle of incidence. For very small λ, θ needs to be small.

- Wolter telescope - the telescope design for X-ray.



- X-ray telescopes
  - Einstein Observatory (1978), aka HEAO-2, was the first orbiting X-ray observatory with a Wolter Type we telescope.
  - Chandra X-ray telescope (1999) returns thousands 0.5 arc-second images and high-resolution spectra of all kinds of astronomical objects in the energy range from 0.5 to 8.0 keV.
  - NuStar (2012) observes radiation in a high-energy range (3–79 keV).

169. (7.) What are the strongest spectral lines seen in each of the following: Integrated light from a typical galaxy, a quasar, a 100 km/s interstellar shockwave, a giant molecular cloud?

- Integrated light from a galaxy: the strongest line is Hα line (Balmer series, transition 3→2). It's from the H II region of ISM where ionized hydrogen recombine to level 3 and cascade to 2.

  For elliptical galaxy, old stars and metallic absorption lines.

- Quasar: Very strong UV lines (Lyα, 122 nm). It's due to the hot accretion disk near the center black hole. Some Balmer lines are also strong. "UV bump"

- 100 km/s interstellar shockwave (caused by supernova explosion): X-ray from bremsstrahlung and synchrotron radiation.

- Giant molecular cloud: low temperature, very thin. Hydrogen molecule is difficult to detect, but transition of rotational levels of CO can be seen (microwave).

170. (8.) Is atmospheric "seeing" better at a wavelength of 1μm, 10μm, or 100m? Why?

- Definition of seeing: the amount of blurring and twinkling of stars due to turbulent mixing in the atmosphere, causing variations of refractive index. It's a limiting factor for ground observatories. The seeing limits the resolution about 1 arcsec. Deformable mirrors, adaptive optics can help produce sharp images as good as 1/20 arcsec.

- Radio waves (> 1 mm) are not affected by Earth atmospheres, because the atmosphere can't be driven to emit radio waves. The Earth ionosphere can perturb radio waves though, so wavelengths longer than 10 m can't be seen (the radio window is 10 cm - 10 m). 10 μm is better than 1 μm.

171. How far away are the following objects? How large and how massive are they?

Name	Distance	Radius	Mass
Earth	0	6300 km	6e24 kg
Sun	1 AU = 1.5e11 m	7e8 m	2e30 kg
Hyades (nearest open cluster)	47 pc	3 pc	400 $M_\odot$
Orion nebula	410 pc	7.4 pc across	2000 $M_\odot$
M13 galaxy	6.8 kpc	26 pc	6e5 $M_\odot$
M31 galaxy	780 kpc	30 pc	1e12 $M_\odot$
Virgo cluster	16 Mpc	2.2 Mpc	1e15 $M_\odot$
Coma cluster	100 Mpc	4 Mpc	1e15 $M_\odot$
3C 273	749 Mpc	60 kpc	1e9 $M_\odot$

172. (6.) What are the brightest extra-solar-system sources at the following wavelengths: 10 cm, 10^{-1} cm, 10^{-3} cm, 10^{-5} cm, 10^{-7} cm, 10^{-9} cm? Which, if any, are isotropic on the sky?

- 10 cm (radio) - radio galaxy (quasar), double-lobed part. Synchrotron radiation.
- 10^{-1} cm (microwave) - CMB (isotropic)
- 10^{-3} cm (IR) - dust around star-forming region, giant molecular cloud
- 10^{-5} cm (UV) - Lyman-α from O, B stars. Things involve accretion like AGN.
- 10^{-7} cm (X-ray) - X-ray binary, supernova remnant, coma clusters
- 10^{-9} cm (gamma-ray) - gamma-ray burst, active galactic nuclei,

173. How massive are the SMBHs in the Milky Way and M31 and how do we know this?

- Milky Way has supermassive black hole of 5 million $M_\odot$. The orbits of stars near the galactic center show something of huge mass (millions of $M_\odot$, but its luminosity is small.
- M31 has a 1e8 $M_\odot$ SMBH.

174. Briefly describe the following kinds of astronomical instruments and their purpose

- A Schmidt camera is a telescope designed to provide wide fields of view with limited aberrations. It's world's first refraction-reflection telescope that combines advantages of each one. It's used in Zwicky Transient Facility (ZTF).
- Echelle spectrograph is a type of diffraction grating which is characterized by a relatively low groove density but is optimized for high diffraction orders. Echelle gratings are used in spectrometers and similar instruments, such as High Accuracy Radial Velocity Planet Searcher (HARPS), and numerous other astronomical instruments.
- CCD is a major technology for digital imaging. In a CCD image sensor, pixels are represented by p-doped MOSFET capacitors. These capacitors are biased above the threshold for inversion when image acquisition begins, allowing the conversion of incoming photons into electron charges at the semiconductor-oxide interface; the CCD is then used to read out these charges. HST and SDSS both uses CCD cameras to take images.
- III aJ emulsion is a photographic plate sensitive to blue light. You lick it to find which side has emulsion.
- The proportional counter is a type of gaseous ionization detector device used to count particles of ionizing radiation. A key feature is its ability to measure the energy of incident radiation. For example, it is used for high energy photon detector (e.g. x-rays and gamma-rays). It's different Geiger counter because Geiger is saturated for even single event. There's no proportion in Geiger.
- A Hydrogen maser is a specific type of maser that uses the intrinsic properties of the hydrogen atom to serve as a precision frequency reference. The successful operation of a VLBI (Very Long Baseline Interferometry) radio telescope system requires that the data recordings be synchronized within a few millionths of a second and that the local oscillator reference signal be stable to better than one part in a trillion. Hydrogen maser frequency standards are used to give a timing accuracy of only a few billionths of a second and a frequency stability of one part in a billion billion.
- Aperture synthesis telescope is a type of interferometry that mixes signals from a collection of telescopes to produce images having the same angular resolution as an instrument the size of the entire collection. At each separation and orientation, the lobe-pattern of the interferometer produces an output which is one component of the Fourier transform of the spatial distribution of the brightness of the observed object. The image (or " map") of the source is produced from these measurements. Astronomical interferometers are commonly used for high-resolution optical, infrared, submillimeter and radio astronomy observations.

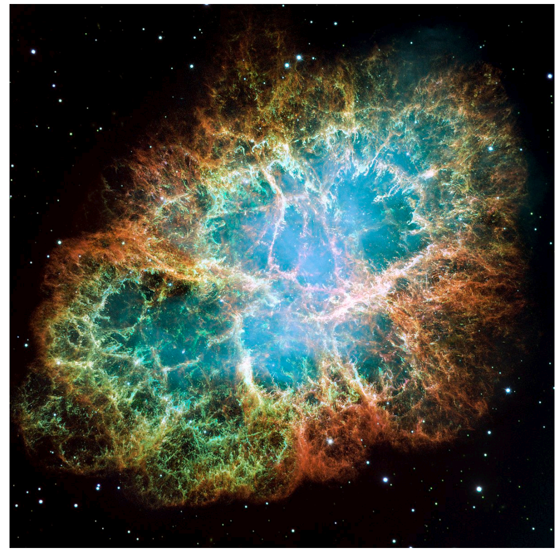
175. (4.) Briefly Describe the following famous astronomical objects: The Crab, M3, M31, M87, W51, Cyg X-1, 3C 273, Cyg A, SS 433, LMC, Orion, M31, 1983 TB, Boötes Void, Virgo, h and χ Persei

- The Crab

  - The Crab Nebula (M1) is a supernova remnant and pulsar wind nebula. It corresponds to a bright supernova recorded by Arab, Chinese and Japanese astronomers in 1054.

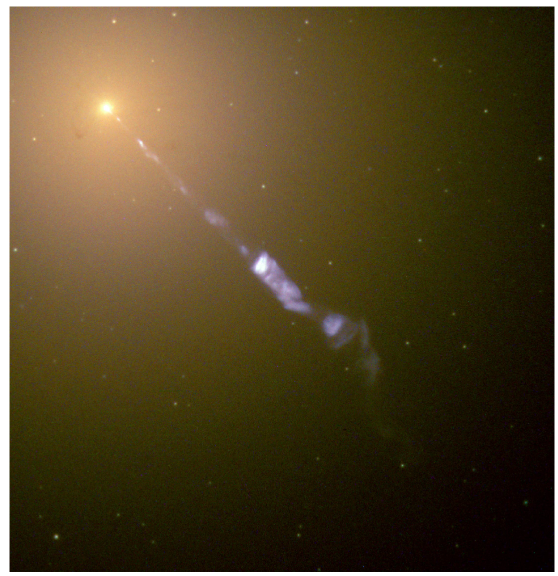
  - Located at a distance of about 2 kpc from Earth, the nebula has a diameter of 3.4 pc and expands at a rate of about 1,500 kilometers per second.

  - At the center of the nebula lies the Crab Pulsar, a neutron star (or spinning ball of neutrons), 28 - 30 km across, which emits pulses of radiation from gamma rays to radio waves with a spin rate of 30.2 times per second. The nebula was the first astronomical object identified with a historical supernova explosion.



- M87

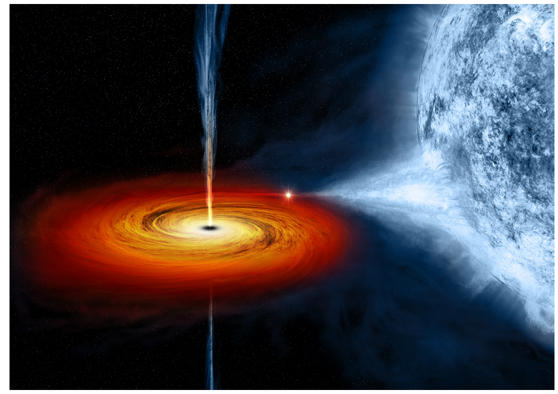
  - Messier 87 (also known as M87, Virgo A or NGC 4486) is a supergiant elliptical galaxy, whose nucleus provides the strongest observational evidence for the existence of a black hole. It is located about 16.4 Mpc from Earth. It is the most powerful known source of radio energy among the thousands of galactic systems comprising the so-called Virgo Cluster. It is also a powerful X-ray source, which suggests the presence of very hot gas in the galaxy. A luminous gaseous jet projects outward from the galactic nucleus. Both the jet and the nucleus emit synchrotron radiation, a form of non-thermal radiation released by charged particles that are accelerated in magnetic fields and travel at speeds near that of light.



- W51

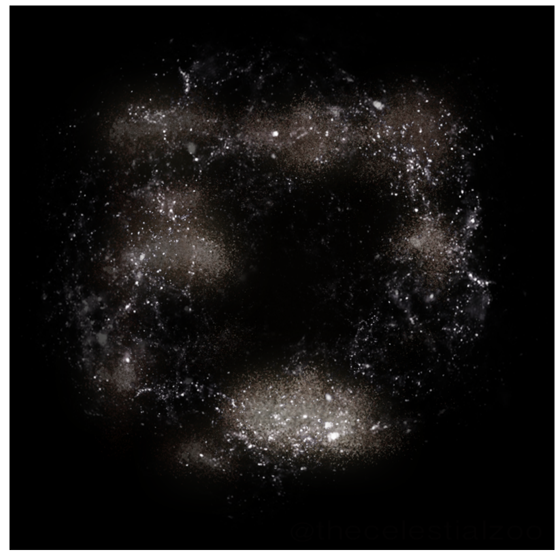
  W51 is a giant molecular cloud and massive star formation region that includes a giant HII region, several smaller HII regions and masers, and a supernova remnant.

- Cygnus X-1
  - Cygnus X-1 (abbreviated Cyg X-1) is a galactic X-ray source in the constellation Cygnus. It is one of the strongest X-ray sources seen from Earth. Cygnus X-1 was the first X-ray source widely accepted to be a black hole candidate.
  - Cygnus X-1 belongs to a high-mass X-ray binary system about 1.8 kpc from the Sun that includes a blue supergiant variable star which it orbits at about 0.2 AU. A stellar wind from the star provides material for an accretion disk around the X-ray source. Matter in the inner disk is heated to millions of degrees, generating the observed X-rays.



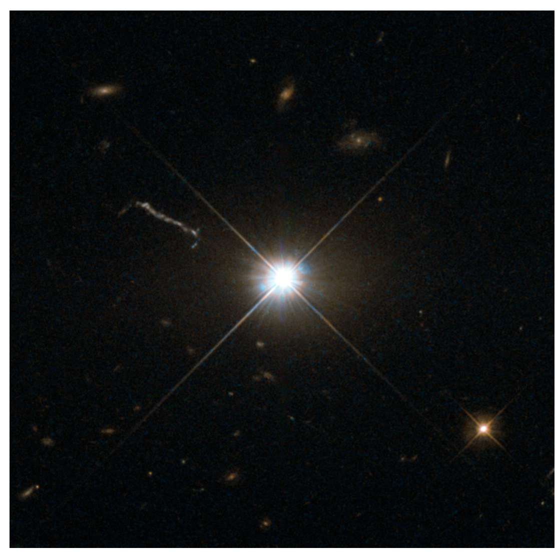
- Boötes void

  The Boötes void or the Great Void is a huge and approximately spherically shaped region of space, containing very few galaxies, located in the vicinity of the constellation Boötes. At nearly 80 Mpc in diameter (approximately 0.27% of the diameter of the visible universe), or nearly 236,000 Mpc^3 in volume, the Boötes void is one of the largest known voids in the universe, and is referred to as a supervoid.



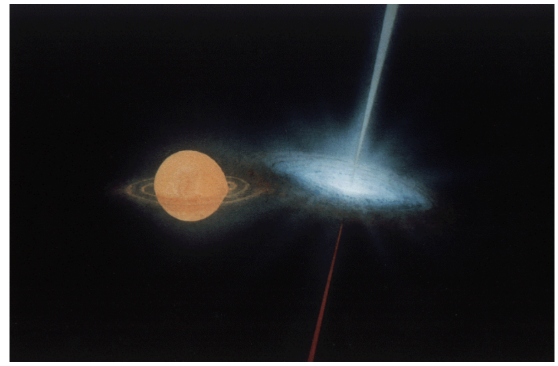
- 3C 273

  3C 273 is the first quasar ever to be identified. It is the optically brightest quasar in our sky, and one of the closest with a redshift, z, of 0.158. It is also one of the most luminous quasars known. The quasar has a large-scale visible jet, which measures 60 kpc long.



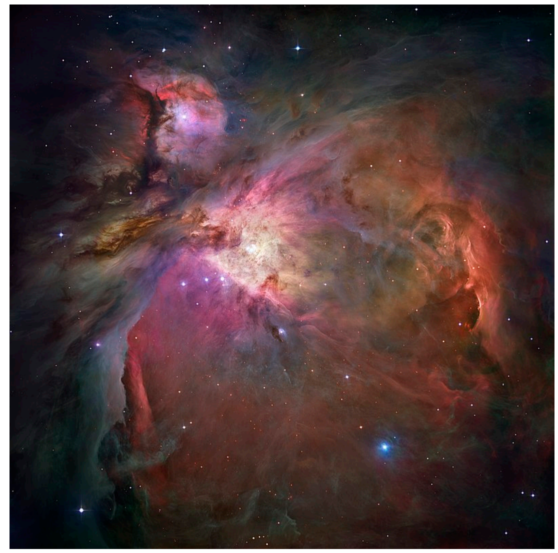
SS 433”

SS 433” is an eclipsing X-ray binary system, with the primary most likely a black hole, or possibly a neutron star. SS 433” is a microquasar, the first discovered. The compact central object is consuming the companion star which rapidly loses mass into an accretion disc formed around the central object.



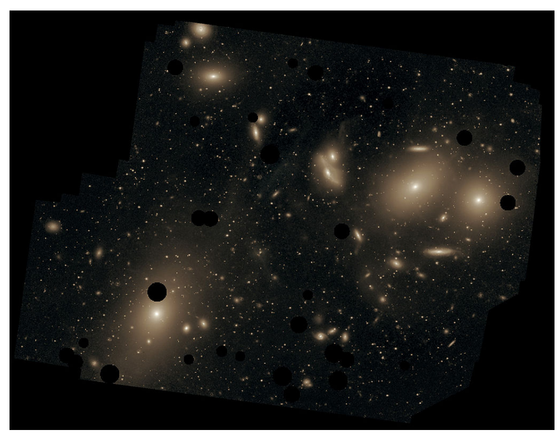
Orion

- The Orion Nebula (also known as Messier 42, M42, or NGC 1976) is a diffuse nebula situated south of Orion’ s Belt in the constellation of Orion. It is one of the brightest nebulae, and is visible to the naked eye in the night sky. M42 is located at a distance of 400 pc and is the closest region of massive star formation to Earth.



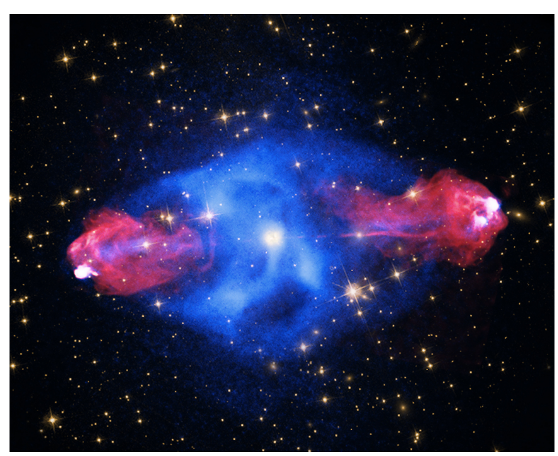
Virgo

- Virgo is one of the constellations of the zodiac and it is the second largest constellation in the sky (after Hydra). Virgo is also home to the quasar 3C 273 which was the first quasar ever to be identified.
- Virgo cluster is the closest large cluster of galaxies. Although spirals are more numerous, the four brightest galaxies are giant ellipticals, among them M87.



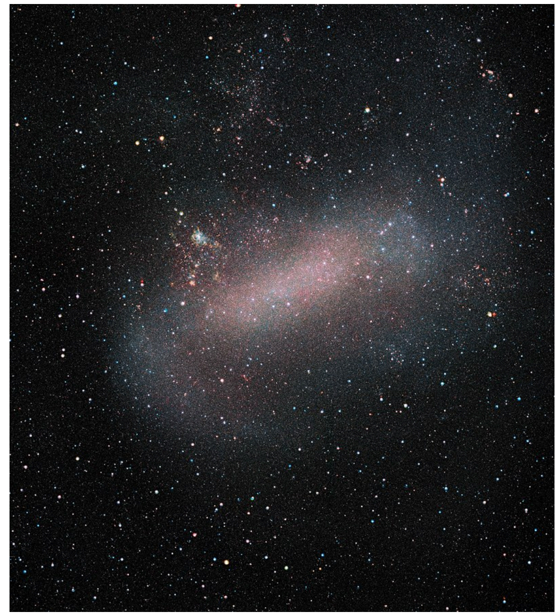
Cygnus A

- Cygnus A (3C 405) is one of the most famous radio galaxies, and among the strongest radio sources in the sky. Like all radio galaxies, it contains an active galactic nucleus.
- Images of the galaxy in the radio portion of the electromagnetic spectrum show two jets protruding in opposite directions from the galaxy’ s center.



LMC

The Large Magellanic Cloud (LMC) is a nearby irregular galaxy, and is a satellite galaxy of the Milky Way. At a distance of slightly less than 50 kpc, the LMC is the third closest galaxy to the Milky Way. The LMC is the fourth largest galaxy in the Local Group, after the Andromeda Galaxy (M31), our own Milky Way Galaxy, and the Triangulum Galaxy (M33). Like many irregular galaxies, the LMC is rich in gas and dust, and it is currently undergoing vigorous star formation activity.



M31

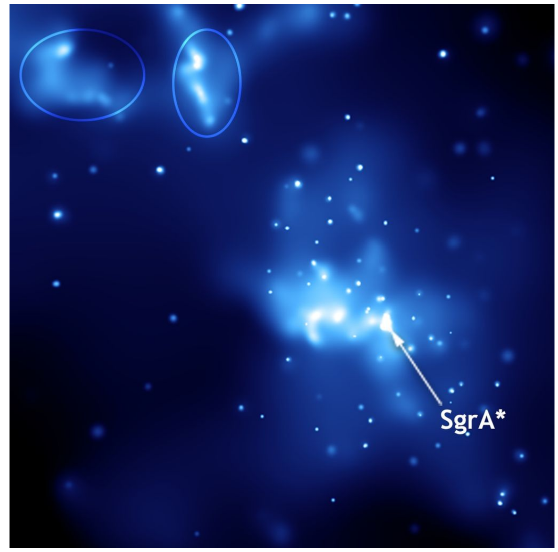
The Andromeda Galaxy (also known as Messier 31, M31, or NGC 224) is a spiral galaxy approximately 0.8 Mpc from Earth in the Andromeda constellation. The Andromeda Galaxy is the largest galaxy of the Local Group, which also contains the Milky Way, the Triangulum Galaxy, and about 30 other smaller galaxies.

h and χ Persei

h and χ Persei, aka Double Clusters, is the two young open clusters in the constellation Perseus. χ Persei (NGC884) and h Persei (NGC869) are about 2 kpc away and less than 31 pc apart. Although open clusters are quite common, this pair is exceptional due to the large number of young bright O and B stars in each, and their closeness whilst still being clearly distinguished. The clusters are also blueshifted, with NGC 869 approaching Earth at a speed of 22 km/s and NGC 884 approaching at a similar speed of 21 km/s.

Sagittarius A*

Sagittarius A* is a bright and very compact astronomical radio source at the Galactic Center of the Milky Way. It's a SMBH and related to 2020 Nobel Prize.



176. Briefly describe the main objectives and capabilities of the following astronomical observatories Voyager, Einstein, IRAS, HST, Magellan, COBE, Compton, GRO, Rossi XTE, AXAF (Chandra), WMAP, Planck, ROSAT, Spitzer, TESS, LIGO & LISA.

Voyager

- The Voyager program is an American scientific program that launched two unmanned space missions, the probes Voyager 1 and Voyager 2.
- On 15 June 2012, scientists at NASA reported that Voyager 1 might be very close to entering interstellar space and becoming the first man-made object to leave the Solar System.

Einstein

- Einstein was the first fully imaging X-ray telescope put into space.
- It carries
  - Grazing incidence telescope
  - Proportional Counter
  - Grating Spectrometer

IRAS

The Infrared Astronomical Satellite, or IRAS, was the first mission to put a telescope in space to survey the sky in infrared.

HST

The Hubble Space Telescope can see apparent magnitude of 30. It has three key goals:

- A study of the nearby intergalactic medium.
- A medium deep survey using the Wide Field Camera.

Magellan

It's a 6-m telescope in Chile.

- COBE

  The COBE satellite was developed to measure the diffuse infrared and microwave radiation from the early universe to the limits set by our astrophysical environment. The angular resolution is not good enough to reach the first peak.

- Compton GRO

  The Compton Gamma-Ray Observatory was launched on April 5, 1991. CGRO has four instruments that cover an unprecedented six orders of magnitude in energy, from 30 keV to 30 GeV. Over this energy range CGRO has an improved sensitivity over previous missions of a full order of magnitude. Some scientific highlights:

  - The Discovery of an isotropic distribution of the Gamma-ray burst events.
  - Discovery of Blazar Active Galactic Nuclei as primary source of the highest energy cosmic Gamma-rays.

- Rossi XTE

  The Rossi X-ray Timing Explorer. No grazing incidence on it. It's super fast.

- AXAF

  NASA' s Advanced X-ray Astrophysics Facility, (AXAF), renamed the Chandra X-ray Observatory in honor of Subrahmanyan Chandrasekhar, was launched in 1999. The combination of high resolution, large collecting area, and sensitivity to higher energy X-rays will make it possible for Chandra to study extremely faint sources. Spatial resolution was less than 1 arcsec.

- WMAP

  The Wilkinson Microwave Anisotropy Probe (WMAP) mission reveals conditions as they existed in the early universe by measuring the properties of the cosmic microwave background radiation over the full sky. They also reveal the primordial structure that grew to form galaxies and will test ideas about the origins of these primordial structures. Measuring the temperature of the microwave sky to an accuracy of one millionth of a degree requires careful attention to possible sources of systematic errors.

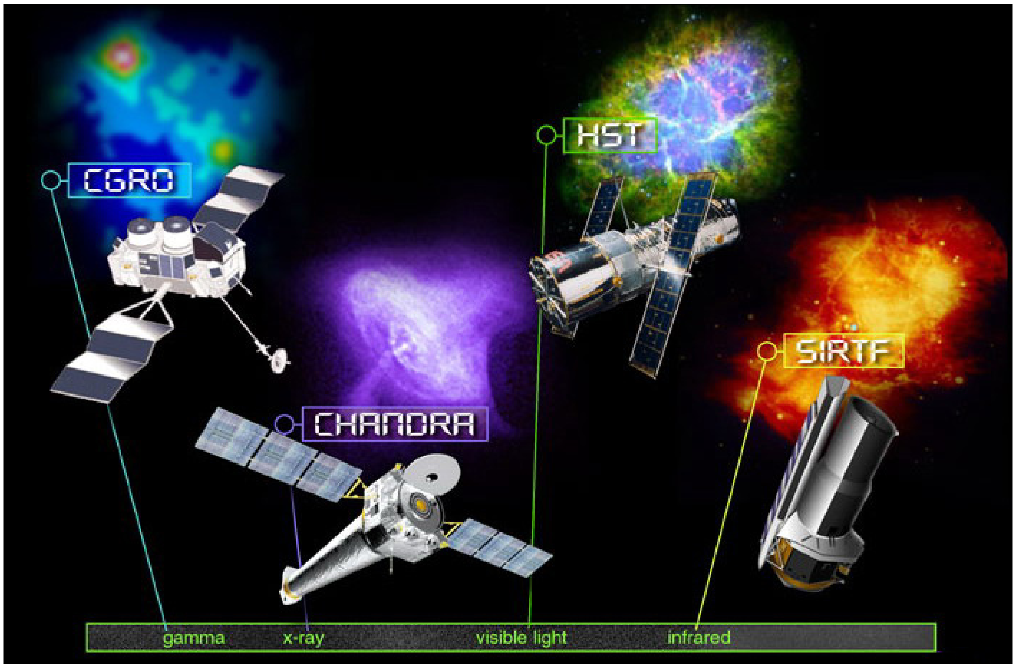
- Planck

  Planck is designed to observe the anisotropies of the cosmic microwave background (CMB) over the entire sky, at a high sensitivity and angular resolution. Some goals:

  - High resolution detections of both the total intensity and polarization of the primordial CMB anisotropies
  - Creation of a catalogue of galaxy clusters through the Sunyaev-Zel' dovich effect.
  - Observations of the gravitational lensing of the CMB, as well as the integrated Sachs-Wolfe effect

- Spitzer Space Telescope

  The Spitzer Space Telescope is designed to study the early universe in infrared light. It's one of the Great Observatories Program. It requires cryogenic cooling.



- Fermi Telescope

  The Fermi Gamma-ray Space Telescope is an international and multi-agency space observatory that will study the cosmos in the photon energy range of 8 keV to greater than 300 GeV.

- Swift

  Swift is a multi-wavelength observatory dedicated to the study of gamma-ray burst (GRB) science.

- James Webb Space Telescope

  The James Webb Space Telescope (sometimes called JWST) will be a large infrared telescope with a 6.5-meter primary mirror. It's going to observe high redshift ($z \sim 9$) where everything is redshifted 10 times. It will be launched at L2 point farther away from Sun, which is 1/100 AU.

- ROSAT

  It's a German X-ray telescope, named after Röntgen. After Einstein, before Chandra. ROSAT is an upgrade from Einstein, and it did a full-sky survey and made a catalogue.

- Kepler

  Kepler Space Telescope is launched to discover Earth-size exoplanets.

177. Describe in detail a large extragalactic radio source such as Cygnus A.

- Radio galaxies and their relatives, radio-loud quasars and blazars, are types of active galaxy that are very luminous at radio wavelengths, with luminosities up to 1e39 W between 10 MHz and 100 GHz. The radio emission is due to the synchrotron process. The observed structure in radio emission is determined by the interaction between twin jets and the external medium, modified by the effects of relativistic beaming. The host galaxies are almost exclusively large elliptical galaxies.

178. (3.) What kind of objects would you guess the following were?

- HD 128 220

  Henry Draper Catalogue (HD) is a listing of the positions, magnitudes, and spectral types of stars in all parts of the sky. It's 100 years old.

- Abell 426

  The Abell catalog of rich clusters of galaxies is an all-sky catalog of 4,073 rich galaxy clusters of nominal redshift $z \leq 0.2$. This guy did a sky survey in 1950.

- QSO 1015 +277

  QSO (quasi-stellar object; also known as quasar, or quasistellar radio source if the object is radio-loud) is a very energetic and distant active galactic nucleus.

- α Crucis

  α Crucis (HD 108 248) is the brightest star in the constellation Crux, the Southern Cross, and, at a combined visual magnitude 0.77, is the twelfth brightest star in the night sky. α always means the brightest.

- 3C 234

  The Third Cambridge Catalogue of Radio Sources (3C) is an astronomical catalogue of celestial radio sources detected originally at 159 MHz, and subsequently at 178 MHz. It was published in 1959 by members of the Radio Astronomy Group of the University of Cambridge. There are 1C, 2C, etc.

- W51

  Westerhout' s Catalogue of 82 Discrete Sources contains positions of 82 galactic radio sources. W51 is a giant molecular cloud and massive star formation region that includes a giant HII region, several smaller HII regions and masers, and a supernova remnant.

- Mk 509

  The Markarian galaxies are a class of galaxies that have nuclei with excessive amounts of UV emissions compared with other galaxies. Mk 509 is a large galaxy that has an Active Galactic Nucleus (AGN) with a supermassive black hole at the core.

- NGC 2808

  The New General Catalogue of Nebulae and Clusters of Stars is a catalogue of deep sky objects. The NGC contains 7,840 objects, known as the NGC objects. It is one of the largest comprehensive catalogues, as it includes all types of deep space objects and is not confined to, for example, galaxies. NGC 2808 is a globular cluster in the constellation Carina. The cluster belongs to the Milky Way, and is one of our home galaxy' s most massive clusters, containing more than a million stars. It is estimated to be 12.5- billion years old.

- M3

  Messier 3 is a globular cluster of stars in the northern constellation of Canes Venatici. It was discovered by Charles Messier on May 3, 1764, and has an apparent magnitude of 6.2.

- PSR 0950+08

  The pulsar PSR B0950+08 seems to have come from a supernova that occurred 1.8 million years ago. The remnant of this supernova may be the nearest besides the Local Bubble, and the supernova would have been as bright as the moon.

- BD +61 1211

  Bonner Durchmusterung (BD), English Bonn Survey, star catalog showing the positions and apparent magnitudes of 324,188 northern stars. BD +61 1211 is a RS CVn binary (variable star). It's older than Draper catalog.

- 1983 TB

  An asteroid.

X *Exoplanets*

179. Describe how one determines the mass and radius of an exoplanet found by TESS.

- Transiting Exoplanet Survey Satellite
  - A space telescope launched in 2018 by SpaceX's Falcon 9 rocket. It has over 2000 candidates with over 100 confirmed.
- Transit method
  - As the exoplanet transits, the change of flux from the primary star is

    $$\frac{\Delta f}{f} = \frac{r_p^2}{r_*^2}$$

    where r_* is the radius of the star that can be determined by its spectral types.
  - The distance a can be found by measuring the period

    $$\omega = \frac{2\pi}{T} = \sqrt{\frac{G(M_p + M_*)}{a^3}}$$

  - The inclination angle i ($i = 0$ when orbit plane is vertical to line of sight) can also be easily derived. Suppose $M_p \ll M_*$ and $r_p \ll r_*$, let's define the projected distance b to the star center while the exoplanet is at mid-transit. Parametrize orbit by $(a\cos\theta, a\sin\theta\cos i, a\sin\theta\sin i)$, the transit happens when

    $$(a\cos\theta)^2 + (a\sin\theta\cos i)^2 \le r_*^2$$

    So the angle interval during transit is

    $$|\cos\theta| \le \sqrt{\frac{r_*^2/a^2 - \cos^2 i}{\sin^2 i}}$$

    and the time duration is thus

    $$\frac{t_{transit}}{T} = \frac{\tau}{2\pi} \times 2\sqrt{\frac{r_*^2/a^2 - \cos^2 i}{\sin^2 i}}$$

    By measuring the transit time, you can find inclination angle. You can also do it by measuring $\cos i = b/a$, where b is the distance between star center and exoplanet when it's at the mid-transit point.

- The mass can be found by measuring the velocities of planets and stars by Doppler shift, $v_{doppler} = v \sin i$. Reverse $\sin i$ to get real velocity and the mass.

180. With what velocity precision must one measure the reflect motion of a sun-like star to detect an Earth-like planet in orbit about it? What astrophysical processes complicate the measurement of radial velocities with this precision?

- The Kepler's third law

$$\omega = \sqrt{\frac{G(M_p + M_*)}{a^3}}$$

The radius has this relation for circular orbit

$$\frac{r_*}{r_p} = \frac{M_p}{M_*}$$

So the star has smaller radius. The radial velocity in line-of-sight is

$$v_* = \sin(i = \pi/2)\omega r_* = M_p \left(\frac{G\omega}{M_{tot}^2}\right)^{1/3}$$

So for Earth around Sun has $v_* = 0.1$ m/s. The corresponding Dopper shift is $\Delta\lambda/\lambda = v/c = 10^{-10}$. A Jupiter mass (1/1000 of Sun) has something like 30 m/s speed (10^{-7} change in wavelength).

- Noise (pure guess)
  - Inclination angle
  - Noise from ISM radiation

181. What is meant by the "obliquity" of an exoplanet orbit about its host star? How can obliquity be measured, and what are typical values?

- Obliquity is the axial tilt, the angle between the self-rotating axis and the orbit axis. It causes season change over a year.
- Measurement
  - Rossiter–McLaughlin effect:
    the rotating object causes blueshift at the edge rotating towards you, and redshit at the other edge rotating away from you. During "transit", the front star can block one of the edge of the planet behind, so the radial velocity of the spinning planet can be detected.
- Obliquity evolution

  The tide force from the moon of the planet can change the obliquity.

182. What is a Hot Jupiter? What is meant by tidal locking between a Hot Jupiter and its host star? What implications might this have for exobiology on these planets?

- Hot jupiter is a jupiter-mass exoplanet with a very short period (< 10 days). It's easily detected by radial-velocity method. It's hot because it's very close to the star.
- Tidally locked means there is one face consistently facing the star.
- Some water vapor in hot Jupiter's atmosphere might be blocked by the opaque cloud above.

183. Describe three methods for detecting an exoplanet, and the potential biases in the inferred populations of planets associated with each method. Roughly how many planets have been detected using each?

- Radial velocity or Doppler method

  As a planet orbits a star, the star also moves in its own small orbit around the system' s center of mass. Variations in the star' s radial velocity can be detected from displacements in the star' s spectral lines due to the Doppler effect. Extremely small radial-velocity variations can be observed, of 1 m/s or even somewhat less. This has been by far the most **productive method** of discovering exoplanets. It has the advantage of being applicable to stars with a wide range of characteristics. One of its disadvantages is that it cannot determine a planet' s true mass, but can only set a lower limit on that mass. However if the radial-velocity of the planet itself can be distinguished from the radial-velocity of the star then the true mass can be determined.

- Transit method (e.g. Kepler)

  If a planet crosses (or transits) in front of its parent star' s disk, then the observed brightness of the star drops by a small amount. The amount by which the star dims depends on its size and on the size of the planet, among other factors. This has been the second most productive method of detection, though it suffers from a substantial rate of **false positives** and confirmation from another method is usually considered necessary. The transit method reveals the radius of a planet, and it has the benefit that it sometimes allows a planet' s atmosphere to be investigated through spectroscopy.

- Astrometry

  Astrometry consists of precisely measuring a star' s position in the sky and observing the changes in that position over time. The motion of a star due to the gravitational influence of a planet may be observable. Because the motion is so small, however, this method has not yet been very productive. It has produced only a few disputed detections, though it has been successfully used to investigate the properties of 48 planets found in other ways.

184. Describe a sequence of observations to identify an exoplanet, to determine that it has a rocky composition, and to measure the chemical constituents of its atmosphere

- Use density of exoplanet to determine if it's rocky or gaseous.
- Spectroscopy to see it's chemical stuff.

185. Why are M Dwarfs thought to be promising candidates for identifying habitable planets?

- M dwarf, aka red dwarf, is the smallest and coolest kind of star on the main sequence. Reasons:
  - The odds of finding a planet in the habitable zone around any specific M dwarf are slim, but the total amount of habitable zone around all M dwarfs combined is equal to the total amount around sun-like stars, because there are so many M dwarfs in the Milky Way.
  - The best argument against the UV rays and the X-rays in particular is that an M dwarf emits most of that in the first billion years of its life. And once you' re through that, then it gets pretty nice.

🚀 🌌 🌞 Past Questions 🪐 🌏 🌚

Before 2019

Taweewat (Champ) Qualified Questions Prepared Questions: Galaxy cluster observation - Frequency of the SZ effect cutoff [Max]- What do we expect to observe from theory in the mass as a function of redshift - Angular scale of the galaxy cluster: fully resolved - the richness of the galaxy cluster as a function of redshift Dark Matter detection - galaxy cluster detection - galaxy dark matter: rotational curve - draw the rotational curve and how we measure it - what is the dark matter density of this room Fermi bubble - what is it has to do with the dark energy - what is the excess of the fermi bubble come from - what is the excess of the gamma ray at the center of the milky way comes from. Supermassive black hole - what is the mass of the supermassive black hole of the milky way, M31 - How do they measure the mass of the SMBH in Milky Way and other galaxies - How are we planning to image the SMBH: event horizon telescope and how it works - relation between mass of the supermassive black hole and the host galaxy (M- sigma relation) Problem of the cosmology that can be solved with inflation - horizontal problem (flatness problem) - isotropic problem - seed of structure - monopole problem