

# A Framework for Development of Universal Rules for Microgrids Stability and Control

Petr Vorobev<sup>1,2</sup>, Po-Hsu Huang<sup>1</sup>, Mohamed Al Hosani<sup>3</sup>, James L. Kirtley<sup>1</sup>, and Konstantin Turitsyn<sup>1</sup>

<sup>1</sup>Massachusetts Institute of Technology, Cambridge, MA

<sup>2</sup>Skolkovo Institute of Science and Technology, Moscow, Russia

<sup>3</sup>Masdar Institute, Khalifa University of Science and Technology, Abu Dhabi, UAE

**Abstract**—Inverter based microgrids are becoming a viable and attractive choice for future power distribution systems with substantial renewable penetration. The control architectures of such microgrids are currently designed to mimic conventional power systems with droop-based control, coupling active power to frequency and reactive power to voltage. However, the dynamic behavior of low-voltage microgrids is very different from large-scale power systems. In particular, the electromagnetic network dynamics plays an unexpectedly important role despite its very small timescales. This makes simplified swing-type equations inappropriate for microgrid stability assessment requiring an explicit consideration of line currents dynamics. In the present work we elucidate the role of network dynamics in such microgrids and uncover the major factors affecting microgrids stability. We then present a systematic approach for proper accounting for the network dynamics by a special model-order reduction procedure. Based on this reduced-order model we derive a set of stability certificates each based only on local parameters, namely the settings of each pair of interconnected inverters and parameters of the corresponding interconnection line. This set of constraints establishes a natural foundation for plug-and-play interconnection standards. Possible applications of the derived stability criteria are discussed ranging from microgrid network planning to multi-microgrid interconnection/reconfiguration decision tools.

**Index Terms**—Droop control, small-signal stability, Lyapunov functions, microgrids.

## I. INTRODUCTION

With the increasing penetration of small-scale generation, especially PV panels, a question is raised for reconfiguring the existing distribution grids to make them able to run autonomously in the case of disconnection from the feeding substation thus forming a microgrid [1]. For such a configuration, even if the total generation is sufficient for satisfying the load demand, the issue of proper control of multiple small generation units becomes dominant. Ideally, one can think about realizing a control hierarchy for proper power sharing between the grid participants based on only local control actions, similar to large-scale power systems. A natural approach is to organize power and voltage control in microgrids by implementing a droop-based response, based on

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active power-frequency and reactive power-voltage coupling. Such a control scheme can be very flexible in terms of generation sharing between participating units and, at the same time, it is rather robust. On the other hand, as known from the large-scale power systems experience, inaccurate settings of control parameters (in particular, droop coefficients) can cause the system to become unstable at certain operating points [2]; therefore for practical realization it is essential to know the regions of allowed droop coefficients/gains to maintain system stability.

A detailed simulation of microgrid stability is a rather difficult problem [3] with complexity rapidly growing with the increase of the network size. While it is possible to perform a detailed stability analysis of a microgrid during the stage of system planning in order to determine the allowed control settings, any later reconfiguration, such as network expansion/upgrade and/or multiple microgrid interconnection, will require similar calculations to be executed again. Such an approach of performing a full-scale stability assessment on a regular basis is justified for large-scale power systems, where reconfiguration and/or significant operating point changes are done in centralized manner, however, it could be problematic for microgrids where the expected number of reconfigurations is much larger. Moreover, while the system configuration (transmission network parameters, settings of individual generators) for large-scale power systems is usually known to a required degree, details of microgrid configuration are likely to be uncertain (due to different practical reasons, eg. unknown line parameters on distribution level, unknown settings of individual units etc.). It is, therefore, desirable to develop stability assessment methods that rely only on the knowledge of local parameters (settings of a unit in question and lines connected to it) without the need to access all the network configuration to perform the full-scale calculations for every action of reconfiguration/interconnection. In the present paper we develop such methods for inverter-based microgrids and formulate simple and practical rules for small-signal stability of such microgrids.

Contrary to large-scale power systems, where the network impedance is substantial due to long transmission lines and machines internal reactances, inverter-based microgrids typically have small values of network impedance. Under such conditions, the network dynamics, i.e. fast electromagnetic

transients in lines, start to have significant influence [4]–[6] on slower system modes - a somewhat counter-intuitive fact that is confirmed by direct analysis of microgrids dynamic equations [7]. This leads to significant changes in the form of dynamic equations - one has to explicitly account for line dynamics in addition to bus dynamics which does not allow to reduce the system to a set of algebraically connected swing-type equations for voltages and angles - one also has line currents as system states. Recently we have proposed [7] an efficient model-order reduction technique which addresses this issue. Based on singular perturbation theory it is possible to formulate microgrid dynamic equations with bus states only (i.e. voltages and angles) where network dynamics is properly accounted for in coefficients of these equations. Such a model allows to represent a microgrid by a system of equations very similar to coupled oscillators thus enabling the possibility for development of advanced methods for stability assessment. In the present paper, based on the model from [7], we develop a method for formulation of stability criteria as a set of concise conditions each containing only the settings of a pair of interconnected inverters and parameters of the interconnection line. The derived stability criteria can have numerous potential practical applications from serving as a guide for system planning to being a tool for development of microgrid advanced control strategies.

The rest of the paper is structured as follows: In section II we present a model of microgrid and describe a procedure of model-order reduction together with derivation of the system dynamic equations. Section III is dedicated to the construction of Lyapunov function based on the reduced-order model; Section IV presents a derivation and discussion of the set of decentralized stability criteria - the main contribution of the paper. Conclusions are drawn in Section V.

## II. MICROGRID MODEL AND DYNAMIC EQUATIONS

In this section we present a mathematical model for stability assessment of microgrids with droop-controlled inverters. We consider a microgrid with  $N$  inverters interconnected by  $M$  lines. We will use latin subscripts to refer corresponding quantities to a particular inverter (i.e.  $U_i$  is the voltage magnitude on inverter  $i$  bus), lines will be referred to by two indices corresponding to buses which the line interconnects - eg.  $R_{ik}$ , and  $L_{ik}$  are respectively the resistance and inductance of the line between buses  $i$  and  $k$ . We assume that the system at an equilibrium is operating at a certain frequency  $\omega_0$  (not necessarily equal to nominal 50 Hz) with balanced 3-phase currents in lines and voltages at inverter buses. Also, we represent bus voltages and line currents with respect to the common reference frame, rotating with the equilibrium frequency  $\omega_0$ :

$$v_i(t) = \text{Re}[V_i(t)e^{j\omega_0 t}]; \quad i_{ik}(t) = \text{Re}[I_{ik}(t)e^{j\omega_0 t}] \quad (1)$$

Both  $V_i(t)$  and  $I_{ik}(t)$  can be arbitrary functions of time, not necessarily slowly varying. It is further convenient to represent

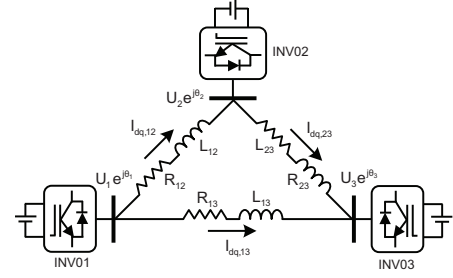


Fig. 1. Example of a microgrid described by equations (3)-(7)

them as phasors with  $d$  and  $q$  components:

$$V_i(t) = V_{d,i} + jV_{q,i} = U_i(t)e^{j\theta_i(t)} \quad (2a)$$

$$I_{ik}(t) = I_{d,ik} + jI_{q,ik} \quad (2b)$$

The inverter-based microgrid (such as the one shown on Fig.1) with droop-based frequency and voltage controls can be described with the required level of accuracy by the following dynamic model [4], [5]:

$$\frac{d\theta_i}{dt} = \omega_i - \omega_0 \quad (3)$$

$$\tau_i \frac{d\omega_i}{dt} = \omega_i^* - \omega_i - n_{pi} P_i \quad (4)$$

$$\tau_i \frac{dU_i}{dt} = U_i^* - U_i - n_{qi} Q_i \quad (5)$$

$$L_{ik} \frac{dI_{d,ik}}{dt} = U_i \cos \theta_i - U_k \cos \theta_k - R_{ik} I_{d,ik} + \omega_0 L_{ik} I_{q,ik} \quad (6)$$

$$L_{ik} \frac{dI_{q,ik}}{dt} = U_i \sin \theta_i - U_k \sin \theta_k - R_{ik} I_{q,ik} - \omega_0 L_{ik} I_{d,ik} \quad (7)$$

where  $P_i$  and  $Q_i$  are the real and reactive powers leaving inverter  $i$ ,  $n_{pi}$ ,  $n_{qi}$ ,  $\omega_i^*$  and  $U_i^*$  being the frequency and voltage droop coefficients and frequency and voltage setpoints respectively for inverter  $i$ . The time-scale  $\tau_i$  is the power controller filtering time, which is inverse to the filter cut-off frequency  $\omega_{ci}$ , the latter is typically around 31.41 rad/s (for 50Hz networks) [3]. We will also use the dimensionless droop coefficients  $k_{pi}$  and  $k_{qi}$  related to inverter power rating  $S_i$  and nominal frequency  $\omega_0$  and voltage  $U_0$  respectively, i.e.  $k_{pi} = S_i n_{pi} / \omega_0$  and  $k_{qi} = S_i n_{qi} / U_0$ .

Equations (3)-(5) are written for every inverter in the system and equations (6)-(7) - for every line, so that the total number of equations for the whole system is  $3N+2M$ . For low voltage grids the timescale of dynamics of currents is typically of the order of few milliseconds (determined by the  $L/R$  value) which is much smaller than the voltage and angle dynamics timescale ( $\sim \tau$ ). This fact allows for model order reduction, where explicit currents dynamics is excluded leaving only the set of equations for voltages and angles. The simplest way is to neglect the left-hand side of equations (6)-(7) assuming that currents (and, consequently, real and reactive power components) are algebraic functions of voltages and

angles - such quasi-stationary approximation is universally accepted for large-scale power systems and also was widely used for analysis of microgrids stability recently [8]–[11]. In fact such an approach is only applicable if the grid network impedance is sufficiently large in relative units - this condition is always satisfied for large-scale power systems due to long transmission lines and large machine internal inductance, but microgrids typically have lines with much smaller overall impedance, so this approach has very limited applicability (we give more precise criteria after presenting a reduced order model). Moreover, microgrids usually have rather low values of  $X/R$  ratio, which also contributes negatively to quasi-stationary model accuracy.

For microgrids the quasi-stationary approximation leads to prediction of incorrect stability region [5]–[7] which becomes more pronounced with decrease of the overall impedances of the lines and with decrease in  $X/R$  ratio, so that one needs to explicitly account for the influence of electromagnetic phenomena. This fact might seem counter-intuitive at first because the increase in line resistance leads to a simultaneous decrease in the characteristic electro-magnetic timescale, making current relaxation (given by equations (6) and (7)) even faster. However, the time-scale alone is not a valid indicator for the influence of fast degrees of freedom on slow modes, a more careful analysis using singular perturbation theory is required which we briefly present here, more details can be found in [7].

The most straightforward way to derive the reduced set of equations is to start from impedances of lines in Laplace's domain as  $Z_{ij} = R_{ij} + jX_{ij} + sL_{ij}$  and use them to construct the  $N \times N$  network admittance matrix  $Y(s)$  following standard rules. Each non-diagonal element  $Y_{ij}$  is equal to the negative of the admittance of the corresponding line (or zero, if buses  $i$  and  $j$  are not directly connected), whereas each diagonal element  $Y_{ii}$  is equal to sum of admittances of all the lines connected to bus  $i$ . Such admittance matrix links the vectors of inverter terminal currents and voltages:  $\underline{I}(s) = \mathbf{Y}(s)\underline{V}(s)$  (both are phasors). The next step is to expand the admittance matrix in power series in respect to  $s$  up to the first order - this is justified since the power controller characteristic timescale is much larger than the electro-magnetic one, which leads us to (switching back from Laplace domain):

$$\underline{I}(t) \approx \mathbf{Y}_0 \underline{V}(t) + \mathbf{Y}_1 \dot{\underline{V}}(t) \quad (8)$$

where we introduced the following denotations:

$$\mathbf{Y}_0 = \mathbf{Y}(s)|_{s=0} \quad (9)$$

$$\mathbf{Y}_1 = \left. \frac{\partial \mathbf{Y}(s)}{\partial s} \right|_{s=0} \quad (10)$$

These equations can now be used to express the real and reactive power components in terms of voltage and angle and their time derivatives. Substituting these expressions to (3)–(5) we finally arrive to the following set of  $2N$  equations (for details on their derivation see [7]):

$$\tau \mathbf{\Lambda}_p \ddot{\vartheta} + (\mathbf{\Lambda}_p - \mathbf{B}') \dot{\vartheta} + \mathbf{B} \vartheta + \mathbf{G} \varrho - \mathbf{G}' \dot{\varrho} = 0 \quad (11a)$$

$$(\tau \mathbf{\Lambda}_q - \mathbf{B}') \dot{\varrho} + (\mathbf{\Lambda}_q + \mathbf{B}) \varrho - \mathbf{G} \vartheta + \mathbf{G}' \dot{\vartheta} = 0 \quad (11b)$$

where  $\vartheta$  and  $\varrho$  are the vectors (of dimension  $N$ ) of inverter bus angles and voltages and all the coefficients in bold are matrices.  $\mathbf{\Lambda}_p$  and  $\mathbf{\Lambda}_q$  are diagonal with following elements:  $(\mathbf{\Lambda}_p)_{ii} = \mu_{pi} = 1/n_{pi}$ ,  $(\mathbf{\Lambda}_q)_{ii} = \mu_{qi} = 1/n_{qi}$  - inverse of frequency and voltage droop coefficients of corresponding inverters. In deriving the system of equations (11) we used the fact that operating point of microgrids corresponds to small angular differences between inverters and small deviations of inverter voltages from nominal [4] - the system is straightforward to generalized to arbitrary operating points.

Matrices  $\mathbf{B}$ ,  $\mathbf{G}$ ,  $\mathbf{B}'$  and  $\mathbf{G}'$  are defines as follows:

$$\mathbf{B} = -U_0^2 \text{Im} \{ \mathbf{Y}_0 \}, \quad \mathbf{G} = U_0^2 \text{Re} \{ \mathbf{Y}_0 \} \quad (12)$$

$$\mathbf{B}' = U_0^2 \text{Im} \{ \mathbf{Y}_1 \}, \quad \mathbf{G}' = -U_0^2 \text{Re} \{ \mathbf{Y}_1 \} \quad (13)$$

We will refer to matrices  $\mathbf{B}$  and  $\mathbf{G}$  as susceptance and conductance respectively, while matrices  $\mathbf{B}'$  and  $\mathbf{G}'$  will be referred to as *transient* susceptance and conductance. Due to the fact that all of them are obtained by linear operations with the initial admittance matrix  $\mathbf{Y}$  they share the same symmetry property described above. Contributions to these matrices from interconnected lines, which are denoted with small letters with index  $ij$ , are shown as follows:

$$b_{ij} = \frac{X_{ij}}{Z_{ij}^2}; \quad g_{ij} = \frac{R_{ij}}{Z_{ij}^2}; \quad (14)$$

$$b'_{ij} = \frac{2\omega_0 R_{ij} L_{ij}^2}{Z_{ij}^4}; \quad g'_{ij} = \frac{L_{ij}(R_{ij}^2 - X_{ij}^2)}{Z_{ij}^4} \quad (15)$$

where matrices  $\mathbf{B}$ ,  $\mathbf{G}$  and  $\mathbf{B}'$  are positive semi-definite, while matrix  $\mathbf{G}'$  is singular and sign-indefinite. We note, that the use of a simple quasi-stationary approximation neglecting the electromagnetic transients corresponds to terms  $\mathbf{B}'$  and  $\mathbf{G}'$  being set to zero in equations (11), which allows us to give a simple illustration of the effect of these transients on system stability.

From equation (11a) (we will talk about one inverter for simplicity) one infers that electromagnetic transients manifest themselves in the form of effective positive feedback corresponding to a  $b'$  term in front of  $\vartheta$ . Thus the effects of these fast transients become important when the  $b'$  becomes comparable to inverse droop  $1/n_p$  - which is typically around  $\sim 10 - 100 p.u.$ . As is seen from relations (15),  $b'$  is inversely proportional to line impedance, and can become large ( $\sim 100$ ) for microgrids with short lines so that the overall positive feedback is observed. On the other hand, for large-scale power systems, the effective impedance of the network is always around unity (and  $R/X$  ratio is also small), so that the corresponding  $b'$  term is almost always negligible compared to inverse droop. This explains why it is important to account for network dynamics when assessing microgrids stability.

### III. LYAPUNOV FUNCTION CONSTRUCTION

Equations (11) can be directly analyzed for stability by regular method of demanding all the eigenvalues of the state matrix to have negative real part. This being done by direct numerical analysis provides no insight into the system and does not show which parameters are to be tuned in case one needs to achieve systematic enhancement of stability. It is possible, however, to derive a set of stability criteria each containing only parameters of a single pair of interconnected inverters; that is, fulfillment of these criteria for every pair of inverters sufficiently certifies the stability of the entire system. This is achieved by constructing a positive definite Lyapunov function  $W$  and demanding its decay rate  $V$  to be positive semi-definite, i.e.  $\partial_t W(\underline{x}(t)) = -V(\underline{x}(t))$ .

The easiest way to arrive to the needed Lyapunov candidate is to start from the system of dynamic equations (11). The first step is to multiply equation (11a) by  $(\vartheta + 2\tau\dot{\vartheta})^T$  and equation (11b) by  $(\varrho + 2\tau\dot{\varrho})^T$  and then sum the results. The obtained expression represents a quadratic form in which the cross-derivative terms can be transformed to full derivatives - eg.  $2\dot{\vartheta}\vartheta = \partial_t \vartheta^2$ . After some rearrangement of the terms (we skip the cumbersome but rather straightforward algebraic operations, they will be reported in subsequent publications) we get the following representation for quadratic Lyapunov function  $W$ :

$$W = \frac{1}{2} \underline{y}^T \mathbf{P} \underline{y}, \quad \underline{y} = [\vartheta, \varrho, \vartheta + 2\tau\dot{\vartheta}]^T \quad (16)$$

where

$$\mathbf{P} = \begin{bmatrix} \frac{\Lambda_p}{2} + 2\tau\mathbf{B} - \mathbf{B}' & 2\tau\mathbf{\Gamma} & 0 \\ 2\tau\mathbf{\Gamma} & 3\tau\Lambda_q + 2\tau\mathbf{B} - \mathbf{B}' & 0 \\ 0 & 0 & \frac{\Lambda_p}{2} \end{bmatrix} \quad (17)$$

and its decay rate  $V$ :

$$V = \underline{z}^T \mathbf{Q} \underline{z}, \quad \underline{z} = [\vartheta, \tau\dot{\vartheta}, \varrho, \tau\dot{\varrho}]^T \quad (18)$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B} & -2\mathbf{\Gamma} & 0 & 0 \\ -2\mathbf{\Gamma} & 2\Lambda_q - \frac{2\mathbf{B}'}{\tau} & 0 & 0 \\ 0 & 0 & \Lambda_q + \mathbf{B} & 0 \\ 0 & 0 & 0 & \frac{(\Lambda_p - 2\mathbf{B}')}{\tau} \end{bmatrix} \quad (19)$$

and  $\mathbf{\Gamma} = \mathbf{G} + \mathbf{G}'/(2\tau)$ . We note that each block in the above expressions for  $\mathbf{P}$  and  $\mathbf{Q}$  represent symmetric  $N \times N$  matrices, and thus  $\mathbf{P}$  has dimensions  $3N \times 3N$  and  $\mathbf{Q}$  of  $4N \times 4N$ .

### IV. STABILITY ASSESSMENT

By Lasalle's Invariance Principle, the solutions of the autonomous system,  $\dot{\underline{x}}(t) = f(\underline{x}(t))$ , asymptotically converge to the largest invariant set if a scalar function  $W(\underline{x}(t)) \rightarrow \infty$  as  $\|\underline{x}(t)\| \rightarrow \infty$  and  $\dot{W}(\underline{x}(t)) \leq 0$  over the entire space [12]. Therefore, in order to obtain sufficient stability criteria for the system one can demand the positive definiteness of  $\mathbf{P}$  and the positive semi-definiteness of  $\mathbf{Q}$ . In fact, the largest invariant set of the linear system described in (11) are the solutions

of the form  $[\dot{\vartheta}_e, \vartheta_e, \varrho]^T = \alpha[0, \underline{1}, 0]^T$  with  $\underline{1}$  and  $\underline{0}$  being column vectors filled with 1's and 0's respectively, which is the expression of the fact that the simultaneous shift of all the bus angles by a constant value does not correspond to any change of physical state of the system.

As was mentioned previously, our goal is to obtain a set of stability criteria local for every pair of interconnected inverters, so we first split both  $\mathbf{P}$  and  $\mathbf{Q}$  into a sum of  $M$  (equal to the total number of lines in the microgrid) sparse matrices each related to one pair of interconnected inverters, denoted as  $\mathbf{p}_{lm}$  and  $\mathbf{q}_{lm}$  (we use small letters in bold to denote the all matrices related to inverter pairs), and demand the positive (semi-)definiteness for every such sparse matrix - this would be a sufficient condition for establishing the sign definiteness of full matrices  $\mathbf{P}$  and  $\mathbf{Q}$ . Since matrices related to different inverter pairs are similar, in our further calculations we can, without the loss of generality, consider only one pair of inverters. Further, since  $\mathbf{p}_{lm}$  and  $\mathbf{q}_{lm}$  will only contain non-zero elements for indexes  $l$  and  $m$  they will be equivalent to  $6 \times 6$  and  $8 \times 8$  matrices respectively. Moreover, every such matrix is composed from  $2 \times 2$  blocks, i.e.:

$$\mathbf{p}_{lm} = \begin{bmatrix} \frac{\lambda_p}{4} + \tau\mathbf{b} - \frac{\mathbf{b}'}{2} & \tau\gamma & 0 \\ \tau\gamma & \frac{3\tau\lambda_q}{2} + \tau\mathbf{b} - \frac{\mathbf{b}'}{2} & 0 \\ 0 & 0 & \frac{\lambda_p}{4} \end{bmatrix} \quad (20)$$

$$\mathbf{q}_{lm} = \begin{bmatrix} \mathbf{b} & -2\gamma & 0 & 0 \\ -2\gamma & 2\lambda_q - \frac{2\mathbf{b}'}{\tau} & 0 & 0 \\ 0 & 0 & \lambda_q + \mathbf{b} & 0 \\ 0 & 0 & 0 & \frac{\lambda_p}{\tau} - \frac{2\mathbf{b}'}{\tau} \end{bmatrix} \quad (21)$$

here matrices  $\mathbf{b}$ ,  $\mathbf{g}$ ,  $\mathbf{b}'$  and  $\mathbf{g}'$  are corresponding  $2 \times 2$  admittance matrices for pair of interconnected inverters (we omit using indexes  $l$  and  $m$ ), for example for  $\mathbf{b}$ :

$$\mathbf{b} = \begin{bmatrix} b_{lm} & -b_{lm} \\ -b_{lm} & b_{lm} \end{bmatrix} = b \cdot \boldsymbol{\sigma} \quad (22)$$

where  $\boldsymbol{\sigma}$  is defined as:

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (23)$$

and  $b_{lm}$  given by equation (14) (we also will omit the indexes  $l$  and  $m$  in  $b_{lm}$  further on). Matrices  $\mathbf{g}$ ,  $\mathbf{b}'$  and  $\mathbf{g}'$  are defined in similar way ( $\gamma = \mathbf{g} + \mathbf{g}'/(2\tau)$ ). A convenient property of all these  $2 \times 2$  admittance matrices is that they all are proportional to  $\boldsymbol{\sigma}$  matrix.

Matrix  $\lambda_p$  is defined as follows:

$$\lambda_p = \begin{bmatrix} \mu_{pl}/\nu_l & 0 \\ 0 & \mu_{pm}/\nu_m \end{bmatrix} \quad (24)$$

where  $\nu_l$  and  $\nu_m$  are the total number of connections for inverters  $l$  and  $m$  respectively ( $\mu_{pl} = 1/n_{pl}$  and  $\mu_{qm} = 1/n_{qm}$ ). Matrix  $\lambda_q$  is defined in a similar way.

Formulating positive definiteness condition for matrices  $\mathbf{p}$  and  $\mathbf{q}$  is now straightforward. For matrix  $\mathbf{q}$  one can analyze its upper left and lower right  $4 \times 4$  sub-matrices for positivity separately. For the latter, condition of positive definiteness reduces to a simple:

$$\lambda_{\mathbf{p}} - 2\mathbf{b}' = \begin{bmatrix} \mu_{p1}/\nu_1 - 2b' & 2b' \\ 2b' & \mu_{p2}/\nu_2 - 2b' \end{bmatrix} \succeq 0 \quad (25)$$

from which we can get stability condition for frequency droop coefficients.

**Stability condition 1:**

$$\frac{k_{p1}\nu_1}{S_1} + \frac{k_{p2}\nu_2}{S_2} \leq \frac{1}{U_0^2} \frac{(R^2 + X^2)^2}{4RX^2} \quad (26)$$

where  $R$  and  $X$  refer to resistance and reactance of the line interconnecting inverters 1 and 2. If we introduce the length of the line  $a$  and per unit length resistance and reactance - respectively  $r$  and  $x$ , condition (26) can be rewritten as:

$$\frac{k_{p1}\nu_1}{S_1} + \frac{k_{p2}\nu_2}{S_2} \leq \frac{1}{U_0^2} \frac{(r^2 + x^2)^2}{4rx^2} a \quad (27)$$

One important general rule can be observed right away from (27): decreasing the length of interconnection length (while keeping the  $x/r$  ratio constant) reduces the stability region in terms of inverter droop coefficients. Practically this means that there exist a *minimum* limit for the length of line for interconnection of two inverters. This result, while it might appear counter-intuitive at first, is specific to microgrids and has no analogy in large-scale power systems. To avoid confusion we note here, that the values  $R$  and  $X$  in the initial condition (26) can also contain contribution from any virtual impedance that can be present in control loops of inverters. In this case one can use condition (26) to assess the required amount of this virtual impedance for ensuring the small-signal stability under the given droop coefficients and network parameters. We also remind that condition (26) should be satisfied for every pair of interconnected inverters in microgrid.

Derivation of criterion (26) allows also to highlight the influence of electro-magnetic transients on microgrids stability boundary. We notice, that neglecting those transients in microgrid model (i.e. using the quasi-stationary approach) is equivalent to setting both  $b'$  and  $g'$  to zero in both Lyapunov function and decay rate. In this case condition (25) is always satisfied and there are no limitations on the allowed values of frequency droop coefficients. This is a clear illustration of the importance of microgrid network dynamics despite it's fast timescale.

We next proceed to analyse for positivity the upper left submatrix of  $\mathbf{q}$  in (21):

$$\begin{bmatrix} \mathbf{b} & -2\gamma \\ -2\gamma & 2\lambda_{\mathbf{q}} - 2\mathbf{b}'/\tau \end{bmatrix} \succeq 0 \quad (28)$$

This condition can be equivalently rewritten using generalized Schur complement:

$$2\lambda_{\mathbf{q}} - 2\mathbf{b}'/\tau \succeq 0; \quad 2\lambda_{\mathbf{q}} - 2\mathbf{b}'/\tau - 4\gamma\mathbf{b}_g\gamma \succeq 0 \quad (29)$$

Out of these two conditions the second one is stronger, so it is the only one that we need to consider. Here  $\mathbf{b}_g$  denotes a generalized inverse of  $\mathbf{b}$  which also should satisfy additional constraint  $(\mathbf{I} - \mathbf{b}\mathbf{b}_g)\gamma = 0$ . One can check that  $\mathbf{b}_g = (4b)^{-1}\boldsymbol{\sigma}$  is an appropriate candidate. Finally, we have the following condition for voltage droop coefficients:

$$n_{q1}\nu_1 + n_{q2}\nu_2 \leq \left( \frac{b'}{\tau} + \frac{2\gamma^2}{b} \right)^{-1} \quad (30)$$

we can simplify this relation by noticing that for any practical values of network parameters the following conditions are satisfied:  $g' \ll \tau g$  and  $b' \ll \tau b$ .

**Stability condition 2:**

$$\frac{k_{q1}\nu_1}{S_1} + \frac{k_{q2}\nu_2}{S_2} \leq \frac{1}{U_0^2} \frac{X(R^2 + X^2)}{2R^2} \quad (31)$$

For Lyapunov function matrix  $\mathbf{p}$  we formulate the positivity criteria by directly considering quadratic form it generates and separating full squares. After cumbersome but straightforward calculations we find that if the connection line is mostly inductive, i.e.  $x > r$ , the Lyapunov function is always positive. If the line is resistive, i.e.  $r > x$ , the following condition is sufficient for Lyapunov function being positive:

$$\frac{k_{q1}\nu_1}{S_1} + \frac{k_{q2}\nu_2}{S_2} \leq \frac{1}{U_0^2} \frac{3X(R^2 + X^2)}{2(R^2 - X^2)} \quad (32)$$

This condition is milder than (31) so it does not bring any additional constraints on top of what (26) and (31) impose. Thus we have derived the **key result** of the present paper:

*An inverter-based microgrid with arbitrary structure described by equations (3)-(7) is small-signal stable if conditions (26) and (31) are satisfied for every pair of interconnected inverters in the grid.*

There could be numerous possible practical applications of these conditions. One can use them as a convenient tool for system planning starting from assigning the target droop coefficients for every participant of the microgrid - in this case conditions (26) and (31) read in reverse will give the *minimum* network lines impedances for safe operation. In the opposite case, when the network configuration is fixed, these equations can provide a guideline for readjusting some droop coefficients for ensuring stability. Another application is the procedure of network reconfiguration or interconnection of multiple microgrids. In this case equation (26) and (31) provide a powerful tool for fast assessment of the interconnection requirements, which could be the minimum interconnecting line length or the required adjustment in droop coefficients of terminal inverters (or both done simultaneously). This is especially convenient since one does not need to know the parameters and settings of the whole network to make a decision on interconnection - only the knowledge of terminal inverters settings is required. Finally, since the region defined

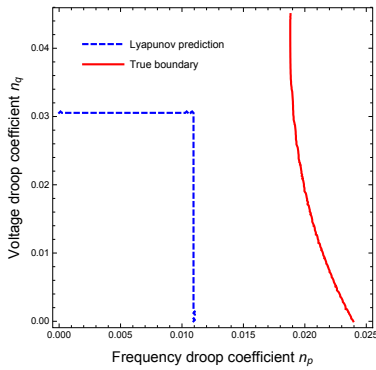


Fig. 2. Stability boundary in the space of frequency and voltage droop coefficients of one of the inverters

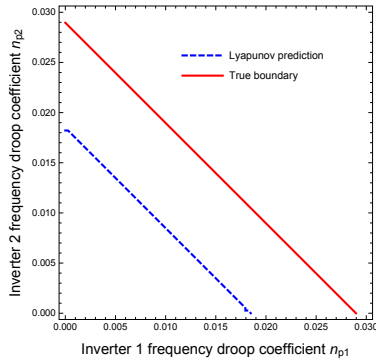


Fig. 3. Stability boundary in the space of frequency droop coefficients of two interconnected inverters

by equations (26) and (31) in the space of all the droop coefficients is a convex polytop, these equations represent an especially convenient set of constraints for any optimization-related problem concerning microgrids (both planning and operation).

Both stability conditions (26) and (31) for a pair of inverters are illustrated on Fig.2 in the space of inverter 1 frequency and voltage droop coefficients with the settings of inverter 2 fixed (within the acceptable region). The true stability boundary calculated using the initial model (3)-(7) is also shown. We note the modest degree of conservativeness that the derived conditions possess. Fig.3 shows the stability boundary in the space of frequency droop coefficients of inverters 1 and 2 provided their voltage droop coefficients are fixed.

## V. CONCLUSION

We derived a set of decentralized sufficient conditions (given by equations (26) and (31)) for the characterization of small-signal stability of microgrids, with each condition depending only on the settings of a pair of inverters which are directly interconnected. Stability of the microgrid is certified if those criteria are satisfied for *every* pair of such inverters in the system. Moreover, the derived criteria represent very simple expressions in terms of power droop coefficients and line resistance and reactance. Such a representation provides a natural means for fast stability assessment of microgrids

with uncertain structure. Moreover, these criteria can serve as a foundation for future decentralized control and real-time reconfiguration of microgrids and multi-microgrid networks. The simple polyhedral structure of the stability conditions in the inverter droop space allows to leverage powerful linear convex optimization based algorithms for control synthesis, while their local nature can be exploited in the design of plug-and-play and ad-hoc architectures [13]. Another important and promising application is the problem of microgrid interconnection and synchronization. The key advantage of the derived criteria is their locality, as the stability of the final configuration can be certified based only on the information about the terminal inverters and interconnection line. Numerical simulation have shown that the derived conditions are not overly conservative in practical situations. However, the conservativeness can be reduced even further by considering different reweighted Lyapunov functions and characterizing their unions. Further work also will include generalization of the results to microgrids with active load types (most importantly induction motors), hybrid microgrids, where synchronous machines are present along with inverters, study of the possible influence of microgrid operating point on stability.

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