Robustness against Disturbances in Power Systems under Frequency Constraints

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***Abstract*—Wide deployment of renewable generation and gradual decrease of the overall system inertia make the modern power grids more vulnerable to transient instabilities and unacceptable frequency fluctuations. Time-domain simulation based assessment of the system security to uncertain and stochastic disturbances is extremely time-consuming. In this paper, we develop an alternative approach allowing for a computationally efficient and mathematically rigorous certification of the power grid stability with respect to external disturbances. The derived sufficient condition is constructed via convex optimization and is shown to be non-conservative for several IEEE cases.**

***Index Terms*—Input to output stability, small gain analysis, local sector bound, power grid, dynamic security assessment**

I. Introduction

Transient stability assessment is one of the most computationally challenging security assessment procedures carried out by the system operators [1]–[3]. In addition to the transient stability, the operators are required to maintain the system frequency close to the nominal 50 or 60 Hz levels [4]. The grid is equipped with under-frequency load shedding (UFLS) relays, under and over-frequency generation protection relays to ensure the frequency regulation is met [5]. Traditionally, the deviations of frequency during faults were suppressed by the turbine speed governors and the natural inertia of the generators. However, in recent years, the primary frequency response capabilities of the grid has declined steadily in many power grids, like the Eastern Interconnection [6] of the US. The decline in response results in deeper frequency nadir, which in turn increases the risk of unintended disconnection of units and cascading outages.

Moreover, the levels of ambient fluctuations in the grid have also increased together with the higher penetration of renewables. Traditional time-domain simulations give a high fidelity assessment of stability when the disturbance and the operating conditions are known exactly. However, the disturbances are often uncertain and impossible to predict by its nature. Typical disturbances could include nearly-instant switching events, such as load shedding and generation tripping, or continuous changes, such as varying power output from wind turbines [5], [7]. For example, one of the most common causes of frequency rise is the near simultaneous tripping of more than one pumped storage unit. With these disturbances to the grid, there is a question concerning the critical disturbance levels that the grid can withstand at the given operating condition.

This paper proposes an efficient estimation on the the maximum acceptable magnitude of disturbance that the grid can handle without losing stability and violating frequency constraints. In other words, it allows the operator to certify that the grid is secure with respect to set of properly bounded disturbances. The disturbance is characterized by only its magnitude and instant step changes such as switching or tripping can be included in the analysis. This result goes beyond the typical certificates established by the energy function and Lyapunov function based methods that are not naturally designed to address the non-autonomous systems with time-dependent disturbances.

Our approach is based on the analysis of Lurie system with sector bound [8], [9]. The small gain theorems has been the key tool for analysis of these system [10]–[12]. Extension of small gain theorem to a locally bounded nonlinearity has been recognized [13]. More recent work has been focusing on the input to state stability (ISS) [14], [15]. Similar to local small gain theorem, the local input to state stability (LISS) has been studied [16], [17]. They were considered for network system [16], [18] where the system gain was extended to gain matrix.

In this paper, we study the system from more traditional framework focusing on the input-output stability rather than ISS. The stability of the output is directly linked to the severity of the disturbance, and there is a upper-bound threshold in absolute magnitude of disturbance that the grid can be guaranteed to be stable. The analysis extends the traditional notion of input-output stability or Bounded Input Bounded Output (BIBO) stability and considers the problem of appropriately constraining the input so that the output is also constrained by operational requirements on the generator frequencies. Constrained Input Constrained Output (CICO) of the system is defined to identify appropriate constraints on the disturbance so that the system response is also constrained by operational limits. The term Constrained Input Constrained Output appeared previously in [19] for imposing constraint on output in a filter design. The condition for this stability is derived in a similar way to the small-gain condition in nonlinear system [20], [21].

Recently, there has been growing applications of local sector bound on the nonlinearities of power flow equation [22], [23]. The performance of these methods is dependent on the how the system states are bounded, but computing the optimal bound for the given problem is often difficult due to the complexity of these problems. In our formulation, we are able to optimize the region for the local sector bound since our sufficient condition is a simple nonlinear inequality constraint. The result of the optimization computes the upper bound on the magnitude of disturbance so that the grid is guaranteed to remain stable. The feasible region of the problem turns out to be convex and thus can be solved efficiently in a scalable way [24]. For the purpose of dynamic security assessment, computational tractability and scalability are essential features, which our method meets these requirements.

The rest of the paper is organized as follows. Chapter II describes the system model and its Lurie representation, as well as formulating the stability problem. Section III introduces our main result with input to output stability analysis and stability condition under disturbances. In Section IV, we present the sector bound of the nonlinear power flows and an optimization procedure to determine the maximum acceptable disturbance magnitude. Section V presents the case studies on several IEEE cases including discussions on how to avoid computational-heavy gain computation step in operation. Finally, Section VI concludes the paper.

II. System Model and Problem Formulation

The power grid is represented as an undirected graph $\mathcal{A}(\mathcal{N},\mathcal{E})$, where $\mathcal{N}=\{1,2,...,n\}$ is the set of buses, and $\mathcal{E}\subseteq\mathcal{N}\times\mathcal{N}$ is the set of transmission lines connecting the buses. The indices $\mathcal{G}=\{1,...,m\}$ denote the generators, and $\mathcal{L}=\{m+1,...,m+n\}$ denote the loads. Let ℓ denote the number of transmission lines, and $E\in\mathbb{R}^{n\times\ell}$ the incidence matrix of the graph. $\mathbb{0}$ and I denotes the zero matrix and the identity matrix of appropriate dimensions. Moreover, given a matrix $A\in\mathbb{R}^{n\times n}$, we denote by $\rho(A)$ its spectral radius, i.e., the maximum norm of its eigenvalues.

A. Power System Model

The structure-preserving second order swing equation is used to model the power system dynamics, given by

$$\begin{aligned} M_k\ddot{\delta}_k + D_k\dot{\delta}_k + \sum_{(k,j)\in\mathcal{E}} B_{kj}\sin(\delta_{kj}) = p_k,\ \forall k\in\mathcal{G} \\ D_k\dot{\delta}_k + \sum_{(k,j)\in\mathcal{E}} B_{kj}\sin(\delta_{kj}) = P_{L,k},\ \forall k\in\mathcal{L} \end{aligned} \tag{1}$$

where M_k and D_k are the inertia and damping coefficients of the generator k, respectively. p_k and $P_{L,k}$ are the mechanical power at generator k and load k, respectively. Moreover, $B_{kj}=b_{kj}V_kV_j$, where b_{kj} is the susceptance of the transmission line (k,j), and V_k is the voltage magnitude at bus k, which we assume constant. Finally, δ_{kj} denotes the phase difference between bus k and bus j, i.e., $\delta_{kj}=\delta_k-\delta_j$.

In addition to the grid dynamics, the first order turbine governor dynamics are considered, which provides the primary frequency control of the synchronous machines, given by

$$T_k\dot{p}_k + p_k + \frac{1}{R_k}\dot{\delta}_k = P_{G,k},\ k\in\mathcal{G}, \tag{2}$$

where $P_{G,k}$ is the scheduled power injection at bus k, T_k is the governor time constant, and R_k is the droop coefficient. To write the system model (1) and (2) in a vector form, the following notations are introduced. Let δ_G and δ_L be the vectors obtained by stacking the scalars δ_k, for $k\in\mathcal{G}$ and δ_k, for $k\in\mathcal{L}$, respectively. Moreover, let $\delta=\begin{bmatrix}\delta_G^T & \delta_L^T\end{bmatrix}^T$. Similarly, let p, P_G and P_L be the vectors obtained by stacking the scalars p_k, $P_{G,k}$, for $k\in\mathcal{G}$, and $P_{L,k}$ for $k\in\mathcal{L}$, respectively. Let M, D_G, D_L and B be the diagonal matrices containing the elements M_k, D_k, for $k\in\mathcal{G}$, D_k, for $k\in\mathcal{L}$, and B_{kj}, for $(k,j)\in\mathcal{E}$, on their diagonal, respectively. Finally, let $E=\begin{bmatrix}E_G^T & E_L^T\end{bmatrix}^T$, where the subscripts G and L correspond to the generator and load buses, respectively.

Consider now the disturbance vector $u=\begin{bmatrix}u_G^T & u_L^T\end{bmatrix}^T$. The system model (1) and (2) can be rewritten in the following form:

$$\begin{aligned} M\ddot{\delta}_G + D_G\dot{\delta}_G + E_GB\sin(E^T\delta) = p \\ D_L\dot{\delta}_L + E_LB\sin(E^T\delta) = P_L + u_L \\ T\dot{p} + p + R^{-1}\dot{\delta} = P_G + u_G \end{aligned} \tag{3}$$

Notice that this simple formulation of the disturbance could incorporate a rich variety of uncertainty scenarios, such as load shedding, generation tripping, and stochastic fluctuations in the power output from wind turbines.

B. Lurie System Representation

In the following, the system (3) will be rewritten as a Lurie system, which is represented as interconnection of linear dynamical system and a nonlinear "diagonal" state feedback operator. As it will be shown in this paper, the Lurie system, together with the efficient bounding of the nonlinearity by two linear functions, heavily simplifies the analysis of nonlinear systems.

The model system (3) can be written in a state space representation. Let $\delta=\delta^*$ and $\dot{\delta}=\mathbb{0}$ represent the equilibrium point of (3), with generator power injection $p=p^*$. Then, we define the state of the system as $x=\begin{bmatrix}x_1^T & x_2^T & x_3^T & x_4^T\end{bmatrix}^T$, with $x_1=\delta_G-\delta_G^*$, $x_2=\dot{\delta_G}$, $x_3=\delta_L-\delta_L^*$, and $x_4=p-p^*$.

Now let $w=E^T\delta-E^T\delta^*$ be the phase difference on each transmission line subtracted by its equilibrium, and y be vector containing the frequencies of the generators $y=\dot{\delta_G}$. Finally, let $\theta^*=E^T\delta^*$, and $v=\sin(\theta^*+w)-\sin(\theta^*)-\text{diag}(\cos(\theta^*))w$. With these new variables, the system (3) can be written in the Lurie form $\dot{x}=Ax+B_vv+B_uu$ as follows:

$$\begin{aligned} \dot{x} = &\begin{bmatrix} \mathbb{0} & I & \mathbb{0} & \mathbb{0} \\ A_{21} & -M^{-1}D_G & A_{23} & M^{-1} \\ A_{31} & \mathbb{0} & A_{33} & \mathbb{0} \\ \mathbb{0} & -R^{-1}T^{-1} & \mathbb{0} & -T^{-1} \end{bmatrix} x \\ &+ \begin{bmatrix} \mathbb{0} \\ -M^{-1}E_G^TX^{-1} \\ -D_L^{-1}E_L^TX^{-1} \\ \mathbb{0} \end{bmatrix} v + \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \\ \mathbb{0} & D_L^{-1} \\ T^{-1} & \mathbb{0} \end{bmatrix} u \end{aligned} \tag{4}$$

with

$$\begin{aligned} A_{21} &= -M^{-1}E_G^TB\text{diag}(\cos\theta^*)E_G \\ A_{23} &= -M^{-1}E_G^TB\text{diag}(\cos\theta^*)E_L \\ A_{31} &= -D_L^{-1}E_L^TB\text{diag}(\cos\theta^*)E_G \\ A_{33} &= -D_L^{-1}E_L^TB\text{diag}(\cos\theta^*)E_L. \end{aligned}$$

The complete model can be rewritten as

$$\dot{x} = Ax + B_vv + B_uu \tag{5a}$$
$$v = \sin(\theta^*+w) - \sin(\theta^*) - \text{diag}(\cos(\theta^*))w \tag{5b}$$
$$y = [\mathbb{0}\ I\ \mathbb{0}\ \mathbb{0}]x = C_yx \tag{5c}$$
$$w = [E_G^T\ \mathbb{0}\ E_L^T\ \mathbb{0}]x = C_wx. \tag{5d}$$

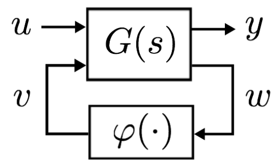

Fig. 1. Lurie system representation of the swing equation with the linearized dynamics in $G(s)$ and the nonlinear components in $\varphi(\cdot)$.

The matrix A in (4) was obtained by linearizing the system (3) around the equilibrium point with respect to x and v. Notice that the vector v represents the nonlinear feedback of the state x, i.e., $v = \varphi(w) = \varphi(C_w x)$. To examine the stability of the system, the linearization around the equilibrium has to be stable, thus the matrix A being a Hurwitz is a necessary condition.

Let the transfer function matrix $G(s)$ represent the linear dynamics in Laplace domain. Then the Lurie system (4) can be graphically represented as in Figure 1. Following this representation of the system, the transfer function matrix G can be divided into four blocks:

$$G(s) = \begin{bmatrix} G_{y,u}(s) & G_{y,v}(s) \\ G_{w,u}(s) & G_{w,v}(s) \end{bmatrix} \tag{6}$$

where each block of transfer matrix can be computed by $G_{i,j}(s) = C_i(sI - A)^{-1}B_j$, with $i \in \{y, w\}$ and $j \in \{u, v\}$. Given the system model described in this section, the problem can be formulated as follows.

C. Problem Formulation

Consider the power system model (4), containing the additive magnitude-bounded disturbance u. The analysis carried out in this paper concentrates on finding the maximum bound on the magnitude of the disturbance such that the system remains inside the operational constraints, which corresponds to the generators remaining synchronized, and the generator frequencies constraints are never violated. In order to quantify the magnitude of the disturbance u, we propose the following element-wise infinite norm.

Definition 1. Let $u(t) \in \mathbb{R}^n$. Its element-wise infinity norm, $\|u\|_\infty \in \mathbb{R}^n$, is defined as

$$\left[\|u\|_\infty\right]_i = \sup_{t \geq 0} |u_i(t)| \tag{7}$$

where $\left[\|u\|_\infty\right]_i$ and u_i are the i-th entry of $\|u\|_\infty$ and u respectively.

Remark. The element-wise infinity norm is different from the standard $\mathcal{L}_\infty$ norm of u, defined as $||u||_{\mathcal{L}_\infty} = \max_i(\sup_{t\geq 0} |u_i(t)|)$. The proposed element-wise norm enables the optimization of each entry in the disturbance vector u, rather than optimizing all the entries uniformly. This important feature will be employed to compute adaptive bound over the state of the system to yield the optimal result.

The problem can be now mathematically formulated as follows.

Problem formulation. Consider the power system (1) written in Lurie form (5). The objective of our problem is to find the maximum bound $\bar{u}$ on the disturbance such that if $\|u\|_\infty \leq \bar{u}$, the following two conditions hold:

(i) $\|w\|_\infty < \pi$, and
(ii) $\|y\|_\infty \leq \bar{y}$.

Notice that the first condition prevents angular separation of the generators during the transient, and the second constraint ensures the frequency constraint to prevent any under or over-frequency emergency control actions.

III. Input-Output Stability Analysis

In this section, the mathematical framework for the analysis and assessment of the system stability under the additive disturbance u will be established. The proposed framework combines the input-output stability approach with the sector bounds on the nonlinearity v in the Lurie system to propose a novel small-gain theorem.

Let $y = Hu$ define an input-output relation , where H is an operator that specifies the output y in terms of the input u, and define the input-output stability of the operator H with respect to the element-wise infinity norm, which we denote $\|\cdot\|_\infty$, as follows.

Definition 2 (**Bounded Input Bounded Output**)**.** The operator H is BIBO if its output is bounded for every bounded input, i.e., there exists a non-negative constant matrix γ_H such that

$$\|y\|_\infty \leq \gamma_H \|u\|_\infty. \tag{8}$$

If the system is BIBO, then the system will be referred as an input-output stable system. For an input-output stable linear system, where the operator H corresponds to the transfer function $G(s)$ in equation (6), the gain matrix φ_G can be computed using the following lemma.

Lemma 1. Given an input-output stable linear system with transfer function $G(s)$, the ij element of the gain matrix γ_G can be computed as

$$\gamma_{G,ij} = ||G_{ij}(t)||_1 \tag{9}$$

where $||G_{ij}(t)||_1 = \int_{-\infty}^{\infty} |h_{ij}(\tau)| d\tau$, with h_{ij} being the impulse response of the G_{ij}.

Proof. For the i-th element of the output vector,

$$\begin{aligned} |y_i(t)| &= \left| \sum_j \int_{-\infty}^{\infty} h_{ij}(\tau) u_j(t-\tau) d\tau \right| \\ &\leq \sum_j \bar{u}_j \int_{-\infty}^{\infty} |h_{ij}(\tau)| d\tau \\ &= \sum_j ||G_{ij}(t)||_1 \bar{u}_j = \sum_j \gamma_{ij} \bar{u}_j. \end{aligned}$$

□

Remark. The linear dynamics $G(s)$ in the Lurie system is input-output stable if the power system is small-signal stable.

The matrix γ_G, can be divided, according to (6), into

$$\gamma_G = \begin{bmatrix} \gamma_{y,u} & \gamma_{y,v} \\ \gamma_{w,u} & \gamma_{w,v} \end{bmatrix}. \tag{10}$$

Consider now the nonlinear component, given by $v = \varphi(w)$, and notice that it is decentralized, i.e., $v_i = \varphi_i(w_i)\ \forall i \in \{1, \dots, \ell\}$. As a consequence, if the nonlinear map $\varphi(\cdot)$ is input-output stable, then the gain matrix γ_φ is a diagonal matrix. The diagonal element of γ_φ in the position $\{i,i\}$ is equal to:

$$\gamma_{\varphi,ii} = \sup_{w_i} \left| \frac{v_i}{w_i} \right|, \tag{11}$$

and direct substitution of equation (5b) results in

$$\gamma_{\varphi,ii} = \sup_{w_i} \left| \frac{\sin(\theta_i^* + w_i) - \sin\theta_i^*}{w_i} - \cos\theta_i^* \right| \tag{12}$$

which is finite for bounded phase angles. Therefore the nonlinear component $\varphi(\cdot)$ is input-output stable. Given the computed gain matrices of the system, the following inequalities hold by definition:

$$\|y\|_\infty \leq \gamma_{y,u}\|u\|_\infty + \gamma_{y,v}\|v\|_\infty \tag{13a}$$
$$\|w\|_\infty \leq \gamma_{w,u}\|u\|_\infty + \gamma_{w,v}\|v\|_\infty \tag{13b}$$
$$\|v\|_\infty \leq \gamma_\varphi\|w\|_\infty \tag{13c}$$

The gain matrices are non-negative matrices ($\gamma_{i,j} \geq 0\ \forall i,j$). Using this property, the next lemma is stated and will be used in the proofs for the results in this paper.

Lemma 2. Given positive matrices, $\gamma_{w,v}$ and γ_φ, the following three conditions are equivalent:

(i) $\rho(\gamma_{w,v}\gamma_\varphi) < 1$
(ii) $(I - \gamma_{w,v}\gamma_\varphi)^{-1} \geq 0$
(iii) There exists $x \geq 0$ such that $(I - \gamma_{w,v}\gamma_\varphi)x > 0$

Proof. We skip the details of the proof for this lemma in this paper and will use the result from [25]. The proof is based on the properties of Z and M matrices. A matrix is a Z-matrix if its off-diagonal elements are on-positive, and M-matrix if it is a Z-matrix and its eigenvalues have nonnegative real parts. First notice that the matrix $I - \gamma_{w,v}\gamma_\varphi$ is a Z-matrix since the gain matrices are non-negative matrices. Now notice that $\rho(\gamma_{w,v}\gamma_\varphi) < 1$ if and only if the eigenvalues of $I - \gamma_{w,v}\gamma_\varphi$ have nonnegative real parts, which is the definition of M-matrix. Given $I - \gamma_{w,v}\gamma_\varphi$ being a M-matrix, condition (i), (ii), and (iii) are equivalent [25]. □

Remark. Since the matrix $\gamma_{w,v}\gamma_\varphi$ is nonnegative, it has a real eigenvalue equal to its spectral radius $\rho(\gamma_{w,v}\gamma_\varphi)$ [26].

The M-matrix appears in other problems in power systems such as voltage stability [27]. In the next theorem, we present the condition under which the power system is input-output stable.

Theorem 1 (Small Gain Theorem for element-wise infinity norm). The system (5) is input-output stable if γ_G and γ_φ are finite and $\rho(\gamma_{w,v}\gamma_\varphi) < 1$.

Proof. By substituting Equation (13b) into Equation (13c) and rearranging, we have

$$(I - \gamma_{w,v}\gamma_\varphi)\|w\|_\infty \leq \gamma_{w,u}\|u\|_\infty.$$

Since $\rho(\gamma_{w,v}\gamma_\varphi) < 1$, Lemma 2 results in $(I - \gamma_{w,v}\gamma_\varphi)^{-1} \geq 0$. As such,

$$\|w\|_\infty \leq (I - \gamma_{w,v}\gamma_\varphi)^{-1}\gamma_{w,u}\|u\|_\infty$$

The output can be bounded by

$$\begin{aligned} \|y\|_\infty &\leq \gamma_{y,u}\|u\|_\infty + \gamma_{y,v}\|v\|_\infty \\ &\leq \gamma_{y,u}\|u\|_\infty + \gamma_{y,v}\gamma_\varphi\|w\|_\infty \\ &\leq \left[\gamma_{y,u} + \gamma_{y,v}\gamma_\varphi(I - \gamma_{w,v}\gamma_\varphi)^{-1}\gamma_{w,u}\right]\|u\|_\infty. \end{aligned}$$

Therefore, the system is input-output stable. □

Notice that this is small-gain theorem for our newly-defined element-wise infinity norm from Definition 1.

Remark. The condition in Theorem 1 is not satisfied for arbitrary nonlinear gain matrix γ_φ. Indeed, since $\rho(\gamma_{w,v}\gamma_\varphi) < 1$, it results that for fixed linear gain matrix $\gamma_{w,v}$, there exists a limit on the magnitude of γ_φ such that our system is input-output stable.

The small gain condition ensures input-output stability, which ensures the output is bounded for every bounded input. However, for a system that is not globally stable, the bounded output cannot be guaranteed for every bounded input. On the other hand if the input is constrained by some $\bar{u}$, the output could be bounded. The following definition exploits this fact, and it is also known as local input-output stable system.

Definition 3. (Constrained Input Bounded Output) The operator H is CIBO if there exist $\bar{u}$, such that for every constrained input ($\|u\|_\infty \leq \bar{u}$), the output is bounded ($\|y\|_\infty < \infty$).

For a locally stable system or a system with strong nonlinearity, the small gain condition may not be satisfied. However, if the input is bounded, then w could be also bounded, which leads to tighter sector bound over the nonlinearity.

Recall that w corresponds to the phase differences deviation from the phase differences at the equilibrium, i.e., $w = E^T\delta - E^T\delta^*$. Let $\bar{w}$ be some magnitude bound on w, i.e., $\|w\|_\infty \leq \bar{w}$. Now notice that γ_φ is function of $\bar{w}$, i.e., $\gamma_\varphi = \gamma_\varphi(\bar{w})$, and that larger $\bar{w}$ results in larger $\gamma_\varphi(\bar{w})$ (see Figure 2). As a consequence, the condition in Theorem 1 could be satisfied for some $\bar{w}$. $\bar{w}$ is some fictitious bound over the phase differences deviation and is used only within the analysis.

This observation is exploited in the following theorem, where a sufficient condition for the stability of our system is presented by considering phase difference bound, $\bar{w}$, to the condition as an auxiliary variable.

Theorem 2. Let $\bar{u} \in \mathbb{R}^n$ be the bound on the input such that $\|u\|_\infty \leq \bar{u}$. If γ_G and γ_φ are finite, and if there exists $\bar{u} \geq 0$ and $\bar{w} \geq 0$ satisfying

$$\gamma_{w,u}\bar{u} < (I - \gamma_{w,v}\gamma_\varphi(\bar{w}))\bar{w}, \tag{14}$$

then the system is CIBO and $\|w\|_\infty \leq \bar{w}$.

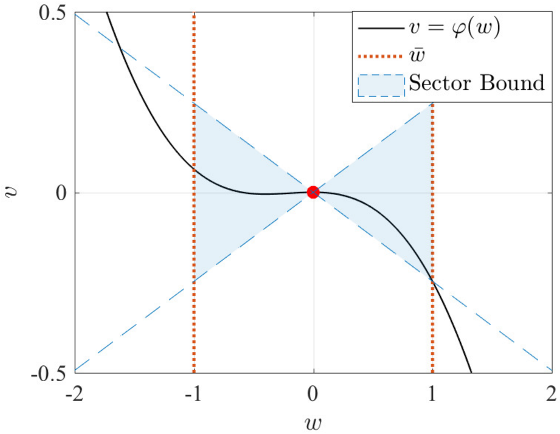

Fig. 2. Sector bound for $v = \varphi(w) = \sin(w + E^T\delta^*) - \cos(E^T\delta^*)w$.

Proof. Since gain matrix is a positive matrix and $\bar{u} \geq 0$, $(I - \gamma_{w,v}\gamma_\varphi(\bar{w}))\bar{w} > \gamma_{w,u}\bar{u} \geq 0$ with $\bar{w} \geq 0$. From Lemma 2, $\rho(\gamma_{w,v}\gamma_\varphi) < 1$, and the system is input-output stable from Theorem 1. Substituting condition (14) and Equation (13c) into Equation (13a), we have

$$\begin{aligned}\|w\|_\infty &\leq \gamma_{w,u}\|u\|_\infty + \gamma_{w,v}\|v\|_\infty \\ &\leq (I - \gamma_{w,v}\gamma_\varphi(\bar{w}))\bar{w} + \gamma_{w,v}\gamma_\varphi\|w\|_\infty.\end{aligned}$$

By rearranging,

$$(I - \gamma_{w,v}\gamma_\varphi)\|w\|_\infty \leq (I - \gamma_{w,v}\gamma_\varphi(\bar{w}))\bar{w}$$

$I - \gamma_{w,v}\gamma_\varphi$ is inverse-positive from Lemma 2, so $\|w\|_\infty \leq \bar{w}$. □

Remark. If condition (14) satisfies for all $\bar{w}$, then this is equivalent condition to the small gain condition in Theorem 1.

This remark can be directly observed from Lemma 2. This inequality condition is a different representation of small gain condition that exploits the fact that γ_φ can be a function of $\bar{w}$. There is a natural trade-off based on the value of $\bar{w}$. The nonlinear gain γ_φ increases as $\bar{w}$ increases, which makes it difficult to meet the small gain condition. On the other hand, small $\bar{w}$ imposes more strict bound on the phase difference on transmission lines. This trade-off is represented as the product of $I - \gamma_{w,v}\gamma_\varphi(\bar{w})$ and $\bar{w}$, which are monotonically decreasing and linearly increasing function of $\bar{w}$.

In power grid, the output is bounded due to the dissipation from damping. In order to prevent continuous angular separation and enforce generator frequency constraints, we need to impose additional conditions. This brings the following definition of constrained input constrained output.

Definition 4. (Constrained Input Constrained Output) The operator H is CICO if there exist $\bar{u}$, such that for every constrained input ($\|u\|_\infty \leq \bar{u}$), the output is constrained ($\|y\|_\infty \leq \bar{y}$).

The condition on the constrained input is extend to incorporate the output constraints.

Theorem 3. Let $\bar{u} \in \mathbb{R}^n$ be the bound on the input such that $\|u\|_\infty \leq \bar{u}$. If γ_G and γ_φ are finite, and there exist $\bar{u} \geq 0$ and $\bar{w} \geq 0$ such that

$$\begin{aligned}&\gamma_{w,u}\bar{u} < (I - \gamma_{w,v}\gamma_\varphi(\bar{w}))\bar{w} \\ &\gamma_{y,u}\bar{u} + \gamma_{y,v}\gamma_\varphi(\bar{w})\bar{w} \leq \bar{y}\end{aligned} \tag{15}$$

then the system is CICO, and $\|w\|_\infty \leq \bar{w}$ and $\|y\|_\infty \leq \bar{y}$.

Proof. From Theorem 2, the first condition in (15) ensures $\|w\|_\infty \leq \bar{w}$. Moreover, substitution of the condition in this theorem and Equation (13c) into Equation (13a) results in

$$\|y\|_\infty \leq \gamma_{y,u}\|u\|_\infty + \gamma_{y,v}\gamma_\varphi\|w\|_\infty \leq \gamma_{y,u}\bar{u} + \gamma_{y,v}\gamma_\varphi\bar{w} \leq \bar{y}.$$

□

The condition proposed in Theorem 3 will be used in the next section to efficiently compute the maximum acceptable disturbance magnitude.

IV. Computation of the Disturbance Bound

In the following, an optimization problem is formulated to find the bound on the disturbance, $\bar{u}$, such that the frequencies of the generator remain inside its operational limit. Given a potential disturbance u, the system operator only needs to check $u \leq \bar{u}$ to ensure generator frequency constraint. The input-output stability framework developed in Theorem 3 will be used as a constraint to the optimization problem.

The first step in doing so is to derive an explicit expression for the gain of nonlinear component γ_φ. Recall that γ_φ is function of $\bar{w}$:

$$\gamma_{\varphi,ii}(\bar{w}_i) = \sup_{|w_i| \leq \bar{w}_i} \left| \frac{\sin(w_i + \theta_i^*) - \sin(\theta_i^*)}{w_i} - \cos(\theta_i^*) \right| \tag{16}$$

where $\theta^* = E^T\delta$.

In the following corollary, an analytical expression is derived for the gain of the nonlinear components $\gamma_{\varphi,ii}(\bar{w}_i)$.

Corollary 1. Let $\bar{w} \in \mathbb{R}^l$ and $\theta^* = E^T\delta^* \in \mathbb{R}^l$ be such that $\forall i : |\theta_i^*| + \bar{w}_i \leq \pi, |\theta_i^*| \leq \frac{\pi}{2}$. Then,

$$\gamma_{\varphi,ii}(\bar{w}_i) \leq \cos|\theta_i^*| - \frac{\sin(|\theta_i^*| + \bar{w}_i) - \sin|\theta_i^*|}{\bar{w}_i}. \tag{17}$$

Proof. From Equation (12) and given $|\theta_i^*| \leq \frac{\pi}{2}$, we have

$$\begin{aligned}\gamma_{\varphi,ii}(\bar{w}_i) &= \sup_{|w_i| \leq \bar{w}_i} \left| \frac{\sin(w_i + \theta_i^*) - \sin(\theta_i^*)}{w_i} - \cos(\theta_i^*) \right| \\ &= \sup_{|w_i| \leq \bar{w}_i} \left| \frac{\sin w_i - w_i}{w_i}\cos\theta_i^* + \frac{\cos w_i - 1}{w_i}\sin\theta_i^* \right| \\ &\leq \sup_{|w_i| \leq \bar{w}_i} \frac{|w_i| - \sin|w_i|}{|w_i|}\cos|\theta_i^*| + \frac{1 - \cos|w_i|}{|w_i|}\sin|\theta_i^*|\end{aligned}$$

Moreover, the function inside the supremum monotonically increases with respect to w_i for $|\theta_i^*| + \bar{w}_i \leq \pi$. Therefore, inequality (17) holds true. □

At this point, by using the analytic expression for the gain of the nonlinearity (16), the implicit bound on the maximum disturbance magnitude can be established from Theorem 2, which also guarantees the input-output stability of the system.

The frequency constraints condition from Theorem 3, compute the bound on the acceptable disturbance can be formulated as the following optimization problem:

$$\begin{aligned} \underset{\bar{w}\geq 0,\ \bar{u}\geq 0,\ \mu}{\text{maximize}} \quad & \mu \\ \text{subject to} \quad & \gamma_{w,u}\bar{u} < (I - \gamma_{w,v}\gamma_{\varphi}(\bar{w}))\bar{w} \\ & \gamma_{y,u}\bar{u} + \gamma_{y,v}\gamma_{\varphi}(\bar{w})\bar{w} \leq \bar{y} \\ & |\theta^*| + \bar{w} < \pi,\ \mu \leq c^T\bar{u} \end{aligned} \tag{18}$$

where the vector $c \in \mathbb{R}^n$ can be used to fix the ratio of the perturbation at each bus. Notice that this procedure allows us to find the maximum disturbance magnitude at a particular bus, or at a combination of buses.

Propostion 1. The feasible space of the optimization problem (18) is convex.

Proof. Using the explicit expression for $\gamma_{\varphi}(\bar{w})$ in the constraint $\gamma_{w,u}\bar{u} \leq (I - \gamma_{w,v}\gamma_{\varphi}(\bar{w}))\bar{w}$, we obtain the following constraint:

$$\begin{aligned} \gamma_{w,u}\bar{u} \leq (I - \gamma_{w,v}&\text{diag}(\cos\theta^*))\bar{w} \\ - \gamma_{w,v}\sin|\theta^*| &+ \gamma_{w,v}\sin(|\theta^*| + \bar{w}) \end{aligned} \tag{19}$$

This constraint is valid in the region defined by $0 \leq |\theta_i^*| + \bar{w}_i \leq \pi$, which appears as a constraint. The sinusoidal term is with this bound is concave, and therefore the constraint in equation (19) forms a convex region. Similarly, the constrained output condition is similarly bounded to a convex region of a sinusoidal function. As a consequence, the feasible region of the optimization problem (18) is convex. □

V. SIMULATIONS

In this section, we present an illustrative example on a single machine infinite bus and more standard case studies on IEEE 9-bus and 39-bus systems.

A. Single Machine Infinite Bus (SMIB)

The procedure and plot of the condition is illustrated on SMIB example with second order swing equation with lossless line. The dynamic equation is given by

$$m\ddot{\delta} + d\dot{\delta} + a\sin\delta = P_0 + u \tag{20}$$

with its equilibrium at $\delta_0 = \arcsin(P_0/a)$. Let the output be the frequency in Hertz, $y = \dot{\delta}/2\pi$. Substituting $w = \delta - \delta_0$, and $v = \sin(\delta) - \cos(\delta_0)w$,

$$m\ddot{x} + d\dot{x} + a\cos(\delta_0)x + av = u \tag{21}$$

where $m = 1$, $D = 1.2$, $Pm = 0.2$ and $a = 0.8$ is used in this study. In frequency domain,

$$\begin{aligned} W &= \frac{1}{ms^2 + ds + a\cos(\delta_0)}\left[U - aV\right] \\ &= G_{w,u}U + G_{w,v}V, \end{aligned} \tag{22}$$

and $Y = sW/2\pi$. The gain of the matrix was computed $G_{y,u} = 0.178$, $G_{y,v} = 0.142$, $G_{w,u} = 1.434$, and $G_{w,v} = 1.148$. Following our procedure, the nonlinear gain can be plotted as a function of bound on the phase difference, which is plotted in Figure 3 (a). Since our gain matrix is a scalar, the condition for stability is $\gamma_{w,v}\gamma_{\varphi} < 1$. In Figure 3 (b), we plot the main condition in Theorem II, which is $\gamma_{w,u}\bar{u} \leq (I - \gamma_{w,v}\gamma_{\varphi}(\bar{w}))\bar{w}$ in blue.

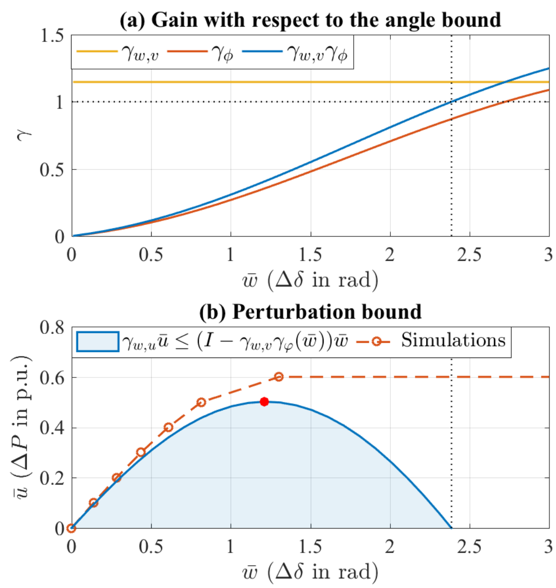

Fig. 3. Maximum perturbation allowed as a function of sector bound for 2 bus system.

The estimation of the upper bound on the acceptable disturbance was computed by time-domain simulation of a step response. A step disturbance of size, $\bar{u}$, was applied and the maximum phase difference deviation, $\bar{w}$, was recorded, and each simulation point was plotted with red points connected by a dashed line in Figure 3.

The simulation points are only upper-bound on the acceptable disturbance, and it does not give a sufficient condition to state the disturbance is acceptable. Monte-Carlo simulation could be used to find tighter upper-bound, but it is extremely time-consuming. On the other hand, our approach uses optimization to efficiently compute the lower-bound of the maximum acceptable disturbance. The result shows that the gap between the upper-bound from step response and lower-bound based on our method is very tight. The maximum perturbation allowed occurs when the $\bar{w}$ is about 0.9 rad, which can be computed with optimization. The small-gain theorem condition in Figure 3 (a) is violated when the deviation of angle is about 2.4. The feasible disturbance disappears at the same $\bar{w}$, which illustrates the equivalence of condition (i) and (iii) Lemma 2. In Figure 4, the upper-bound on the frequency deviation is computed with the output gain. Similarly, the lower bound on the frequency deviation was computed using simulation.

B. 9-bus and 39-bus systems

This section presents experimental case studies on IEEE 9-bus and 39-bus systems. The nonlinear optimization was computed on PC laptop with an Intel Core I7 3.3 GHz CPU and 16GB of memory with interior point method with IPOPT [24]. Figure 5 plots a graphical representation of the computed

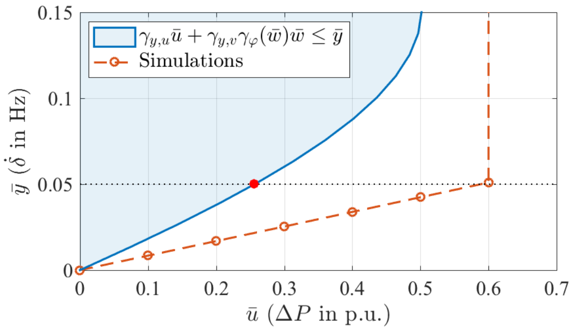

Fig. 4. Maximum frequency deviation for 2 bus system.

maximum bound on the disturbance. The maximum acceptable disturbance was computed assuming the disturbance is applied at a single bus. The result shows that the disturbance tend to be more acceptable on the buses that have many neighbors to distribute the impact. On the generator nodes, the second order dynamics plus the governor reduces the damping ratio, and only small disturbances are acceptable.

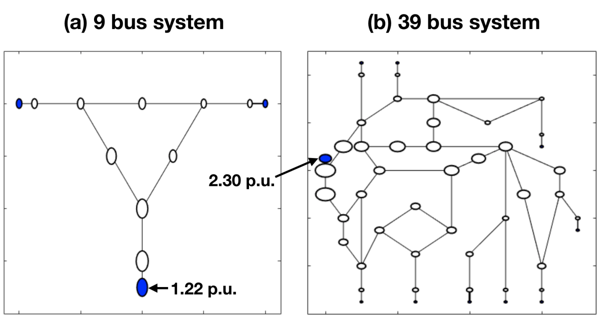

Fig. 5. Result of analysis for 9 bus and 39-bus systems. A disturbance on individual node is considered and the resulting maximum bound is represented as the size of circle on each node. For both systems, a reference circle (bus 1) is labeled with its value.

For the 9 bus system, the gain matrix took 1.86 seconds and optimization took 0.017 seconds. For the 39 bus system, the computation time for the gain matrix was 166.9 seconds, and optimization time was 0.148 seconds. The most computational intense step in our method is computation of gain matrix of the linear component, which requires simulation of impulse response and numerical integration. One of the approach for mitigating the long computation time is building the data base for the gain matrix. Since the generator locations and types are fixed, the operating point is frequent at certain region. When the system runs under a similar operating point to the one stored in the database, the stored gain matrix can be reused. Moreover, the computation could be accelerated by exploiting the fact that the modes of the each entry in the gain matrix is common and could be bounded [28]. Only the optimization has to be carried online, which takes in the range of milliseconds.

For the 39-bus case study, two scenarios were considered which introduced step and continuous disturbances.

1) Simultaneous Distributed Generators tripping: Near simultaneous tripping of loads on bus 3, 15 and 27 are considered in this scenario. The active power load at those buses are 3.22 p.u., 3.2 p.u. and 2.81 p.u. respectively. The maximum perturbation was solved with the frequency deviation bounded by 0.5 Hz. The result bounded the tripped load magnitude to be less than 0.939 p.u. Without frequency constraint, the maximum perturbation allowed is computed to be 2.29 p.u. at each load.

2) Wind generation: In this scenario, varying power output from wind generation at bus 1, 9 and 16 was considered. This gives quick assessment of the grid to plan for curtailment of renewables. The result gives the bound of 1.305 p.u. deviation of renewable from its nominal generation. Without the frequency constraint, 2.02 p.u. deviation in active power is allowed at each wind generator.

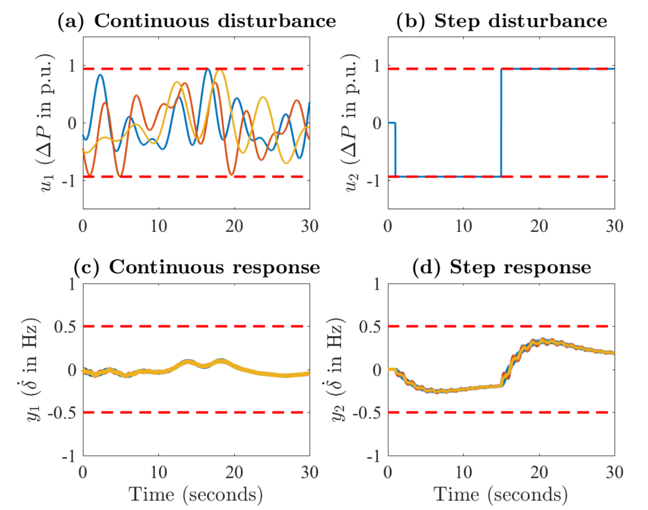

Fig. 6. Simulation results for simultaneous generation tripping (b) and wind generation (a) and their frequency response for 39 bus system.

VI. Conclusion

The input to output stability analysis technique presented in this paper provides a novel and practical solution to quickly estimate disturbances withstandable by the electric power grids. The system operators often need to quickly estimate the risk and take action in a pressing and unexpected environment where offline studies have not covered. Conventionally, the consideration of operational constraints on frequency deviation has been dealt separately from studying the transient stability. Our formulation offers a simple way to unify these considerations and utilize well-developed optimization methods to perform stability assessment. Our method efficiently divide computationally heavy step, which can be done in an offline environment, and the optimization procedure, which can be quickly done in an online environment. Our studies shows that our technique is not very conservative can include wide range of disturbances.

In the future work, we plan to extend our result to additional characterizations of disturbance such as ramping rate constraint or duration of disturbance. While our approach allows inclusion of any disturbance that is only bounded by its magnitude, the disturbance may exhibit specific characteristics. By exploiting additional knowledge about the nature of disturbance, our method can be extended to adapted to more specific applications.

References

[1] P. Kundur, N. J. Balu, and M. G. Lauby, *Power system stability and control*. McGraw-hill New York, 1994, vol. 7.

[2] M. Pavella, D. Ernst, and D. Ruiz-Vega, *Transient stability of power systems: a unified approach to assessment and control*. Springer Science & Business Media, 2012.

[3] J. Machowski, J. Bialek, and J. Bumby, *Power system dynamics: stability and control*. John Wiley & Sons, 2011.

[4] P. Kundur, J. Paserba, V. Ajjarapu, G. Andersson, A. Bose, C. Canizares, N. Hatziargyriou, D. Hill, A. Stankovic, C. Taylor *et al.*, "Definition and classification of power system stability ieee/cigre joint task force on stability terms and definitions," *IEEE transactions on Power Systems*, vol. 19, no. 3, pp. 1387–1401, 2004.

[5] N. Miller, K. Clark, and M. Shao, "Frequency responsive wind plant controls: Impacts on grid performance," in *Power and Energy Society General Meeting, 2011 IEEE*. IEEE, 2011, pp. 1–8.

[6] J. W. Ingleson and E. Allen, "Tracking the eastern interconnection frequency governing characteristic," in *Power and Energy Society General Meeting, 2010 IEEE*. IEEE, 2010, pp. 1–6.

[7] F. Wilches-Bernal, J. H. Chow, and J. J. Sanchez-Gasca, "A fundamental study of applying wind turbines for power system frequency control," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1496–1505, 2016.

[8] P. Kokotović and M. Arcak, "Constructive nonlinear control: a historical perspective," *Automatica*, vol. 37, no. 5, pp. 637–662, 2001.

[9] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems part one: Conditions derived using concepts of loop gain, conicity, and positivity," *IEEE transactions on automatic control*, vol. 11, no. 2, pp. 228–238, 1966.

[10] D. J. Hill, "A generalization of the small-gain theorem for nonlinear feedback systems," *Automatica*, vol. 27, no. 6, pp. 1043–1045, 1991.

[11] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for iss systems and applications," *Mathematics of Control, Signals and Systems*, vol. 7, no. 2, pp. 95–120, 1994.

[12] I. Karafyllis and Z.-P. Jiang, "A vector small-gain theorem for general non-linear control systems," *IMA Journal of Mathematical Control and Information*, vol. 28, no. 3, pp. 309–344, 2011.

[13] H. Bourles, "A local small gain theorem for discrete-time systems," in *Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on*, vol. 3. IEEE, 1994, pp. 2137–2138.

[14] E. D. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and optimal control theory*. Springer, 2008, pp. 163–220.

[15] E. D. Sontag and Y. Wang, "Notions of input to output stability," *Systems & Control Letters*, vol. 38, no. 4-5, pp. 235–248, 1999.

[16] S. N. Dashkovskiy and B. S. Rüffer, "Local iss of large-scale interconnections and estimates for stability regions," *Systems & Control Letters*, vol. 59, no. 3-4, pp. 241–247, 2010.

[17] A. Mironchenko, "Local input-to-state stability: Characterizations and counterexamples," *Systems & Control Letters*, vol. 87, pp. 23–28, 2016.

[18] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "An iss small gain theorem for general networks," *Mathematics of Control, Signals, and Systems*, vol. 19, no. 2, pp. 93–122, 2007.

[19] H. Lev-Ari, K.-F. Chiang, and T. Kailath, "Constrained-input/constrained-output stability for adaptive rls lattice filters," *IEEE transactions on circuits and systems*, vol. 38, no. 12, pp. 1478–1483, 1991.

[20] H. K. Khalil, "Noninear systems," *Prentice-Hall, New Jersey*, vol. 2, no. 5, pp. 5–1, 1996.

[21] S. Skogestad and I. Postlethwaite, *Multivariable feedback control: analysis and design*. Wiley New York, 2007, vol. 2.

[22] T. L. Vu and K. Turitsyn, "Lyapunov functions family approach to transient stability assessment," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1269–1277, 2016.

[23] D. Lee and K. Turitsyn, "Dynamic polytopic template approach to robust transient stability assessment," *arXiv preprint arXiv:1705.01189*, 2017.

[24] A. Wächter and L. T. Biegler, "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical programming*, vol. 106, no. 1, pp. 25–57, 2006.

[25] R. Plemmons, "M-matrix characterizations.i—nonsingular m-matrices," *Linear Algebra and its Applications*, vol. 18, no. 2, pp. 175 – 188, 1977.

[26] F. Bullo, *Lectures on Network Systems*. http://motion.me.ucsb.edu/book-lns/, 2017.

[27] L. Aolaritei, S. Bolognani, and F. Dörfler, "A distributed voltage stability margin for power distribution networks," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 13 240–13 245, 2017.

[28] V. Balakrishnan and S. Boyd, "On computing the worst-case peak gain of linear systems," *Systems & Control Letters*, vol. 19, no. 4, pp. 265 – 269, 1992.