Journal of Experimental and Theoretical Physics, Vol. 98, No. 2, 2004, pp. 348–351.
Translated from Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, Vol. 125, No. 2, 2004, pp. 395–398.

NONLINEAR PHYSICS

Effective Equation of Nonlinear Pulse Evolution in a Randomly Anisotropic Medium

I. V. Kolokolov[a,b] and K. S. Turitsyn[b]

[a]*Budker Institute of Nuclear Physics, Siberian Division, Russian Academy of Sciences, pr. Akademika Lavrent'eva 11, Novosibirsk, 630090 Russia*

[b]*Landau Institute for Theoretical Physics, Russian Academy of Sciences, Chernogolovka, Moscow oblast, 142432 Russia*

e-mail: kolokolov@itp.ac.ru, tur@itp.ac.ru

Received May 29, 2003

Abstract—Propagation of a light pulse through a weakly inhomogeneous optical fiber is analyzed. The nonlinear envelope equation describing the evolution of polarized pulses is determined by statistical properties of inhomogeneities in the optical fiber. The isotropic Manakov system of equations is shown to be applicable in the presence of high-frequency small-scale defects in the fiber. In the presence of only large-scale inhomogeneities, the signal dynamics are described by an anisotropic system of equations. *© 2004 MAIK "Nauka/Interperiodica".*

Currently, fiber optic communication systems are considered the most promising in information transfer over long distances. Such a system is a sequence of optical fibers and amplifiers. The amplifiers are required to compensate for losses inside a fiber. In the linear regime (when pulse power is low), the channel capacity is primarily limited by the noise generated by amplifiers. Since the amplitude of spontaneous emission noise is independent of signal power, considerable effort is applied to develop soliton systems, where a sequence of digits is encoded into high-power soliton pulses. These systems are characterized by essentially nonlinear signal dynamics. In the case of an ideal fiber, the dynamics are described by the nonlinear Schrödinger equation [1]. In this study, we analyze the more realistic case of a fiber with random fluctuating index profile and polarization-dependent evolution of electric-field energy density. We show that the form of an averaged large-scale equation describing this system strongly depends on the statistics of fluctuations and their scale distribution.

The light pulses used in information transfer have narrow spectral widths $\delta\omega$ compared to the carrier frequency ω_0. They can be described in terms of the envelope defined by a two-component complex vector $\boldsymbol{\psi} = (\psi_1, \psi_2)$:

$$\mathbf{E} = \boldsymbol{\psi}(z, t)\exp(i\omega_0 t) + \boldsymbol{\psi}^*(z, t)\exp(-i\omega_0 t). \quad (1)$$

Here, $\mathbf{E}$ is the electric field of a pulse; z is the longitudinal coordinate in a fiber; and t is the retarded time related to the physical time $t_{\rm phys}$ as $t = t_{\rm phys} - z/c$, where c is the group velocity of the packet. The evolution equation for the vector $\boldsymbol{\psi}$ is obtained by averaging Maxwell's equations for the electromagnetic field in the fiber medium over the fast-oscillation period $2\pi/\omega_0$. Taking into account the Kerr nonlinearity in chromatic dispersion and choosing appropriate units of $\boldsymbol{\psi}$, z, and t, one can reduce this equation to the following form [2]:

$$-i\partial_z\boldsymbol{\psi} = \partial_t^2\boldsymbol{\psi} + \frac{4}{3}(|\psi_1|^2 + |\psi_2|^2)\boldsymbol{\psi} + \frac{2}{3}(\psi_1^2 + \psi_2^2)\boldsymbol{\psi}^* + \hat{V}(z)\boldsymbol{\psi} + \dots. \quad (2)$$

The matrix $\hat{V}(z)$ describing birefringence effects is a random function of z because the fiber shape is irregular. This irregularity can be caused by static stresses, technological defects, etc. In what follows, we assume that $V_{\alpha\beta} \gg 1$ unless stated otherwise. Physically, this means that the effects due to nonlinearity and chromatic dispersion are much weaker than birefringence for optical pulses of typical width Δ and amplitude A. This condition is satisfied in real communication lines [1]. In the units of measure used in Eq. (2), $\Delta \sim 1$ and $A \sim 1$.

In Eq. (2), we omit the terms containing time derivatives due to the same inhomogeneities, such as $\hat{m}(z)\partial_t\boldsymbol{\psi}$ and $\xi(z)\partial_t^2\boldsymbol{\psi}$, where $\hat{m}(z)$ and $\xi(z)$ are random matrix and scalar functions, respectively. These corrections for random dispersion are small (about $\delta\omega/\omega_0$) as compared to the terms retained in Eq. (2), and their contribution is significant only at large z. The effective deterministic equation describing unperturbed evolution (if this equation exists, see below) is determined by the statistical properties of the matrix $\hat{V}(z)$ at $z \leq 1$. The form of this averaged equation may depend on the parameters of the problem. In this paper, we refine the applicability conditions both for specific effective equations and for deterministic description in general.

The term $V(z)\boldsymbol{\psi}$ in evolution equation (2) is responsible for strong dependence of the vector $\boldsymbol{\psi}$ on z. This dependence is eliminated by the transformation

$$\boldsymbol{\psi}(z, t) = \mathcal{T}\exp\left[i\int_0^z \hat{V}(\tau)d\tau\right]\boldsymbol{\Psi}(z, t). \quad (3)$$

The equation of motion for the field $\boldsymbol{\Psi}(z, t)$ contains rapidly oscillating functions of z. However, their amplitudes do not exceed unity, which means that the oscillation scale (about $1/V$) is much smaller than the scale of significant variation of signal amplitude (about 1). Therefore, an averaged description of the system's dynamics is possible.

The matrix $\hat{V}(z)$ is treated as traceless (this can be achieved by a phase transformation of the field $\boldsymbol{\psi}$). Furthermore, we consider fibers that do not exhibit natural optical activity. Therefore, $\hat{V}(z)$ can be represented as

$$\hat{V}(z) = b(\hat{\sigma}_3\cos\theta + \hat{\sigma}_1\sin\theta)$$

(see [3]), where $b(z)$ is the difference of the wave vectors for different polarizations and the angle $\theta(z)$ characterizes the orientation of these polarizations with respect to fixed coordinate axes. It is easy to check that the ordered exponential in Eq. (3) can be represented as

$$\mathcal{T}\exp\left[i\int_0^z \hat{V}(\tau)d\tau\right] = \exp\left[-\frac{i}{2}\hat{\sigma}_2\theta\right]\hat{W}(z), \quad (4)$$

with

$$\hat{W}(z) = \mathcal{T}\exp\left[i\int_0^z\left(b\hat{\sigma}_3 + \frac{\dot{\theta}}{2}\hat{\sigma}_2\right)d\tau\right], \quad (5)$$

where $\dot{\theta} \equiv d\theta/dz$. The matrix $\hat{W}(z)$ is the evolution operator for spin 1/2 in the varying magnetic field $\mathbf{h}(\tau) = (0, \dot{\theta}, b)$. Therefore, the explicit form of $\hat{W}(z)$ strongly depends on the ratio of the amplitude h to the characteristic scale l of its variation ($\dot{\theta}/\theta \sim \dot{h}/h \sim 1/l$). If the fluctuating amplitude $h = \sqrt{\dot{\theta}^2 + b^2}$ is much larger than $1/l$ (which is analogous to the characteristic frequency of the field $\mathbf{h}(\tau)$), then the following estimate for the operator $\hat{W}(z)$ holds up to values of z that are exponentially large in $hl \gg 1$ [4, 5]:

$$\hat{W}(z) = \exp\left[i\int_0^z h(\tau)d\tau + i\Gamma\hat{\sigma}_3\right](1 + \gamma\hat{\sigma}^+ - \gamma^*\hat{\sigma}^-),$$
$$\gamma \sim O\left(\frac{1}{hl}\right), \quad \Gamma \sim 1. \quad (6)$$

Here, Γ is the first correction in the adiabatic expansion for the spinor phase (which is sometimes called the Berry phase [6]) and $\hat{\sigma}^{\pm} = (\hat{\sigma}_1 \pm i\hat{\sigma}_2)/2$. Indeed, the varying profile $\mathbf{h}(\tau)$ can be represented in this case as a superposition of inhomogeneities of characteristic size l. First, consider one such fluctuation localized near the point $z = 0$. For $z \le l$, the off-diagonal elements in the matrix $\hat{W}(z)$ are determined by the "instantaneous" values $\dot{\theta}(z)$, $h(z)$, $\ddot{\theta}(z)$, $\dot{h}(z)$, … and are on the order of $\dot{\theta}(z)/h(z) \sim (hl)^{-1}$. It is easy to see that this parameter is an adiabaticity parameter: the first correction to the adiabatic approximation of $W(z)$ is proportional to $\dot{\theta}(z)/h(z)$. For $z \gg l$, all derivatives $\dot{\theta}(z)$, $\ddot{\theta}(z)$, … vanish and the off-diagonal elements γ are on the order of $\exp(iCh\tau_s)$, where the singular point (or zero) τ_s of the analytic continuation of $h(z)$ into the upper half-plane is nearest to the real axis (for details, see [4, 5]). If this function has no scales other than l, then $\operatorname{Im}\tau_s \sim l$ and $\gamma(z \gg l) \sim \exp(-\text{const} \cdot hl)$. When inhomogeneities are repeatedly encountered by a pulse propagating along the fiber, such exponentially small corrections add up. Therefore, the applicability of estimate (6) is limited with respect to z. The inequality $hl \gg 1$ means that the scale of variation of h is much larger than the length $1/h$. Since $h \gg 1$, we can average over $1/h$-scale oscillations after substituting Eqs. (4) and (6) into Eq. (2). The resulting system of equations,

$$-i\partial_z\Psi_1 = (1 + \xi_1)\partial_t^2\Psi_1 + 2\left(|\Psi_1|^2 + \frac{2}{3}|\Psi_2|^2\right)\Psi_1,$$
$$-i\partial_z\Psi_2 = (1 + \xi_2)\partial_t^2\Psi_2 + 2\left(|\Psi_2|^2 + \frac{2}{3}|\Psi_1|^2\right)\Psi_2, \quad (7)$$

was used in [7] to analyze the effects of small noise terms $\xi_{1,2}$ having a relative order of magnitude h^{-1}.

The above analysis is applicable when the Fourier components of the field $\mathbf{h}(z)$ with $k \sim h \gg 1/l$ are suppressed. For a random field $\theta(z)$, these conditions are satisfied when the correlation function $Q(z) = \langle\dot{\theta}(z)\dot{\theta}(0)\rangle$ is decreasing at $z \gg l$ and analytic at $z \longrightarrow 0$. If there exist regions of rapidly varying $\theta(z)$ (sharp bends, defects of structure, etc.), then the form of the matrix $\hat{W}(z)$ is determined by their statistical properties. For example, expression (6) is applicable at moderate distances z when the amplitudes of inhomogeneities are not too large, but with $\gamma \sim \sqrt{nz}$. Here, n is the number of such microscopic defects per unit length estimated as the asymptotic value of the Fourier transform $Q(z)$ at $k \sim h^{-1}$. A similar "Brownian" increase in γ is characteristic of intervals where the amplitude $h(z)$ is about $1/l$ (i.e., regions of nearly circular fiber cross section). The corresponding n is estimated as the fraction of these intervals in the total distance z. We define z_c as the distance for which the off-diagonal elements of $\hat{W}(z)$ are about unity. For a fiber with weak defects, $z_c \sim 1/n$. For a fiber with sharp bends and jumps in θ, the length z_c is estimated as the characteristic distance between such defects. After averaging over scales exceeding z_c, only the identity representation of the group $SU(2)$ in the tensor product $\hat{W}(z) \otimes \hat{W}(z) \otimes \dots$ is retained. Otherwise, the group $SU(2)$ would contain a subgroup invariant under multiplication by matrices $\hat{W}(z_1, z_2)$ with arbitrary z_1 and z_2. There is no such group unless the amplitude of fluctuations of the direction of $\mathbf{h}(z)$ is zero. This obviously follows from the fact that matrices $\exp(\mathbf{h}_1 \cdot \hat{\boldsymbol{\sigma}})$ and $\exp(\mathbf{h}_2 \cdot \hat{\boldsymbol{\sigma}})$ with noncollinear vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ do not commute. Reduction to an identity representation is equivalent to averaging over an invariant measure on the group $SU(2)$ (e.g., see [8]).

Averaging over $SU(2)$ can be carried out in the equation for varying $\mathbf{\Psi}(z, t)$ if $z_c \ll 1$. In this case, the form of the effective equation is determined by the nonzero averages

$$\langle |W_{11}|^2 |W_{12}|^2 \rangle = 1/6, \quad \langle |W_{11}|^4 + |W_{12}|^4 \rangle = 2/3. \quad (8)$$

We conclude that the evolution of a light pulse in a fiber with a relatively high density of microscopic defects is described by the Manakov equations [9]

$$-i\partial_z \mathbf{\Psi} = (1 + \boldsymbol{\xi})\partial_t^2 \mathbf{\Psi} + \frac{16}{9}(|\Psi_1|^2 + |\Psi_2|^2)\mathbf{\Psi}, \quad (9)$$

where $\boldsymbol{\xi}$ denotes small chaotic perturbations (see above). Equations (9) were derived by various methods by Menyuk and Wai (see [10, 11] and references cited therein). However, these authors erroneously concluded that system (9) is universally applicable as a model of pulse evolution if the correlation length of fluctuations of fiber inhomogeneities is much less than both dispersion and nonlinearity length scales ($l \sim 1/\dot{\theta} \ll 1$ in the present units). It was shown above that one must take into account the relative values of $b \sim h$ and $1/l$, as well as the short-wavelength asymptotic behavior of the correlation function of these fluctuations, which is determined by rare events. The importance of the value of hl in the linear problem of evolution of polarization was emphasized in [12].

In principle, an averaged description based on Eq. (7) of Eq. (9) is applicable if $z_c \gg 1 \gg h^{-1}$ or $z_c \ll 1$, respectively. If $z \sim z_c$, the signal shape is determined by the detailed behavior of the functions $\hat{V}(z)$. Indeed, the averaging over $SU(2)$ can be performed only if the trajectory of $\hat{W}(z)$ has traversed the neighborhood of any point of the group manifold a sufficient number of times. The ratio $1/z_c$ is a good measure of this "covering density" on the nonlinearity scale (i.e., over lengths of about 1). When $z \sim z_c$, fluctuations of the moments of the ordered exponential $\hat{W}(z)$ are also on the order of unity and there is no self-averaging. In the limit of $z_c \gg z_c \sim 1$, fluctuating stresses inside the fiber combined with shape fluctuations destroy the pulse [7, 13]; i.e., the maximum amplitude falls well below its initial value. The values of z and z_c can be compared by measuring the ellipticity of a signal that is linearly polarized along one of the principal axes at $z = 0$ in the linear regime.

Finally, we present the basic conclusions of this work. Since we discuss signal propagation in a random medium, only statistics of various observables are generally meaningful. However, the system can be described by deterministic equations when the z_c (characteristic length of change in wave polarization of order unity) has either of two limit values. If $z \ll z_c$, where z is the fiber length, then polarization adiabatically follows the variations of the principal axes of the fiber and Eqs. (7) are applicable. In the opposite limit of $z \geq 1 \gg z_c$, effective self-averaging associated with uniform distribution of polarization over the Poincaré sphere is obtained, and pulse evolution is described by the Manakov equations (9). If $z_c \sim 1$, the system cannot be described by any deterministic model. We note that the fiber can be deformed intentionally to reduce z_c to $z_c \ll 1$ in the soliton regime of information transfer. The reason is that Manakov system (9) is integrable. This property is very important with regard to interaction between solitons via disorder-induced radiation: this interaction in integrable case (9) is much weaker than that in nonintegrable cases [7, 13, 14], and the signal structure is noticeably distorted at much longer distances.

ACKNOWLEDGMENTS

We are grateful to I.R. Gabitov, V.V. Lebedev, and M.V. Chertkov for numerous discussions and questions that stimulated this work. The work of I.V.K. was supported in part by the Russian Foundation for Basic Research (project no. 03-02-16147a) and the Russian Foundation for Support of Science. The work of K.S.T. was supported by the Dinastiya Foundation.

REFERENCES

1. G. P. Agrawal, *Applications of Nonlinear Fiber Optics* (Academic, New York, 2001).
2. A. L. Berkhoer and V. E. Zakharov, Zh. Éksp. Teor. Fiz. **58**, 903 (1970) [Sov. Phys. JETP **31**, 486 (1970)].
3. P. K. A. Wai and C. R. Menyuk, Opt. Lett. **19**, 1517 (1994).
4. A. M. Dykhne, Zh. Éksp. Teor. Fiz. **41**, 1326 (1961) [Sov. Phys. JETP **14**, 941 (1962)].

5. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 3: *Quantum Mechanics: Non-Relativistic Theory*, 3rd ed. (Nauka, Moscow, 1974; Pergamon, New York, 1977).
6. M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
7. M. Chertkov, I. Gabitov, I. Kolokolov, and V. Lebedev, Pis'ma Zh. Éksp. Teor. Fiz. **74**, 608 (2001) [JETP Lett. **74**, 535 (2001)].
8. M. I. Petrashen' and E. D. Trifonov, *Applications of Group Theory in Quantum Mechanics*, 2nd ed. (UrSS, Moscow, 2000; Butterworths, London, 1969).
9. S. V. Manakov, Zh. Éksp. Teor. Fiz. **65**, 505 (1974) [Sov. Phys. JETP **38**, 248 (1974)].
10. C. R. Menyuk and P. K. A. Wai, J. Light Technol. **14**, 148 (1996).
11. C. R. Menyuk and P. K. A. Wai, J. Opt. Soc. Am. B **11**, 1288 (1994).
12. C. D. Poole, J. H. Winters, and J. A. Nagel, Opt. Lett. **16**, 372 (1991).
13. M. Chertkov, Y. Chung, A. Dyachenko, *et al.*, Phys. Rev. E **67**, 036615 (2003).
14. Y. Chung, V. Lebedev, and S. S. Vergeles, Jr., Opt. Lett. (in press).

Translated by R. Tyapaev