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Lagrangian and Eulerian descriptions of inertial particles in random flows

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We study how the spatial distribution of inertial particles evolves with time in a random flow. We describe an explosive appearance of caustics and show how they influence an exponential growth of clusters due to smooth parts of the flow, leading in particular to an exponential growth of the average distance between particles. We demonstrate how caustics restrict applicability of Lagrangian description to inertial particles.

Keywords: Two-phase flows; Inertial particles; Lyapunov exponents; Caustics

1. Introduction

Random compressible flows generally have regions where contractions accumulate and density grows. Small volume elements expand or contract exponentially which can be characterized by the set of Lyapunov exponents. Since the sum of the exponents is non-positive [1–5], density tends to a singular multi-fractal set with moments growing exponentially. Both the evolution and the final state of density in spatially smooth random flows have been described recently within some models [4–7]. The flow of inertial particles is compressible even when the flow of ambient fluid is incompressible [8] so particles participate in the fractalization and have some of their concentration moments growing exponentially [5]. On the other hand, every time there is a negative velocity gradient exceeding the inverse viscous response time of particles, faster particles from behind catch slower ones creating folds in distribution and caustics [9, 10]. Such breakdowns of distribution lead to finite-time singularities and explosive growth of some density moments. The goal of the present paper is to describe the statistical evolution of concentration from a uniform one to a set of clusters and voids and, in particular, to describe the role of folds in this evolution.

Because of inertia, the velocities of particles are not completely determined by the local velocity of the fluid. In the same small portion of the fluid one can find particles moving with substantially different velocities so that one cannot characterize the motion of droplets in continuous (hydrodynamic) approach which presumes velocity a single-valued function of coordinates. Therefore, the problem of inertial particles in a flow is kinetic rather than hydrodynamic [9, 11, 12]. Analytic approach to a realistic kinetic description does not seem to be feasible yet. On the other hand, the significant progress of analytic Lagrangian methods

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[4] makes it tempting to use them: to follow, for instance, a couple of close particles and to account only for a local velocity gradient. The question is: what can we learn from the Lagrangian approach about the statistics of particle concentration? To answer that, one needs a model that allows us to compare numerical data from kinetics with an analytic Lagrangian solution. For that end we consider here the motion of inertial particles in a one dimensional random flow, which is appropriate for our main goal to describe the role of breakdowns that are one-dimensional in any space dimensionality. This model is a subject of much interest from different perspectives [13, 14].

Here we briefly review what is known and derive new results, in particular, describe the statistics of the inter-particle distances R. We also carry out direct numerical simulation of kinetics in this model and find the growth rates of the moments of concentration n. It is only for smooth flows that one can immediately convert R into n (in 1d simply taking $n = 1/|R|$). Since the flow of inertial particles has discontinuities, any given interval between two chosen particles does not contain the same particles all the time. Particles can enter and leave the interval, i.e. numerous folds appear in particle distributions making nonlocal even the problem of describing single-point density statistics. We show that indeed the growth rates of density moments and inter-particle distances are different.

Particle coordinate $\mathbf{q}$ and velocity $\mathbf{V}$ change according to

$$\mathrm{d}\mathbf{q}/\mathrm{d}t = \mathbf{V}(\mathbf{q}, t), \quad \mathrm{d}\mathbf{V}/\mathrm{d}t = [\mathbf{u}(\mathbf{q}, t) - \mathbf{V}]/\tau, \quad \mathbf{q}(\mathbf{r}, 0) = \mathbf{r}. \tag{1}$$

Here the viscous (response) time is $\tau = (2/9)(\rho_0/\rho)(a^2/\nu)$ with a particle radius and ρ_0, ρ are particle and fluid densities, respectively. We treat the fluid velocity $\mathbf{u}$ as a given random function of time and smooth function of space coordinates. When the product of a typical velocity gradient and τ (called Stokes number, see below) is small, inertia is negligible and $\mathbf{V} \approx \mathbf{u}$.

Let us briefly recall some relevant properties of smooth compressible random flows [4]. The behavior of a small volume is governed by the local matrix of derivatives (called strain matrix) taken in the Lagrangian frame $s_{ik} = \partial u_i/\partial x_k$. Considering the distance between two fluid particles, $\mathbf{R}(t, \mathbf{r}_1 - \mathbf{r}_2) = \mathbf{q}(\mathbf{r}_1, t) - \mathbf{q}(\mathbf{r}_2, t)$ one finds $\langle R^m\rangle \propto \exp(E_m t)$ with E_m being a convex function of m. Angle brackets mean Lagrangian averaging (i.e. over the ensemble of fluid trajectories), unless otherwise specified. Such quantities naturally appear in theoretical description. However, straightforward space averages (called Eulerian) are more accessible experimentally. The relation between the two averages comes from the fact that every trajectory comes with the weight n^{-1} where the density is $n(t) = \det^{-1} \partial R_i(t, \mathbf{r})/\partial r_j$ (provided that the initial distribution is uniform $n_0 = 1$). Therefore, the Lagrangian moments $\langle n^{-m}\rangle$ are related to the Eulerian moments via $\langle n^{-m}\rangle = \langle n^{1-m}\rangle_E \propto \exp(\Gamma_m t)$. Therefore, $\Gamma_0 = 0 = \Gamma_1$ which correspond to conservation of mass and volume (Lagrangian and Eulerian measures), respectively. In one-dimensional (1d) smooth flows, $\Gamma_m = E_m$.

In 1d, one has for the distance $R(t)$ and velocity difference $v(t)$ between two close inertial particles

$$\dot{R} = v, \quad \tau\dot{v} = sR - v \quad \Rightarrow \quad \tau\ddot{R} + \dot{R} = sR. \tag{2}$$

Note in passing that the substitution $R = \Psi \exp(-t/2\tau)$ turns (2) into Schrödinger equation with a random potential (Anderson localization), with space replacing time and localization length replacing the Lyapunov exponent.

2. Appearance of folds

Consider the quantity $\sigma = v/R$ which satisfies the Langevin equation driven by the random noise $s(t)$:

$$\dot{\sigma} = -\sigma^2 - \sigma/\tau + s/\tau \equiv -\mathrm{d}U/\mathrm{d}\sigma + s/\tau, \tag{3}$$

where U is some effective potential. Let us describe the probability of finite-time singularity (explosion) $\sigma \to -\infty$ which corresponds to crossing of particle trajectories. Such probability can be written as a path integral over trajectories with $\sigma(0) = \sigma_0$, $\sigma(T) = -\infty$ [15–17]

$$P(T) = \int \mathcal{D}\sigma \mathcal{D}p \mathcal{D}s \mathcal{P}\{s\} \exp\left\{ \int_0^T \mathrm{i}p\left[\dot{\sigma} + W - \frac{s}{\tau}\right]\mathrm{d}t' \right\}. \tag{4}$$

Here $\mathcal{P}\{s\}$ is the probability functional for s and $W = U' = (\sigma^2 + \sigma/\tau)$. When T is much less than the average time between explosions (defined below), $P(T)$ is given by the single trajectory (optimal fluctuation [18–20]) which maximizes the probability and can be found by a saddle-point integration of (4).

First, consider T which is much less than the correlation time of the air velocity gradient s. Then the optimal fluctuation corresponds to $s = s_0$ which does not change during T because of the long correlation time. In this case, the path integration over the processes $s(t)$ is reduced to the integration over a single parameter s_0 with the measure $P_s(s_0)$, which is a single time statistics of velocity gradient s. The saddle-point integration over the fields p, σ is reduced to solving equation (3) with constant $s(t) = s_0$ and the boundary conditions $\sigma(0) = \sigma_0$, $\sigma(T) = -\infty$. Straightforward integration yields the following relation:

$$\begin{aligned} T &= \tau \int_{\sigma_0}^{-\infty} \frac{\mathrm{d}\sigma}{s_0 - \sigma - \tau\sigma^2} \\ &= \tau(-1 - 4s_0\tau)^{-1/2}\left[\pi - 2\arctan\left(\frac{1 + 2\sigma_0\tau}{\sqrt{-1 - 4s_0\tau}}\right)\right], \end{aligned} \tag{5}$$

which formally gives a relation between the optimal value of s_0 and the collapse time T. It is not possible to find the analytic expression for $s_0(T)$ for an arbitrary value of σ_0, however the situation greatly simplifies for $\sigma_0 = +\infty$. In this case, $P(T)$ can be interpreted as the probability of having time interval T between consequent collapses. It is also a lower bound estimate for a general σ_0: $P(T; \sigma_0) > P(T; +\infty) \equiv P(T)$. Substituting $\sigma_0 = +\infty$ in (5) one obtains

$$T = \frac{2\pi\tau}{\sqrt{-1 - 4s_0\tau}}, \tag{6}$$

or equivalently

$$s_0 = -\frac{1}{4\tau} - \frac{\pi^2\tau}{T^2}. \tag{7}$$

In this case the probability of a collapse is given by

$$P(T) = P_s(s_0)\left|\frac{\mathrm{d}s_0}{\mathrm{d}T}\right| = \frac{2\pi^2\tau}{T^3} P_s\left(-\frac{1}{4\tau} - \frac{\pi^2\tau}{T^2}\right). \tag{8}$$

One can see from this expression that collapses occur only if there is a finite probability of having sufficiently negative flow gradient, $s_0 < -1/4\tau$. In particular for Gaussian gradients, $P_s(x) = (\alpha/\pi)^{1/2}\exp(-\alpha x^2)$, the short-time asymptotics is as follows: $P(T) \propto T^{-3}\exp(-\alpha\pi^4\tau^2/T^4)$.

Consider now the case when the correlation time of s is much shorter than T. In this case, the noise can be effectively considered as white Gaussian, $\langle s(t)s(0)\rangle = 2D\tau^2\delta(t)$, and

$$P(T) = \int \mathcal{D}\sigma \exp\left\{-\frac{1}{4D}\int_0^T [\dot{\sigma} + W]^2 \mathrm{d}t'\right\}. \tag{9}$$

For small enough T (the exact conditions will be found later), it follows from the saddle-point approximation that the probability is given by the optimal fluctuation (also called ‘instanton’ trajectory [18–20]) which satisfies $\ddot{\sigma} = W(\sigma)W'(\sigma)$ with the boundary conditions $\sigma(0) = \sigma_0, \sigma(T) = -\infty$. After one integration one obtains the following equation:

$$\dot{\sigma} = -\sqrt{E + W^2}, \tag{10}$$

where E is an integration constant, characterizing the trajectory. This constant is determined by the boundary conditions

$$T = \int_{-\infty}^{\sigma_0} \frac{\mathrm{d}\sigma}{\sqrt{E + W^2}}. \tag{11}$$

The probability of such fluctuation is given by $P(T) \propto \exp(-A)$, where

$$A = \int_0^T \mathrm{d}t \frac{(\dot{\sigma} + W)^2}{4D} = \int_{-\infty}^{\sigma_0} \frac{\mathrm{d}\sigma}{4D} \frac{(\sqrt{E + W^2} - W)^2}{\sqrt{E + W^2}}. \tag{12}$$

Unfortunately, integrals (11), (12) cannot be expressed through known special functions, so we are able to get analytical results only in some limiting cases. We will consider the case $\sigma_0 = +\infty$ following the same arguments as in the preceding analysis. First, we consider the limit $E\tau^4 \ll 1$. In this case, the main contribution to integral (11) is given by the neighborhood of $\sigma = 0$ where one can neglect the nonlinear term in $W(\sigma)$:

$$T \approx \int_{-\sigma_c}^{\sigma_c} \frac{\mathrm{d}\sigma}{\sqrt{E + \sigma^2/\tau^2}} = 2\tau \operatorname{arcsinh}\left[\frac{\sigma_c}{\sqrt{E\tau^4}}\right]. \tag{13}$$

The value of cut-off σ_c is determined by the nonlinear terms in $W(\sigma)$ and can be estimated as $\sigma_c \approx \tau^{-1}$ This yields the following estimation: $T \propto \tau \log(1/E\tau^4) \gg \tau$. From expression (12) we obtain

$$A = \int_{-\infty}^{\infty} \frac{(|W| - W)^2 \mathrm{d}\sigma}{4D|W|} = \int_{-1/\tau}^{0} \frac{|W|\mathrm{d}\sigma}{D} = \frac{1}{6D\tau^3}. \tag{14}$$

We see that in the main approximation the action does not depend on T, which has a simple interpretation: the collapses are produced by universal tunneling processes, each having a probability $\exp(-1/6D\tau^3)$ and characteristic time-scale τ. In order to find the T dependence of the total probability, we should study the fluctuations around this instanton [23] which would involve some bulky calculations. Note that the average time between tunneling events (an analog of half life for radioactive decay) is exponentially large $\bar{T} = \tau \exp(1/6D\tau^3)$. For short times $T \ll \bar{T}$ one can treat the sequence of tunneling events as a Poisson process and predict the constant behavior $P(T) \propto \partial_T[1 - \exp(-T/\bar{T})] \approx 1/\bar{T}$. Let us stress again that this expression is true only for weak inertia $D\tau^3 \ll 1$ when the action A is large and the saddle-point approximation is applicable. Another limiting case, which can be studied analytically, corresponds to the very high 'energies' $E\tau^4 \gg 1$ where one can neglect the linear σ/τ terms in (11), (12), so that one has

$$T = \frac{\Gamma(1/4)^2}{2\sqrt{\pi}E^{1/4}}, \qquad A = \frac{\Gamma(1/4)^8}{96\pi^2 DT^3} \approx \frac{31.5}{DT^3}, \tag{15}$$

which corresponds to $T \ll \tau$. The crossover between the two regimes happens at $T \approx \tau$. To summarize, for the white $s(t)$ one gets

$$P(T) \propto \begin{cases} \exp(-c/DT^3), & T < \tau, \\ \exp(-1/6D\tau^3) & \tau < T < \tau\exp(1/6D\tau^3), \end{cases} \tag{16}$$

where $c = [\Gamma(1/4)]^8/96\pi^2 \approx 31.5$.

The gradient σ changes the sign after the explosion as the fast particle passes the slow one (we study one-dimensional flow but particles are generally in a three-dimensional space). That is the flux of probability that goes to $\sigma \to -\infty$ returns from $\sigma \to +\infty$. That allows for the steady-state probability density function (PDF) having constant probability flux F equal to the number of breakdowns per unit time. Such PDF must have $P(\sigma) \approx F\sigma^{-2}$ at $\sigma \to \pm\infty$. If, as is usually the case, the initial $P(\sigma, 0)$ does not have power tails, they appear at $t = +0$ according to $P(t, \sigma) \propto P(t)\sigma^{-2}$ and (8), (16). When $\sigma \to -\infty$, $R \to 0$. This means that negative moments of particle distance blow up in a finite time. Appendix A gives the relevant estimates for the parameters.

3. Short-correlated fluid flow

In this section, we approximate the flow gradient $s(t)$ in the particle reference frame by a white noise, which is quantitatively good for fast-falling heavy particles and gives a qualitatively correct description in other cases. In the white case, a variety of analytic results can be obtained, some translated from the localization theory and super-symmetric quantum mechanics [21, 23] and some original that we derive here. The steady-state PDF can be found explicitly [21]

$$P_0 = \frac{F}{D} \exp\left[-\frac{U(\sigma)}{D}\right] \int_{-\infty}^{\sigma} \exp\left[\frac{U(\sigma')}{D}\right] d\sigma' , \tag{17}$$

with the flux $F = D\partial P_0/\partial\sigma + \left(\sigma^2 + \sigma/\tau\right) P_0 \approx (2\pi\tau)^{-1} \exp[-1/(6D\tau^3)]$ for $D\tau^3 \ll 1$ (the dimensionless Stokes number $D\tau^3 = St$ measures the inertia of the particle). At $St \gg 1$, $F \approx 0.2D^{1/3}$ [13, 14], note that the average time between breakdowns is much smaller than τ in this limit. The Lyapunov exponent $\langle\sigma\rangle$ changes the sign at $St_* \approx 0.827$ [13, 14]: $\langle\sigma\rangle \approx -D\tau^2/2$ at $St \ll St_*$ and $\langle\sigma\rangle \propto D^{1/3}$ at $St \gg St_*$. That means that small particles cluster while large ones mix uniformly.

Note that the Gibbs state $\exp(-U/D)$ is non-normalizable in this case. The flux state (17) minimizes entropy production [25]. It can be shown that it is indeed the asymptotic solution at $t \to \infty$ [26].

To describe the joint statistics of σ and R we introduce the generating function $Z_k(\sigma, t) = \langle \delta\left[\sigma(t) - \sigma\right] R^k(t)\rangle$, which must be normalizable with respect to σ (because $\partial\sigma Z_k = \langle R^k\rangle$ is finite) and satisfy the equation

$$\frac{\partial Z_k}{\partial t} = k\sigma Z_k + \frac{\partial}{\partial\sigma}\left(\frac{\sigma}{\tau} + \sigma^2 + D\frac{\partial}{\partial\sigma}\right) Z_k . \tag{18}$$

Substitution $Z_k = \Psi(\sigma, t) \exp[-U/2D]$ turns it into the Schrödinger equation in a double well, which has been a subject of numerous works related to tunneling and instantons (see e.g. [23, 24, 27, 29, 30]. Following [23, 27], we first find (non-normalizable) solutions $\exp(\gamma_k t/\tau - U/D) f_k(\sigma)$ with f_k being polynomials and then the conjugated solutions by the method of variable constants. For example, there are steady states $Z_0 = P_0$ and

$$Z_1(\sigma) = (1 + \sigma\tau) \exp\left[\frac{U(\sigma)}{D}\right] \int_{-\infty}^{\sigma} \exp\left[\frac{U(\sigma')}{D}\right] \frac{d\sigma'}{(1 + \sigma'\tau)^2} .$$

In particular, this solution allows one to obtain the mean velocity difference between two particles at the distance a: $a \int \sigma Z_1(\sigma, t)\, d\sigma$. The growth rates of the moments of inter-particle distance can be obtained from (18) or in a straightforward way by writing

$$\dot{R}_{l,k} = -lR_{l,k}/\tau - (l - k)R_{l+1,k} + l(l - 1)DR_{l-2,k} \tag{19}$$

$$R_{l,k} = \langle R^k \sigma^l \rangle. \tag{20}$$

This allows one to calculate the growth rates of the distance moments for positive integer k since the equations for the higher moments are expressed only via lower ones. Assuming that for a given k all $R_{l,k} \propto \exp(\gamma_k t)$, we get for γ_k the $(k+1)$st-order equation

$$\det \begin{bmatrix} \gamma_k & -k & 0 & 0 & 0 & \dots & 0 \\ 0 & \gamma_k + \tau^{-1} & 1-k & 0 & 0 & \dots & 0 \\ -D & 0 & \gamma_k + 2\tau^{-1} & 2-k & 0 & \dots & 0 \\ 0 & -3D & 0 & \gamma_k + 3\tau^{-1} & 3-k & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -k(k-1)D/2 & 0 & \gamma_k + k\tau^{-1} \end{bmatrix} = 0\,. \qquad (21)$$

It is a three-diagonal matrix. We need its largest eigenvalue. For the second moment, the equation on the eigenvalues,

$$\det \begin{bmatrix} \gamma_2 & -2 & 0 \\ 0 & \gamma_2 + \tau^{-1} & -1 \\ -D & 0 & \gamma_2 + 2\tau^{-1} \end{bmatrix} = \gamma_2 \left(\gamma_2 + \tau^{-1}\right)\left(\gamma_2 + 2\tau^{-1}\right) - 2D = 0\,, \qquad (22)$$

gives $\langle R^2 \rangle \propto \exp(\gamma_2 t)$ with $\gamma_2 \approx D\tau^2$ for $D\tau^3 \ll 1$ and $\gamma_2 \approx (2D)^{1/3}$ for $D\tau^3 \gg 1$.

For arbitrary k, we find asymptotics. If $D\tau^3 k^2 \ll 1$, then $\gamma_k \ll \tau^{-1}$ so that γ_k can be neglected in (21) everywhere except the first element. That gives $\gamma_k \approx D\tau^2 k(k-1)/2$. When $D\tau^3 k^2 \gg 1$, the determinant of (19) is approximately $\gamma_k^{k+1} - \gamma_k^{k-2} Dk(k-1)\sum$, where $\sum = \sum_{i=1}^{k}(k-i) \propto k^2$ and $\gamma_k \propto (Dk^4)^{1/3}$. Let us compare the growth rates of the distance moments for the inertial particles with those for smooth compressible short-correlated flow. For the latter, Gaussianity of the stretching rates is translated into $\gamma_k \propto k(k-1)$ (see e.g. [4]), while for the former the dependence is parabolic only for low-order moments in the low-inertia limit $D\tau^3 k^2 \ll 1$. High moments correspond to high inertia and have $\gamma_k \propto (Dk^4)^{1/3}$ even for $St \ll 1$. Note that conservation requires $\gamma_0 = \gamma_1 = 0$ for inertial particles as well.

Since R is sign-changing for inertial particles, the statistics of $|R|$ deserves separate study, particularly for comparison with the concentration. The equation for the time derivative of $\tilde{R}_{lk} = \langle \sigma^l |R|^k \rangle$ for odd k differs from (19) by the extra term $2\langle \sigma^{l+1} R^{k+1} \delta(R) \rangle$, which is nonzero for $l = k$. As a result, the growth rates $\tilde{\gamma}_k \equiv \tilde{R}_{lk}^{-1} \mathrm{d}\tilde{R}_{lk}/\mathrm{d}t$ may differ from γ_k. The most dramatic new effect can be readily appreciated since $\tilde{\gamma}_k$ are related to the Lyapunov exponent via $\langle \sigma \rangle = (\mathrm{d}\tilde{\gamma}_k/\mathrm{d}k)_{k=0}$. At high inertia, when $St > St_*$ and $\langle \sigma \rangle$ is positive, it is thus evident that $\tilde{\gamma}_1 > 0$ as seen from the sketch in figure 1. The nonzero growth rate of $\langle |R| \rangle$ is a remarkable qualitatively new effect with a clear physical meaning: in every breakdown, extra particles enter the interval between the two particles that we follow; the interval length must

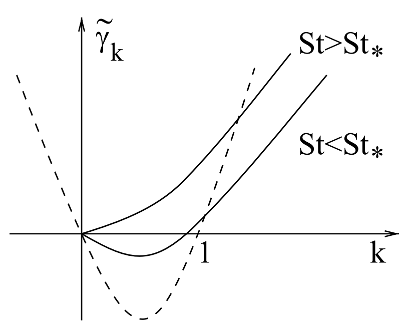

Figure 1. Growth rates of distance moments for a smooth flow (broken line) and inertial particles for different Stokes numbers (two solid lines).

grow to ensure conservation of the total number of particles. From this interpretation, it is clear that the growth rate must be nonzero at low inertia as well, when it must be proportional to the exponentially small rate of explosions: $\tilde{\gamma}_1 \propto F \propto \exp(-1/6St)$. Luckily, one can also establish asymptotically an exact pre-exponential factor. From the dynamical point of view the only difference between R and $|R|$ is in the effect of breakdown events. In order to obtain the explicit expression for $\tilde{\gamma}_1$, we first analyze the dynamical equation $\mathrm{d}|R|/\mathrm{d}t = \sigma|R|$ on the stages between the breakdowns, which formally coincides for both R and $|R|$ and then account for breakdowns explicitly.

First we introduce the Green function $P_t(R, v; R_0, v_0)$ which is the transition probability, i.e. the joint probability of having $v(t) = v$, $R(t) = R$ provided the initial conditions $R(0) = R_0$, $v(0) = v_0$. For delta-correlated velocity field it obeys the Chapman–Kolmogorov relation

$$P_t(R, v, R_0, v_0) = \int \mathrm{d}R'\mathrm{d}v' P_{t-T}(R, v; R', v')P_T(R', v'; R_0, v_0), \tag{23}$$

where $0 < T < t$. This property allows to separate the dynamics between the breakdowns. It is possible to average the transition probability over all possible trajectories which had $N = 0, 1\ldots$ breakdowns at moments $t_1, \ldots, t_N$:

$$P_t(R, v; R_0, v_0) = \tilde{P}_t(R, v; R_0, v_0) + \int \mathrm{d}v_1\mathrm{d}t_1 \tilde{P}_{t-t_1}(R, v; 0, v_1)\Pi(t_1, v_1; R_0, v_0)$$
$$+ \int \mathrm{d}v_1\mathrm{d}v_2\mathrm{d}t_1\mathrm{d}t_2 \tilde{P}_{t-t_2}(R, v; 0, v_2)\Pi(t_2 - t_1, v_2, 0, v_1)\Pi(t_1, v_1; R_0, v_0) \tag{24}$$

where $\tilde{P}_t(R, v; R_0, v_0)$ is the probability of a trajectory $R(t) = R$, $v(t) = v$ given $R(0) = R_0$, $v(0) = v_0$ without any breakdowns in the time interval $(0, t)$. Formally, $\tilde{P}_t(R, v; R_0, v_0) = P\{R(t) = R, v(t) = t, (R(s) \neq 0, s \in [0, t])|R(0) = R_0, v(0) = v_0\}$. It obeys the following Fokker–Planck equation

$$\left[\partial_t + \partial_R v - \partial_v v - DR^2\partial_v^2\right]\tilde{P}_t(R, v; R_0, v_0) = 0, \tag{25}$$

with the initial condition $\tilde{P}_0(R, v; R_0, v_0) = \delta(R - R_0)\delta(v - v_0)$, absorbing boundary conditions at $R = 0$, $v < 0$ and no-flux condition at $R = 0$, $v > 0$. $\Pi(t, v; R_0, v_0)$ is the joint probability of the first breakdown time t and the value of the velocity at the breakdown $v(t) = v$. Breakdown moment corresponds to $R = 0$, so $\Pi(t, v; R_0, v_0)$ can be expressed through $\tilde{P}_t$ in the following way:

$$\Pi(t, v; R_0, v_0) = \lim_{R\to 0} \left|\frac{\partial(R, v)}{\partial(t, v)}\right| \tilde{P}_t(R, v; R_0, v_0) = v\tilde{P}_t(0, v; R_0, v_0). \tag{26}$$

It is convenient to pass to the Laplace transform of all the Green functions: $F^s = \int_0^\infty \mathrm{d}t \exp(-st)F(t)$. In this case, equation (24) turns into

$$P^s(R, v; R_0, v_0) = \tilde{P}^s(R, v; R_0, v_0) + \int \mathrm{d}v_1 \tilde{P}^s(R, v; 0, v_1)\Pi^s(v_1; R_0, v_0)$$
$$+ \sum_{N=2}^{\infty} \int \mathrm{d}v_1 \ldots \mathrm{d}v_N \tilde{P}^s(R, v; 0, v_N) \ldots \Pi^s(v_2; 0, v_1)\Pi^s(v_1; R_0, v_0). \tag{27}$$

Equation (2) requires self-similarity of the Green functions

$$\alpha^2 \tilde{P}^s(\alpha R, \alpha v; \alpha R_0, \alpha v_0) = \tilde{P}^s(R, v; R_0, v_0) \tag{28}$$

$$\alpha \Pi^s(\alpha v; \alpha R_0, \alpha v_0) = \Pi^s(v; R_0, v_0). \tag{29}$$

Using this relation it is possible to simplify greatly the expression for the moment $\langle|R|^k\rangle = \int \mathrm{d}R\mathrm{d}v|R|^k P_t(R, v; R_0, v_0)$. After turning to the variables $x = R/v_N$, $y = v/v_N$, $x_1 =$

$v_1/R_0, x_2 = v_2/v_1, \dots x_N = v_N/v_{N-1}$ multidimensional integrals in (27) are factorized into the product of one-dimensional integrals. This yields the following result:

$$\left\langle \left| \frac{R}{R_0} \right|^k \right\rangle = \int \frac{\mathrm{d}s}{2\pi i} \exp(st) \left[Y_k^s(1, \sigma_0) + Y_k^s(0, 1) M_k^s(1, \sigma_0) \sum_{N=0}^{\infty} \{M_k^s(0, 1)\}^N \right]$$
$$= \int \frac{\mathrm{d}s}{2\pi \mathrm{i}} \exp(st) \left[Y_k^s(1, \sigma_0) + \frac{Y_k^s(0, 1) M_k^s(1, \sigma_0)}{1 - M_k^s(0, 1)} \right], \tag{30}$$

where $\sigma_0 = v_0/R_0$ and the new functions are defined as follows:

$$Y_k^s(a, b) = \int \mathrm{d}r \mathrm{d}v |r|^k \tilde{P}^s(r, v; a, b) \tag{31}$$

$$M_k^s(a, b) = \int \mathrm{d}v |v|^k \Pi^s(v; a, b) = \int \mathrm{d}v |v|^{1+k} \tilde{P}^s(0, v; a, b). \tag{32}$$

Note, that the function $M_k^s(a, b)$ can be interpreted as the average value of $|v|^k$ of the probability current flowing through the line $R = 0, v < 0$. The integration in (30) is performed over the contour which lies on the right of all the singularities in the integrated function. Long-time asymptotics of (30) is determined by the rightmost pole of the integrated functions. Since all four functions Y_k^s, M_k^s are constructed from $\tilde{P}^s(R, v; R_0, v_0)$ they have the same singularities. One can show that these singularities are simple poles $s = -E_{m,k} < 0,\ m = 0, 1 \dots$. Indeed, as follows from the analysis in previous sections there is a finite probability of breakdown in a finite time, therefore the solution of (25) should monotonically decrease with time. Therefore, the long-time asymptotic is determined either by $E_{0,k}$ or the rightmost s solving the equation $M_k^s(0, 1) = 1$. Let us show that the second possibility is realized. As long as the Green functions $\tilde{P}$, Π are determined by the trajectories without breakdowns, it is convenient to use the variable $\sigma = v/R$. One can introduce the new joint PDF $Z_t(\sigma, R; \sigma_0, R_0) = R\tilde{P}_t(R, \sigma R; R_0, \sigma_0 R_0)$ which obeys the following Fokker–Planck equation equation:

$$\left[\partial_t + \partial_R \sigma R - \partial_\sigma(\sigma/\tau + \sigma^2) - D\partial_\sigma^2\right] Z_t(R, \sigma, \sigma_0) = 0, \tag{33}$$

where the boundary and initial conditions are the same as in (25). We impose the no-flux boundary condition at $\sigma = +\infty$ and absorbing boundary conditions at $\sigma = -\infty$. The function $M_k^s \equiv M_k^s(0, 1)$ is given by

$$M_k^s = \lim_{\sigma \to -\infty, \sigma_0 \to \infty} |\sigma|^{2+k} |\sigma_0|^{-k} \int_0^\infty \mathrm{d}t \exp(-st) Z_{t,k}(\sigma; \sigma_0), \tag{34}$$

where $Z_t^k = \int \mathrm{d}R R^k Z_t(R, \sigma, \sigma_0)$. Formally, it is the expectation value of $\exp(k \int_0^t \sigma(t')\mathrm{d}t')$ averaged over the continuous functions $\sigma(t')$ satisfying $\sigma(t) = \sigma, \sigma(0) = \sigma_0$:

$$Z_{t,k}(\sigma; \sigma_0) = \left\langle \exp\left(k \int_0^t \sigma(t')dt'\right) \delta(\sigma - \sigma(t)) \middle| \sigma(0) = \sigma_0 \right\rangle_c. \tag{35}$$

Here the index c indicates that the averaging is performed over continuous trajectories $\sigma(t')$. Using (33), one can show that the Laplace transform of $Z_{t,k}$ obeys the following equation:

$$[s + \hat{\mathcal{L}}_k] Z_k^s(\sigma; \sigma_0) = \left[s - \partial_\sigma \left(\frac{\sigma}{\tau} + \sigma^2 \right) - D\partial_\sigma^2 - k\sigma \right] Z_k^s(\sigma; \sigma_0) = \delta(\sigma - \sigma_0). \tag{36}$$

The operator $\hat{\mathcal{L}}_k$ can be turned into a Hermitian one with the rotation $\hat{\mathcal{L}}_k = \exp[-U(\sigma)/2D] \hat{\mathcal{H}}_k \exp[U(\sigma)/2D]$, where $U(\sigma) = \sigma^2/(2\tau) + \sigma^3/3$ and $\hat{\mathcal{H}}_k$ is the

Hamiltonian of one-dimensional quantum mechanics with asymmetric double-well potential

$$\hat{\mathcal{H}}_k = -D\partial_\sigma^2 + \frac{\sigma^2(\tau^{-1}+\sigma)^2}{4D} - \frac{1}{2\tau} - (1+k)\sigma. \tag{37}$$

Therefore, $\hat{\mathcal{L}}_k$, $\hat{\mathcal{H}}_k$ have a bounded from below discrete spectrum $\hat{\mathcal{H}}_k\Psi_{m,k}(\sigma) = E_{m,k}\Psi_{m,k}(\sigma)$ corresponding to normalizable Z_k. The quantity M_k^s can be represented as

$$M_k^s = \sum_m \frac{\Delta_{m,k}}{s + E_{m,k}} \tag{38}$$

$$\Delta_{m,k} = \lim_{\sigma\to-\infty,\sigma_0\to\infty} |\sigma|^{2+k}|\sigma_0|^{-k} \exp\left[\frac{U(\sigma_0)-U(\sigma)}{2D}\right]\Psi_{m,k}(\sigma)\Psi_{m,k}(\sigma_0). \tag{39}$$

For arbitrary values of $St = D\tau^3$ it is not possible to find analytic expressions for $E_{m,k}$ and $\Delta_{m,k}$. However, for $St \ll 1$ the situation is greatly simplified. As we will show below in this case both $E_{0,k}$ and $\Delta_{m,k}$ are exponentially small in comparison to $E_{m,k}$ for $m > 0$. Therefore, one can leave only the first term in (38) and solve the equation $M_k^s = 1$ explicitly: $s = -E_{0,k} + \Delta_{0,k}$. As we will check *a posteriori*, $\Delta_{0,k} \geq 0$, so this solution indeed corresponds to the most right singularity in (30). Spectral properties of operator (37) in the limit $St \ll 1$ have been extensively studied in [23]. It was shown there that the ground-state energy can be represented as follows:

$$E_{0,k} = E_{0,k}^P + E_{0,k}^N = E_{0,k}^P + \tau^{-1}\cos(\pi k)\frac{\Gamma(1+k)St^k}{2\pi}\exp\left(-\frac{1}{6St}\right), \tag{40}$$

where $E_{0,k}^P$ is the perturbative contribution to the energy which vanishes for $k = 1$ and is given by the solution $E_{0,k}^P = -\gamma_k$, where γ_k is the solution of (21) for integer k. In order to find the difference $\tilde{\gamma}_k - \gamma_k = \Delta_{0,k} - E_{0,k}^N$, we have to find the asymptotics of $\Psi_{0,k}(\sigma)$. For small Stokes numbers this can be done by the semiclassical analysis. The details of these calculations are presented in appendix B. The final result $\Delta_{0,k} = (2\pi\tau)^{-1}\Gamma(1+k)St^k\exp(-1/6St)$ yields

$$\tilde{\gamma}_k - \gamma_k = -E_{0,k}^N + \Delta_{0,k} = \tau^{-1}\sin^2\left(\frac{\pi k}{2}\right)\frac{\Gamma(1+k)St^k}{\pi}\exp\left(-\frac{1}{6St}\right). \tag{41}$$

Note, that this difference vanishes for even k as one could expect. Elsewhere it is exponentially small and is negligible with comparison to γ_k. However, for $k = 1$ we have $\gamma_1 = 0$, and $\tilde{\gamma}_1$ is fully determined by (41).

4. Numerical simulations

We now present the results of numerical simulations of the growth of particle separation $\langle|R|^k\rangle$ in the Lagrangian frame and of negative moments of density $\langle n^{-k}\rangle$ in the Eulerian frame. The method used to obtain the growth rates is the Multicanonical Monte Carlo [31], a technique of adaptive importance sampling which boosts the probability of rare events that determine large negative moments. The Lagrangian results were obtained solving (2). The results presented in figures 2 and 3 confirm an exponential growth of $\langle|R|^k\rangle$. We also observe an exponential growth of the particle separation, $\langle|R|\rangle$. Figure 4 shows a good agreement between the numerics and the theoretical prediction (41) for $k = 1$ up to a fairly large $St \simeq 0.35$.

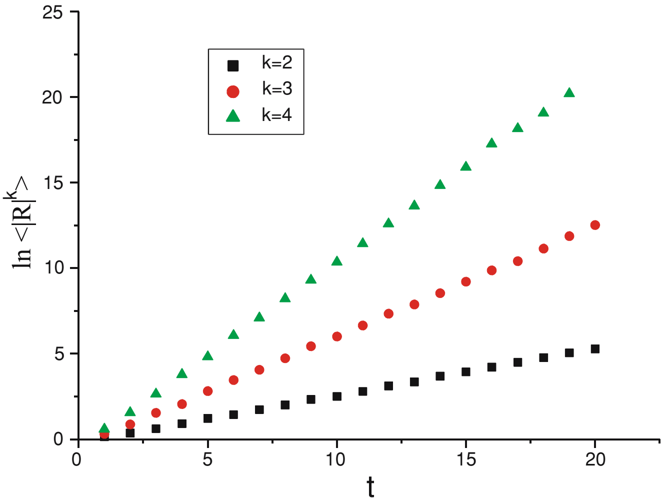

Figure 2. The moments $\langle |R|^k \rangle$ for $k = 2, 3, 4$ for $St = 0.2$. Time is normalized by τ.

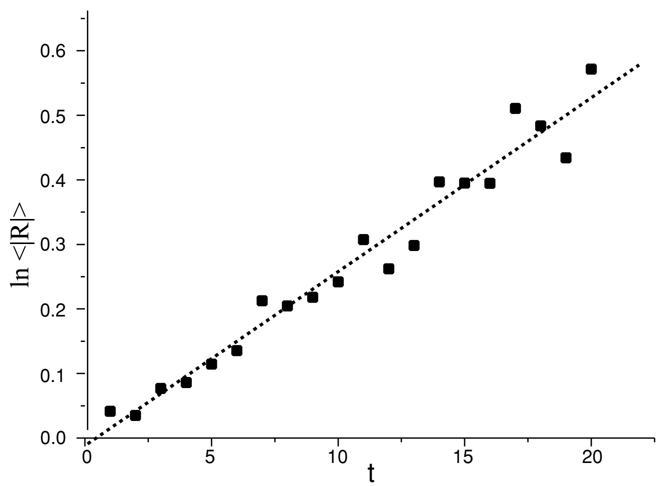

Figure 3. Inertia-caused growth of the modulus of particle separation $\langle |R| \rangle$ ($St = 0.2$).

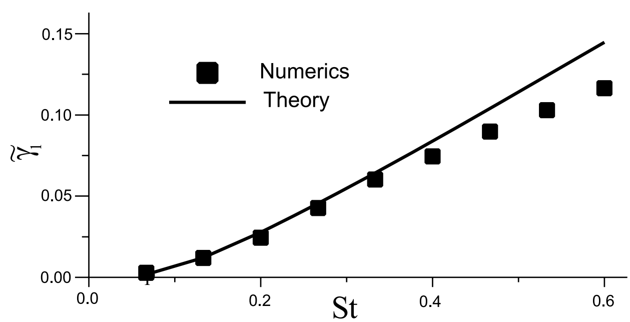

Figure 4. The growth rate $\tilde{\gamma}_1$ versus the Stokes number. The solid curve represents the theoretical prediction.

For the 1D Eulerian simulations, the density field is given by the following expression:

$$n(x,t) = \int \mathrm{d}x_0 n_0(x_0)\delta(x(t|x_0) - x), \tag{42}$$

where $n_0(x_0)$ is an initial Eulerian density distribution (which we assume uniform) and $x(t|x_0)$ is a Lagrangian trajectory of a particle.

This trajectory is obtained from the system of the ODEs (characteristic equations):

$$\frac{\mathrm{d}}{\mathrm{d}t}\, x(t|x_0) = v(t), \quad x(0|x_0) = x_0, \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, v(t) = -\frac{v(t) - u(x(t|x_0), t)}{\tau}, \tag{44}$$

where $u(x,t)$ is the Eulerian Gaussian velocity of the turbulent flow.

We assume that $u(x,t)$ is delta-correlated in time and has a spatial correlation length l_c:

$$\langle u(x,t)u(x',t')\rangle = B(x-x')\delta(t-t'), \quad B(x) = B_0 \mathrm{e}^{-x^2/l_c^2}. \tag{45}$$

The specific form of the correlation function B is not important. Eulerian field $u(x,t)$ is related to the Lagrangian process $s(t)$ (see equation (3)) via $s(t) = \partial u(x(t|x_0), t)/\partial x$. From this it follows that $St \equiv D\tau^3 = (\tau/2)\,|B''(0)| = (\tau/l_c^2)\, B_0$. Prior to solving the system of ODEs (43), (44) one has to generate a 1D Eulerian velocity field $u(x,t)$ with the prescribed correlation function (45). The algorithm for this is fairly standard (see. e.g. [32]). First we note that since the field $u(x,t)$ is delta correlated in time, its temporal regularization is trivial. Introducing discrete temporal step Δt at each time step, n, we now need to generate a spatially distributed Gaussain field $u_n(x)$ with the correlation property $\langle u_n(x)u_m(x')\rangle = B(x-x')\,\delta_{mn}$. In order to generate the field $u_n(x)$ we utilize the Fourier method. Indeed the field $u_n(x)$ can be represented as the following Fourier integral:

$$u_n(x) = \int_0^\infty \cos(2\pi kx)\,[2E(k)]^{1/2}\xi_n(k)\mathrm{d}k + \int_0^\infty \sin(2\pi kx)\,[2E(k)]^{1/2}\eta_n(k)\mathrm{d}k, \tag{46}$$

where $\xi_n(k)$ and $\eta_n(k)$ are independent real Gaussian processes with the following properties:

$$\langle \xi_n(k)\rangle = \langle \eta_n(k)\rangle = 0$$
$$\langle \xi_n(k)\xi_m(k')\rangle = \langle \eta_n(k)\eta_m(k')\rangle = \delta(k-k')\,\delta_{mn} \tag{47}$$

and $E(k)$ is an energy spectrum of the random field u_n; it coincides with the Fourier transform of the correlation function $B(x)$:

$$E(k) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}kx} B(x)\mathrm{d}x = B_0\sqrt{\pi l_c^2}\,\exp\big[-\pi^2 l_c^2 k^2\big]. \tag{48}$$

We then use a discrete version of (46):

$$u_n(x) \approx \sqrt{E(0)\,\Delta k}\,\xi_0^n + \sum_{j=1}^{M} \sqrt{2\,E(k_j)\,\Delta k}\,[\xi_j^n \cos(2\pi k_j\, x) + \eta_j^n \sin(2\pi k_j\, x)]. \tag{49}$$

Here we have partitioned the Fourier space into M intervals, so that the wavevectors $k_j = j\Delta k$ denote the locations of the equispaced grid points. Variables ξ_j^n and η_j^n form a set of independent standard Gaussian variables (mean zero and unit variance). Because of the nature of the Fourier method the synthetically generated field $u_n(x)$ will contain an intrinsic spatial period $\lambda_F = (\Delta k)^{-1}$. Naturally, one wishes to make it much bigger than the characteristic scale

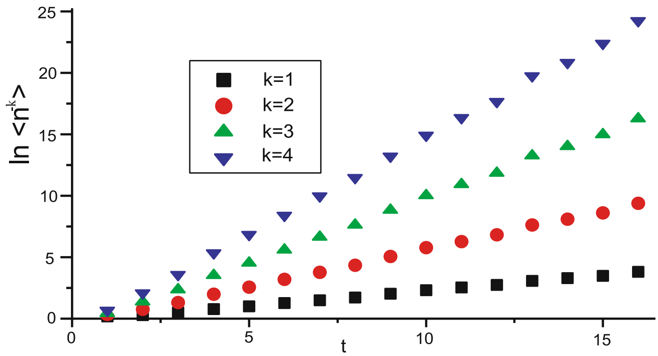

Figure 5. The Eulerian moments $\langle n^{-k}\rangle$ for $St = 0.2$.

of the system L. On the other hand, one has to ensure that we have enough harmonics in (49) to sample the peak of the function $E(k)$. These two requirements can be met assuming $(l_c M)^{-1}\Delta k \ll L^{-1}$.

Once we generated the synthetic Eulerian velocity field $u(x, t)$, we use a method of Lagrangian markers to obtain the Eulerian particle density at each point (using effectively formula (42)). We introduce a chain of N_L representative Lagrangian markers connected by some fictitious 'strings'. Each 'string' contains a large constant number of uniformly distributed real particles. This number is fixed for each string; it does not change during the evolution and is determined by the initial density distribution (note that all the results presented here are independent of the initial distribution as is also the case for a smooth random flow [5]). During the evolution, the strings deform according to the Lagrangian dynamics of the initial markers. In particular, the occurrence of explosions in Lagrangian frame corresponds to the formation of *folds* in the chain of markers. In order to obtain numerically the local Eulerian particle density at a given point, we count the number of strings passing through this point and then for each string determine the contribution to the density as a ratio N_i/l_i, where N_i is the number of particles in the string and l_i is the current length of the string. In figure 5, we plot the first four negative moments of n. Similarly to Lagrangian moments, Eulerian moments also grow exponentially: $\langle n^{-k+1}\rangle \propto \exp(\Gamma_k t)$. The table compares Γ_k and Lagrangian $\tilde{\gamma}_k$ given by (19) for $St = 0.1$ and $St = 0.2$. We see that Lagrangian breakdowns (Eulerian folds) violate $\Gamma_k = \tilde{\gamma}_k$ that one would have for a smooth flow. We do not have a meaningful parametrization for the dependences of $\tilde{\gamma}_k - \Gamma_k$ on k and St. It is likely that rare explosions cannot be completely disentangled from the exponential evolution.

In 1D case, there is a very simple way of visualizing the dynamics of the caustics. At a given time moment, t, one can plot the final displacement of a particle, $x(t)$ versus its initial position x_0. The Eulerian density distribution can be obtained by projecting the plot onto the coordinate axis $x(t)$. At the time moment $t = 0$ the curve is just a straight line at the half the right angle

Table 1. The comparison of Eulerian and Lagrangian growth rates for $St = 0.1$ (left) and $St = 0.2$ (right).

k	$\tilde{\gamma}_k$	Γ_k	$\tilde{\gamma}_k - \Gamma_k$	k	$\tilde{\gamma}_k$	Γ_k	$\tilde{\gamma}_k - \Gamma_k$
1	0.006	–	–	1	0.028	–	–
2	0.158	0.146	0.012 ± 0.003	2	0.274	0.250	0.025 ± 0.002
3	0.393	0.374	0.019 ± 0.005	3	0.643	0.611	0.032 ± 0.005
4	0.695	0.666	0.029 ± 0.006	4	1.098	1.054	0.044 ± 0.008
5	1.054	1.012	0.043 ± 0.009	5	1.627	1.564	0.063 ± 0.009
6	1.459	1.403	0.056 ± 0.010	6	2.223	2.131	0.098 ± 0.012

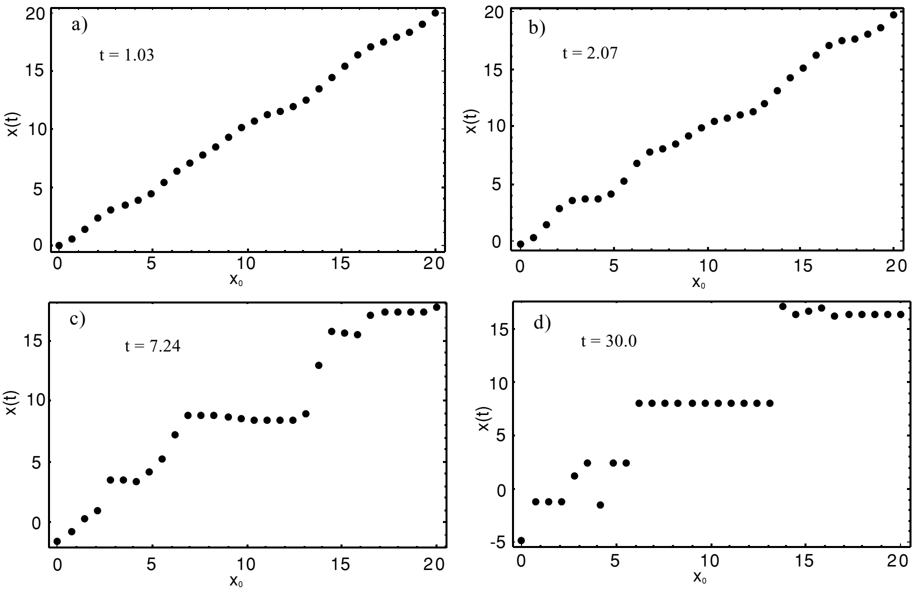

Figure 6. The evolution of 30 Lagrangian markers with time. The time is normalized to τ and increases from a) to d). $St = 0.4$

to the axes. During the evolution it will deform, according to the Lagrangian dynamics of individual particles (43), (44) eventually leading to the formation of folds illustrating the nonlocal nature of the Eulerian density.

In figure 6, we plot the three stages of evolution of particle distribution. We take $N_L = 30$ initially equispaced Lagrangian markers and follow the evolution of the function $x(x_0)$ through time for a particular realization of the velocity field. We observe that at the initial stage (figure 6(a)) the particle displacements are small so that the density distribution is smooth and there is one-to-ne correspondence $x(t) \leftrightarrow x_0$. Figure 6(b) shows the appearance of the first caustic (a particle overtakes another). In figure 6(c) the folds are more pronounced and clearly visible. Finally, at large times (figure 6(d)) one can evidently observe the effect of the clustering of particles.

5. Two-dimensional problem

Now, we must consider the matrix equation

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}t} + \hat{\sigma}^2 + \hat{\sigma}/\tau = \hat{s}, \tag{50}$$

$$\text{with} \quad \hat{s}(t) = \sqrt{D}\begin{pmatrix} \xi_1 & \xi_2 - \sqrt{2}\xi_3 \\ \xi_2 + \sqrt{2}\xi_3 & -\xi_1 \end{pmatrix}$$

and $\langle \xi_i(t)\xi_j(0)\rangle = \delta_{ij}\delta(t)$. One can write the equation in components
$\omega = tr\,\sigma = \sigma_{11} + \sigma_{22}, \quad a = \sigma_{11} - \sigma_{22}, \quad b = \sigma_{12} + \sigma_{21}, \quad c = \sigma_{12} - \sigma_{21}$:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\omega}{\tau} + \frac{c^2 - a^2 - b^2 - \omega^2}{2}, \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{a}{\tau} - \omega a + 2\sqrt{D}\xi_1, \tag{51}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -\frac{b}{\tau} - \omega b + 2\sqrt{D}\xi_2, \qquad \frac{\mathrm{d}c}{\mathrm{d}t} = -\frac{c}{\tau} - \omega c - 2\sqrt{2D}\xi_3. \tag{52}$$

Define $I \equiv c^2 - a^2 - b^2$. Note the exact symmetry relation $\langle \sigma_{ij}\sigma_{kl}\rangle = \langle(\partial v_i/\partial x_j)(\partial v_k/\partial x_l)\rangle = \langle \sigma_{il}\sigma_{kj}\rangle$ which gives

$$\langle \omega^2 + I\rangle = 0\,, \tag{53}$$

when the moment exists.

Lagrangian determinant $V = \mathrm{tr}\,\partial R_i(t)/\partial R_j(0)$ satisfies the equation $\dot V = \omega V$. To describe the growth of the volume moments, we introduce moments $V_m^{k,l} = \langle \omega^k I^l V^m\rangle$ and assume $V_m^{k,l} \propto \exp(\gamma_m t)$. Differentiating $\langle V\rangle$ three times we get the conservation of the Eulerian volume (i.e. the largest γ_1 is zero)

$$\gamma_1(\gamma_1+1)(\gamma_1+2) = 0. \tag{54}$$

For $m = 2$, we differentiate six times and in the basis

$$\langle V^2\rangle,\ \langle \omega V^2\rangle,\ \langle(3\omega^2+I)V^2\rangle,\ \langle \omega(\omega^2+I)V^2\rangle,\ \langle(\omega^2+I)^2V^2\rangle,\ \langle(2c^2+a^2+b^2)V^2\rangle$$

we get the following matrix (compare with (21)):

$$\det\begin{bmatrix} \gamma_2 & -2 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2+\tau^{-1} & -3/2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_2+2\tau^{-1} & -1 & 0 & 0 \\ 0 & 0 & 0 & \gamma_2+3\tau^{-1} & -1/2 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2+4\tau^{-1} & -16D \\ -24D & 0 & 0 & 0 & 0 & \gamma_k+2\tau^{-1} \end{bmatrix} = 0\,. \tag{55}$$

This gives $\gamma_2(\gamma_2+\tau^{-1})(\gamma_2+2\tau^{-1})^2(\gamma_2+3\tau^{-1})(\gamma_2+4\tau^{-1}) = (24D)^2$. At small St, we get $\gamma_2 \approx 12D^2\tau^5$ [33]. At large D, we get $\gamma_2 \approx (24D)^{1/3}$. Similarly to 1d case, one can show that the Lagrangian volume $|V|$ grows exponentially with time (the details will be published elsewhere).

As far as explosions are concerned, it can be argued that at small Stokes numbers it is much more probable to have a large velocity gradient along one direction rather than simultaneously along both directions. Therefore, most of the explosions are locally one dimensional in such a case. Indeed to have $\omega \to -\infty$, the system has to pass through the barrier in the region $-2/\tau < \omega < 0$. For, that, it needs a fluctuation with $I = c^2 - a^2 - b^2$, which is negative and large compared to its average scale during a long enough time. The most probable trajectory which satisfies this condition has $c = 0$. That allows us to rewrite equations (51), (52) in the following way:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\omega}{\tau} - \frac{\omega^2+r^2}{2} \tag{56}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -(1/\tau+\omega)r + \frac{4D}{r} + 2\sqrt{2D}\xi_r\,, \tag{57}$$

where $r = \sqrt{a^2+b^2}$ and $\langle \xi_r(0)\xi_r(t)\rangle = \delta(t)$. As long as we will be interested in trajectories where $r \sim \tau^{-1}$ the second term on the rhs of (57) can be neglected. In this case, it is convenient to turn to the new variables $y_\pm = \omega \pm r$ which obey the following equation:

$$\frac{\mathrm{d}y_\pm}{\mathrm{d}t} = -\frac{y_\pm}{\tau} - \frac{y_\pm^2}{2} \pm \xi_r\,. \tag{58}$$

These are one-dimensional equations similar to (3). As long as the breakdown in ω corresponds to the breakdown in either $y_\pm$ all of the results from section 2 can be applied here. We conclude that the breakdown processes in two dimensions are effectively one dimensional.

6. Conclusion

Let us summarize the peculiarities of the evolution of the distribution of inertial particles that distinguish them from smooth compressible flows

1. Infinite moments of density and inter-particle distance appear non-analytically at $t = +0$, according to $\exp(-C/Dt^3)$.
2. Average distance between particles in 1d (and average Lagrangian volume in higher dimensions) grows exponentially.
3. Negative moments of density in the Eulerian reference frame grow with the rates not reducible to those of distance moments (or volume moments in higher dimensions) in the Lagrangian frame.

Acknowledgement

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Appendix A: the Blow-up criteria for negative moments of *R*

Consider the moments $R_{l,k} = \langle \sigma^l R^k \rangle$ with negative k. Equation for the moments is given by (19). Consider first two equations for the moments

$$\frac{\mathrm{d}R_{0,k}}{\mathrm{d}t} = -|k| R_{1,k}, \tag{A1}$$

$$\frac{\mathrm{d}R_{1,k}}{\mathrm{d}t} = -\frac{R_{1,k}}{\tau} - (1 + |k|) R_{2,k}. \tag{A2}$$

Assuming that k is even, we can use the Cauchy–Schwartz inequality

$$R_{1,k} \leq R_{2,k}^{1/2} R_{0,k}^{1/2}, \tag{A3}$$

and combining these two equations we get

$$\frac{\mathrm{d}^2 R_{0,k}}{\mathrm{d}t^2} + \frac{1}{\tau}\frac{\mathrm{d}R_{0,k}}{\mathrm{d}t} = |k|(1 + |k|) R_{2,k} \geq \frac{R_{1,k}^2}{R_{0,k}} = \frac{1 + |k|}{|k|}\frac{(\mathrm{d}R_{0,k}/\mathrm{d}t)^2}{R_{0,k}}. \tag{A4}$$

Making transform $Z = R_{0,k}^{-1/|k|}$ yields

$$Z_{tt} + Z_t/\tau \leq 0\,. \tag{A5}$$

Equality here would give $Z_t(t_1)/Z_t(t) = \exp[(t - t_1)/\tau$. Inequality allows one to define the function $f(t) = Z(t_1) + \tau Z_t(t_1)(1 - \mathrm{e}^{-(t-t_1)/\tau})$, which bounds $Z(t)$ from above. Therefore, $Z(t) \leq f(t)$. Yet $f(t)$ turns into zero provided that there exists such t_1 that $Z(t_1) + \tau Z_t(t_1) < 0$. To prove that the moments can collapse in a finite time we need to demonstrate that

$$-\frac{Z(t_1)}{\tau Z_t(t_1)} = -\frac{R_{0,k}(t_1)}{\tau R_{1,k}(t_1)} < 1 \tag{A6}$$

at some t_1. It is useful to re-write equations on several first moments in the integral form. In particular,

$$R_{0,k} = R_{0,k}(0) + |k|(|k|+1) \int_0^t \mathrm{d}t' \exp[-t'/\tau] \int_0^{t'} \mathrm{d}t'' \; \exp[t''/\tau]\, R_{2,k}(t'').$$

Similar expressions can be derived for the first and second moments. From these equations one can infer the upper bound estimate for $R_{1,k}$:

$$R_{1,k} \le -D\,\tau^2 R_{0,k}(0)\, h(t),$$

where

$$h(t) = (1+|k|)(1-\exp(-t/\tau))^2 + St(1+|k|)^2(2+|k|)\{1 - 6(t/\tau)\exp(-2t/\tau) - 6(t/\tau)\exp(-t/\tau) - \exp(-3t/\tau) - 9\exp(-2t/\tau) + 9\exp(-t/\tau)\}. \tag{A7}$$

Next we want to estimate $R_{0,k} = R_{0,k}(0) - |k| \int_0^t \mathrm{d}t'\, R_{1,k}(t')$. In order to do so we note that $R_{1,k}\exp(t/\tau)$ is monotonously decreasing function so that $R_{0,k}(t_1) \le R_{0,k}(0) + |k| R_{1,k}(t_1)\left[1 - \exp(t_1/\tau)\right]$. Thus collapse condition (A6) can be rewritten as

$$|k|[\exp(t_1/\tau) - 1] + \frac{1}{St\,h(t_1)} \le 1, \tag{A8}$$

with $St = D\tau^3$.

An analysis shows that $h(t)$ is monotonously increasing function. Therefore, from the first term on the lhs it follows that the sought time t_1 (if exists) should be less than $t_{\max} = \ln(1 + 1/|k|)$, while from the second term one infers the Stokes parameter should be at least $St_{\min} = [(4|k|+9)^{1/2} - 1]/[2(1+|k|)(2+|k|)]$. For small t_1 on the lhs of (A8) the second term dominates while at t_1 close to $t_{\max}$ the first term becomes important. An analysis shows that for $k = -2$ for $St > St_* \approx 12$ the l h s of (A8) can indeed be less than 1 which means that the collapse does occur. The critical value St_* is a function of moment number k. For instance for $k = -4$ one can determine that $St_* \approx 22$. Generally, the higher the $|k|$ the higher is the estimate for St_*.

Appendix B: semiclassical calculations

In this section, we use the semiclassical calculations to obtain the value of $\Delta_{0,k}$ in the limit $St \ll 1$. The potential part of $\hat{\mathcal{H}}_k$ defined in (37) contains two minima close to $\sigma_{1,2} = 0, -\tau^{-1}$. For small enough $k \ll (D\tau^3)^{-1/2}$ in the vicinity of these minima the Hamiltonian can be approximated as

$$\hat{\mathcal{H}}_k = -D\partial_\sigma^2 + \frac{\sigma^2}{4D\tau^2} - \frac{1}{2\tau}, \qquad \sqrt{D\tau} \ll |\sigma| \ll \tau^{-1} \tag{B1}$$

$$\hat{\mathcal{H}}_k = -D\partial_\sigma^2 + \frac{(\sigma+\tau^{-1})^2}{4D\tau^2} + \frac{1+2k}{2\tau}, \qquad \sqrt{D\tau} \ll |\sigma + \tau^{-1}| \ll \tau^{-1}. \tag{B2}$$

In these region, the wavefunction can be approximated as the following:

$$\Psi_{0,k} = A_+ D_\epsilon\left(\frac{\sigma}{\sqrt{D\tau}}\right), \qquad |\sigma| \ll \tau^{-1} \tag{B3}$$

$$\Psi_{0,k} = A_- D_{\epsilon-1-k}\left(-\frac{\sigma+\tau^{-1}}{\sqrt{D\tau}}\right), \qquad |\sigma + \tau^{-1}| \ll \tau^{-1}, \tag{B4}$$

where $\epsilon = E_{0,k}\tau$ and $D_\nu(x)$ is the parabolic cylinder function. In the classically forbidden region away enough from the minima the wavefunction has the following WKB form:

$$\Psi_{0,k} = \frac{C_-}{|\sigma|^{1+\epsilon}|\tau^{-1}+\sigma|^{1+k-\epsilon}} \exp\left[\frac{U(\sigma)}{2D}\right], \quad \sigma < -\tau^{-1} \tag{B5}$$

$$\Psi_{0,k} = \frac{B_-}{|\sigma|^{1+\epsilon}|\tau^{-1}+\sigma|^{1+k-\epsilon}} \exp\left[\frac{U(\sigma)}{2D}\right] + \frac{B_+}{|\sigma|^{-\epsilon}|\tau^{-1}+\sigma|^{\epsilon-k}} \exp\left[-\frac{U(\sigma)}{2D}\right], \quad -1 < \sigma\tau < 0 \tag{B6}$$

$$\Psi_{0,k} = \frac{C_+}{|\sigma|^{-\epsilon}|\tau^{-1}+\sigma|^{\epsilon-k}} \exp\left[-\frac{U(\sigma)}{2D}\right], \quad \sigma > 0. \tag{B7}$$

In order to find the expressions for $A_\pm$, $B_\pm$ and $C_\pm$, we have to use normalization and to match the solutions (B3), (B4) with (B5)–(B7) in the intersection regions $\sqrt{D\tau} \ll |\sigma| \ll \tau^{-1}$ and $\sqrt{D\tau} \ll |\sigma + \tau^{-1}| \ll \tau^{-1}$. The parabolic cylinder functions have the following asymptotic expansions at $x \to +\infty$:

$$D_p(x) \sim x^p \exp(-x^2/4) \tag{B8}$$

$$D_p(-x) \sim \cos(\pi p) x^p \exp(-x^2/4) - \frac{\sqrt{2\pi}}{\Gamma(-p)} x^{-1-p} \exp(x^2/4). \tag{B9}$$

Omitting the tedious but rather straightforward calculations, we present only the final expressions here: $A_- = A_+(2\pi)^{-1/2}\Gamma(1+k)St^{k/2}\exp(-1/12St)$. One can see that the amplitude A_- is exponentially suppressed, so one has to apply the usual quantum-mechanical normalization condition to (B3), which automatically gives $A_+ = \sqrt{D\tau}(2\pi)^{-1/4}$. Finally, we obtain

$$\Delta_{0,k} = C_+C_- = St^{k/2-1}\exp(-1/12St)A_-A_+ = \tau^{-1}\frac{\Gamma(1+k)}{2\pi}St^k \exp\left[-\frac{1}{6St}\right]. \tag{B10}$$

Formally, the same calculations can be performed for arbitrary $\epsilon_m = E_{m,k}\tau$. Although the final expression $\Delta_{m,k}$ will be more complicated, it will always contain the exponential small factor $\Delta_{m,k} \propto \exp(-1/6St)$ which is a direct consequence of a high barrier between the two minima in the potential. Such suppression of $\Delta_{m,k}$ justifies the dropping of $m > 0$ terms in (38) for $St \ll 1$.

References

[1] Ruelle, D., 1996, Positivity of entropy production in non-equilibrium statistical mechanics. *Journal of Statistical Physics*, **85**, 1–23.

[2] Ruelle, D., 1997, Positivity of entropy production in the presence of a random thermostat. *Journal of Statistical Physics*, **86**, 935–951.

[3] Falkovich, G. and Fouxon, A., 2004, Entropy production and extraction in dynamical systems and turbulence. *New Journal of Physics*, **6**, 50.

[4] Falkovich, G., Gawedzki, K. and Vergassola, M., 2001, Particles and fields in fluid turbulence. *Reviews of Modern Physics*, **73**, 913–975.

[5] Balkovsky, E., Falkovich, G. and Fouxon, A., 2001, Intermittent distribution of inertial particles in turbulent flows. *Physical Review Letters*, **86**, 2790–2793.

[6] Bec, J., Gawedzki, K. and Horvai, P., 2004, Multifractal clustering in compressible flows. *Physical Review Letters*, **92**, 224501.

[7] Klyatskin, V. and Gurarie, D., 1999, Coherent phenomena in stochastic dynamical systems. *Soviet Physics—Uspekhi*, **169**, 171–207.

[8] Maxey, M.R., 1987, The gravitational settling of aerosol particles in homogeneous turbulence and random flow field. *Journal of Fluid Mechanics*, **174**, 441–465.

[9] Falkovich, G., Fouxon, A. and Stepanov, M.G., 2002, Acceleration of rain initiation by cloud turbulence. *Nature*, **419**, 151–154.

[10] Wilkinson, M. and Mehlig, B., 2005, Caustics in turbulent aerosols. *Europhysics Letters*, **71**, 186–192.
[11] Bec, J., 2003, Fractal clustering of inertial particles in random flows. *Physical of Fluids*, **15**, L81–L84.
[12] Bec, J., 2005, Multifractal concentrations of inertial particles in smooth random flows. *Journal of Fluid Mechanics*, **528**, 255–277.
[13] Mehlig, B. and Wilkinson, M., 2004, Coagulation by random velocity fields as a Kramers problem. *Physics Review Letters*, **92**, 250602.
[14] Wilkinson M. and Mehlig, B., 2003, Path coalescence transition and its applications. *Physical Reviews E*, **68**, 040101(R).
[15] C. de Dominicis, 1976, *Journal of Physics (Paris)* **37**, C01-247.
[16] H. Janssen, 1976, *Zeitchrift for Physical B*, **23**, 377.
[17] C. de Dominicis and L. Peliti, 1978, *Physical Review B*, **18**, 353.
[18] Falkovich, G. and Lebedev, V., 1997, Single-point velocity distribution in turbulence. *Physical Review Letters*, **79**, 4159–4161.
[19] Chertkov, M., Kolokolov, I., Lebedev, I. and Turitsyn,K., 2005, Polymer statistics in a random flow with mean shear. *Journal of Fluid Mechanics*, **531**, 251–260.
[20] Turitsyn, K.S., 2005, Polymer dynamics in chaotic flows with strong shear component. Available online at: arXiV:nlin.CD/0501025.
[21] Halperin, B.I., 1965, Green's functions for a particle in a one-dimensional random potential. *Physical Reviews*, **139**, A104–A117.
[22] Aoyama, H., Kikuchi, H., Okouchi, I., Sato, M. and Wada S., 1998, Valleys in quantum mechanics. *Physical Letter B*, **424**, 93–100.
[23] Aoyama, H., Kikuchi, H., Okouchi, I., Sato, M. and Wada S., 1999, Valley views: instantons, large order behaviors, and supersymmetry. *Nucl. Phys. B*, **553**, 644–710.
[24] Schomerus, H. and Titov, M., 2002: *Physical Review E*, **66**, 066207.
[25] Feigel'man, M.V. and Tsvelik, A.M., 1982, Hidden supersymmetry of stochastic dissipative dynamics. *Soviet Physics—JETP*, **56**, 823.
[26] Gawedzki, K., 2006, private communication.
[27] Turbiner, A.V., 1988, Quasi-exactly-solvable problems and sl(2) algebra. *Communication in Mathematical Physics*, **118**, 467–474.
[28] Balitsky, I.I. and Yung A.V., 1986, Instanton molecular vacuum in N = 1 supersymmetric quantum mechanics. *Nucl. Phys. B*, **274** 475–508.
[29] Bogomolny, E.B., 1980, Calculation of instanton–anti-instanton contributions in quantum mechanics. *Physical Letter B*, **91**, 431–435.
[30] Witten, E., 1981, Dynamical breaking of sypersymmetry. *Nucl. Phys. B*, **185**, 513–554.
[31] Berg B., 2004, *Markov Chain Monte Carlo Simulations and their Statistical Analysis* (Singapore: World Scientific).
[32] Majda, A.J. and Kramer, P.R., 1999, Simplified models for turbulent diffusion: Theory, numerical modelling, and physical phenomena. *Physical Reports*, **314**, 238–574.
[33] G. Falkovich and M. Stepanov, 2006, private communication.