1 Discrete (weighted) Besov spaces

We denote by $\Lambda_\varepsilon = (\varepsilon\mathbb{Z})^d$ for $\varepsilon = 2^{-N}$, $N \in \mathbb{N}_0$, the rescaled lattice $\mathbb{Z}^d$ and by $\Lambda_{M,\varepsilon} = \varepsilon\mathbb{Z}^d \cap \mathbb{T}^d_M = \varepsilon\mathbb{Z}^d \cap \left[-\frac{M}{2}, \frac{M}{2}\right)^d$ its periodic counterpart of size $M > 0$ such that $M/(2\varepsilon) \in \mathbb{N}$. For notational simplicity, we use the convention that the case $\varepsilon = 0$ always refers to the continuous setting. For instance, we denote by Λ_0 the full space $\Lambda_0 = \mathbb{R}^d$ and by $\Lambda_{M,0}$ the continuous torus $\Lambda_{M,0} = \mathbb{T}^d_M$. With the slight abuse of notation, the parameter ε is always taken either of the form $\varepsilon = 2^{-N}$ for some $N \in \mathbb{N}_0$, $N \geqslant N_0$, for certain $N_0 \in \mathbb{N}_0$ that will be chosen later, or $\varepsilon = 0$. Various proofs below will be formulated generally for $\varepsilon \in \mathcal{A} := \{0, 2^{-N}; N \in \mathbb{N}_0, N \geqslant N_0\}$ and it is understood that the case $\varepsilon = 0$ or alternatively $N = \infty$ refers to the continuous setting. All the proportionality constants, unless explicitly stated, are independent of M, ε.

Denote $\hat{\Lambda}_\varepsilon := (\varepsilon^{-1}\mathbb{T})^d$. For $f \in \ell^1(\Lambda_\varepsilon)$ and $g \in L^1(\hat{\Lambda}_\varepsilon)$ we define the Fourier and the inverse Fourier transform as

$$\mathcal{F}f(k) := \varepsilon^d \sum_{x \in \Lambda_\varepsilon} f(x) e^{-2\pi i k \cdot x}, \qquad \mathcal{F}^{-1}g(x) := \int_{\hat{\Lambda}_\varepsilon} g(k) e^{2\pi i k \cdot x} \mathrm{d}k,$$

where $k \in \hat{\Lambda}_\varepsilon$ and $x \in \Lambda_\varepsilon$. These definitions can be extended to discrete Schwartz distributions in a natural way, we refer to [3] for more details. In general, we do not specify on which lattice the Fourier transform is taken as it will be clear from the context.

Consider a smooth dyadic partition of unity $(\varphi_j)_{j \geqslant -1}$ such that φ_{-1} is supported in a ball around 0 of radius $\frac{1}{2}$, φ_0 is supported in an annulus, $\varphi_j(\cdot) = \varphi_0(2^{-j}\cdot)$ for $j \geqslant 0$ and if $|i - j| > 1$ then $\mathrm{supp}\varphi_i \cap \mathrm{supp}\varphi_j = \emptyset$. For the definition of Besov spaces on the lattice Λ_ε for $\varepsilon = 2^{-N}$, we introduce a suitable periodic partition of unity on $\hat{\Lambda}_\varepsilon$ as follows

$$\varphi^\varepsilon_j(k) := \begin{cases} \varphi_j(k), & j < N - J, \\ 1 - \sum_{j < N - J} \varphi_j(k), & j = N - J, \end{cases} \tag{1}$$

where $k \in \hat{\Lambda}_\varepsilon$ and the parameter $J \in \mathbb{N}_0$, whose precise value will be chosen below independently on $\varepsilon \in \mathcal{A}$, satisfies $0 \leqslant N - J \leqslant J_\varepsilon := \inf\{j : \mathrm{supp}\varphi_j \nsubseteq [-\varepsilon^{-1}/2, \varepsilon^{-1}/2)^d\} \to \infty$ as $\varepsilon \to 0$. We note that by construction there exists $\ell \in \mathbb{Z}$ independent of $\varepsilon = 2^{-N}$ such that $J_\varepsilon = N - \ell$.

Then (1) yields a periodic partition of unity on $\hat{\Lambda}_\varepsilon$. The reason for choosing the upper index as $N - J$ and not the maximal choice J_ε will become clear in Lemma ? below, where it allows us to define suitable localization operators needed for our analysis. The choices of parameters N_0 and J are related in the following way: A given partition of unity $(\varphi_j)_{j \geqslant -1}$ determines the parameters J_ε in the form $J_\varepsilon = N - \ell$ for some $\ell \in \mathbb{Z}$. By the condition $N - J \leqslant J_\varepsilon$ we obtain the first lower bound on J. Finally, the condition $0 \leqslant N - J$ implies the necessary lower bound N_0 for N, or alternatively the upper bound for $\varepsilon = 2^{-N} \leqslant 2^{-N_0}$ and defines the set $\mathcal{A}$.

We stress that once the parameters J, N_0 are chosen, they remain fixed. Then (1) yields a periodic partition of unity on $\hat{\Lambda}_\varepsilon$.

Reason for choosing the upper index as $N-J$ and not the maximal choice J_ε: it allows us to define suitable localization operators needed for our analysis.

The choices of parameters N_0 and J are related in the following way: A given partition of unity $(\varphi_j)_{j\geqslant -1}$ determines the parameters J_ε in the form $J_\varepsilon = N-\ell$ for some $\ell \in \mathbb{Z}$. By the condition $N-J \leqslant J_\varepsilon$ we obtain the first lower bound on J. The condition $0 \leqslant N-J$ implies the necessary lower bound N_0 for N, or alternatively the upper bound for $\varepsilon = 2^{-N} \leqslant 2^{-N_0}$ and defines the set $\mathscr{A}$. We stress that once the parameters J, N_0 are chosen, they remain fixed.

Remark that according to our convention, $(\varphi_j^0)_{j\geqslant -1}$ denotes the original partition of unity $(\varphi_j)_{j\geqslant -1}$ on $\mathbb{R}^d$, which can be also read from (1) using the fact that for $\varepsilon = 0$ we have $J_\varepsilon = \infty$.

Now we may define the Littlewood–Paley blocks for distributions on Λ_ε by

$$\Delta_j^\varepsilon f := \mathscr{F}^{-1}(\varphi_j^\varepsilon \mathscr{F} f),$$

which leads us to the definition of weighted Besov spaces. In the sequel, ρ denotes a polynomial weight of the form

$$\rho(x) = \langle hx \rangle^{-\varsigma} = (1+|hx|^2)^{-\varsigma/2} \tag{2}$$

for some $\varsigma > 0$ and $h > 0$. The constant h shall be fixed in order to produce a small bound for certain terms. Such weights satisfy the admissibility condition $\rho(x)/\rho(y) \lesssim \rho^{-1}(x-y)$ for all x, $y \in \mathbb{R}^d$. For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\varepsilon \in [0,1]$ we define the weighted Besov spaces on Λ_ε by the norm

$$\|f\|_{B^{\alpha,\varepsilon}_{p,q}(\rho)} = \left(\sum_{-1\leqslant j\leqslant N-J} 2^{\alpha j q} \|\Delta_j^\varepsilon f\|^q_{L^{p,\varepsilon}(\rho)} \right)^{1/q} = \left(\sum_{-1\leqslant j\leqslant N-J} 2^{\alpha j q} \|\rho\, \Delta_j^\varepsilon f\|^q_{L^{p,\varepsilon}} \right)^{1/q},$$

where $L^{p,\varepsilon}$ for $\varepsilon \in \mathscr{A} \setminus \{0\}$ stands for the L^p space on Λ_ε given by the norm

$$\|f\|_{L^{p,\varepsilon}} = \left(\varepsilon^d \sum_{x\in\Lambda_\varepsilon} |f(x)|^p \right)^{1/p}$$

(with the usual modification if $p = \infty$). Analogously, we may define the weighted Besov spaces for explosive polynomial weights of the form ρ^{-1}. Note that if $\varepsilon = 0$ then $B^{\alpha,\varepsilon}_{p,q}(\rho)$ is the classical weighted Besov space $B^\alpha_{p,q}(\rho)$. In the sequel, we also employ the following notations

$$\mathscr{C}^{\alpha,\varepsilon}(\rho) := B^{\alpha,\varepsilon}_{\infty,\infty}(\rho), \qquad H^{\alpha,\varepsilon}(\rho) := B^{\alpha,\varepsilon}_{2,2}(\rho).$$

We will frequently use the following auxiliary results whose proofs can be found in Appendix A.1 in [1].

Lemma 1. *Let $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$. Fix $n > |\alpha|$ and assume that ρ is a weight such that*

$$\|\rho\|_{B^{n+1,\varepsilon}_{\infty,\infty}(\rho^{-1})} + \|\rho^{-1}\|_{B^{n+1,\varepsilon}_{\infty,\infty}(\rho)} \lesssim 1$$

uniformly in ε. Then

$$\|f\|_{B^{\alpha,\varepsilon}_{p,q}(\rho)} \sim \|\rho f\|_{B^{\alpha,\varepsilon}_{p,q}},$$

where the proportionality constant does not depend on ε.

This is useful to transfer the results for the unweighted setting to the weighted one.

Lemma 2. *Let $\alpha \in \mathbb{R}$, $p, p', q, q' \in [1, \infty]$ such that p, p' and q, q' are conjugate exponents. Let ρ be a weight as in Lemma 1. Then*

$$\langle f, g \rangle_\varepsilon \lesssim \| f \|_{B_{p,q}^{\alpha,\varepsilon}(\rho)} \| g \|_{B_{p',q'}^{-\alpha,\varepsilon}(\rho^{-1})}$$

with a proportionality constant independent of ε. Consequently, $B_{p',q'}^{-\alpha,\varepsilon}(\rho^{-1}) \subset (B_{p,q}^{\alpha,\varepsilon}(\rho^{-1}))^*$.

Lemma 3. *Let $\varepsilon \in \mathscr{A}$. Let $\alpha, \alpha_0, \alpha_1, \beta, \beta_0, \beta_1 \in \mathbb{R}$, $p, p_0, p_1, q, q_0, q_1 \in [1, \infty]$ and $\theta \in [0, 1]$ such that*

$$\alpha = \theta\, \alpha_0 + (1 - \theta)\, \alpha_1, \quad \beta = \theta\, \beta_0 + (1 - \theta)\, \beta_1, \quad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1 - \theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1 - \theta}{q_1}.$$

Then

$$\| f \|_{B_{p,q}^{\alpha,\varepsilon}(\rho^\beta)} \leqslant \| f \|_{B_{p_0,q_0}^{\alpha_0,\varepsilon}(\rho^{\beta_0})}^\theta \| f \|_{B_{p_1,q_1}^{\alpha_1,\varepsilon}(\rho^{\beta_1})}^{1-\theta}.$$

2 Setup for stochastic quantization of $\Phi^4_{2,3}$

Based on arguments from the theory of PDEs, we intend to construct the Euclidean Φ^4 quantum field theory on $\mathbb{R}^3$. This is a probability measure ν on the space of Schwartz distributions $\mathscr{S}'(\mathbb{R}^3)$ which is formally represented by

$$\nu(\mathrm{d}\varphi) \sim \exp\left\{ -2 \int_{\mathbb{R}^3} \left[\frac{\lambda}{4} |\varphi(x)|^4 + \frac{m^2}{2} |\varphi(x)|^2 + \frac{1}{2} |\nabla \varphi(x)|^2 \right] \mathrm{d}x \right\} \mathrm{d}\varphi,$$

where $m^2 \in \mathbb{R}$ is the mass and $\lambda > 0$ is the coupling constant. The above expression is only formal, because powers of distributions are not well defined in general. The measure ν is associated to the corresponding stochastic quantization equation

$$(\partial_t + m^2 - \Delta) \varphi + \lambda \varphi^3 - \infty \varphi = \xi, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \tag{3}$$

where ξ is a space-time white noise on $\mathbb{R}^3$ and the term $-\infty\varphi$ stands for the so-called mass renormalization. This permits to make sense of the nonlinearity in (3) and in turn to obtain an honest probability measure ν as the invariant measure of (3).

For notational simplicity, we let $m^2 = \lambda = 1$ in the sequel. For $d \in \{2, 3\}$, let $\Lambda_\varepsilon = (\varepsilon \mathbb{Z})^d$ for $\varepsilon = 2^{-N}$, $N \in \mathbb{N}_0$, be the rescaled lattice $\mathbb{Z}^d$ and let $\Lambda_{M,\varepsilon} = ((\varepsilon \mathbb{Z}) / (M \mathbb{Z}))^d$ be the periodic lattice with mesh size ε and side length M with $M / (2\varepsilon) \in \mathbb{N}$. We denote by Δ_ε the lattice Laplacian on Λ_ε

$$\Delta_\varepsilon f(x) := \varepsilon^{-2} \sum_{i=1}^d (f(x + \varepsilon e_i) - 2 f(x) + f(x - \varepsilon e_i)), \qquad x \in \Lambda_\varepsilon,$$

where $(e_i)_{i=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$. Let $\mathscr{Q}_\varepsilon := 1 - \Delta_\varepsilon$, $\mathscr{L}_\varepsilon = \partial_t + \mathscr{Q}_\varepsilon$ and we write $\mathscr{L}$ for the continuum analogue of $\mathscr{L}_\varepsilon$.

We approximate the stochastic quantization equation (3) by the finite dimensional lattice model

$$\mathscr{L}_\varepsilon \varphi_{M,\varepsilon} + \varphi_{M,\varepsilon}^3 - 3c_{M,\varepsilon}\varphi_{M,\varepsilon} = \xi_{M,\varepsilon}, \qquad x \in \Lambda_{M,\varepsilon}. \tag{4}$$

Here $c_{M,\varepsilon}$ are renormalization constants diverging as $M \to \infty$ and $\varepsilon \to 0$ and $\xi_{M,\varepsilon}$ is the discrete and periodic approximation of a space-time white noise ξ on $\mathbb{R}^d$

$$\xi_{M,\varepsilon}(t,x) := \varepsilon^{-d}\langle \xi_M(t,\cdot), \mathbf{1}_{|\cdot - x| \leqslant \varepsilon/2}\rangle, \qquad (t,x) \in \mathbb{R} \times \Lambda_{M,\varepsilon},$$

with ξ_M being the periodization of ξ

$$\xi_M(h) := \xi(h_M), \qquad \text{where} \quad h_m(t,x) := \mathbf{1}_{[-\frac{M}{2},\frac{M}{2})^d}(x) \sum_{y \in M\mathbb{Z}^d} h(t, x+y), \quad h \in L^2(\mathbb{R} \times \mathbb{R}^d).$$

Then (4) is a finite-dimensional SDE in a gradient form and it has a unique invariant measure $\nu_{M,\varepsilon}$ given by

$$\nu_{M,\varepsilon}(\mathrm{d}\varphi) \sim \exp\left\{-2\varepsilon^d \sum_{x \in \Lambda_{M,\varepsilon}} \left[\frac{1}{4}|\varphi_x|^4 + \frac{-3c_{M,\varepsilon}+1}{2}|\varphi_x|^2 + \frac{1}{2}|\nabla_\varepsilon \varphi_x|^2\right]\right\} \prod_{x \in \Lambda_{M,\varepsilon}} \mathrm{d}\varphi_x, \tag{5}$$

where ∇_ε denotes the discrete gradient

$$\nabla_\varepsilon f(x) := \left(\frac{f(x+\varepsilon e_i) - 2f(x)}{\varepsilon}\right)_{i=1,\ldots,d}$$

satisfying the corresponding integration by parts

$$\langle \Delta_\varepsilon f, g\rangle_{M,\varepsilon} = -\langle \nabla_\varepsilon f, \nabla_\varepsilon g\rangle_{M,\varepsilon},$$

where by $\langle \cdot,\cdot\rangle_{M,\varepsilon}$ we denoted the duality associated to $L^2(\Lambda_{M,\varepsilon})$.

Global existence of solutions to (4) can be proved along the lines of Khasminskii nonexplosion test [2, Theorem 3.5] whereas invariance of the measure (5) follows from Itô's formula together with the integration by parts.

Our goal is to show that there exists a choice of the renormalization constants $(c_{M,\varepsilon})_{M,\varepsilon}$ such that the family of probability measures $(\nu_{M,\varepsilon})_{M,\varepsilon}$ extended to $\mathscr{S}'(\mathbb{R}^d)$ is tight. Consequently, there exists a subsequence converging weakly in the sense of probability measures to some ν and this is the candidate for the Euclidean quantum field theory on $\mathbb{R}^d$. It can be additionally proved that every such accumulation point ν is translation invariant, reflection positive and non-Gaussian and satisfies a stretched exponential integrability. In order words, it satisfies part of the Osterwalder–Schader axioms for a Euclidean quantum field theory. Furthermore, an integration by parts formula can be established and it leads to the hierarchy of Dyson–Schwinger equations for Euclidean correlation functions.

3 On the regularity of the renormalized cube & C.

Recall our discrete setting:

$\varepsilon = 2^{-N}, M = 2^{N'}$ $\Lambda_{\varepsilon,M} = ((\varepsilon\mathbb{Z}) \cap [-M/2, M/2))^d$, $\Lambda^*_{\varepsilon,M} = ((\mathbb{Z}/M) \cap [-1/2\varepsilon, 1/2\varepsilon))^d$.

And the definition of the process X, which is the stationary solution to

$$\mathscr{L}_\varepsilon X_{\varepsilon,M} = \xi_{\varepsilon,M}.$$

It has the random Fourier series representation (with correct factors of M):

$$X(t,x) = \frac{1}{M^{d/2}} \sum_{k\in\Lambda^*_{\varepsilon,M}} \exp(2\pi i k\cdot x) \underbrace{\int_{-\infty}^t e^{-(m^2+\hat{k}^2)(t-s)} \mathrm{d}\beta_s^{(k)}}_{=:\hat{X}(t,k)}$$

where (computation with the discrete Laplacian)

$$\hat{k}^2 = \sum_{i=1}^d [2\varepsilon^{-1}\sin(\pi\varepsilon k_i)]^2 \approx (2\pi)^2|k|^2$$

for $|k| \ll \varepsilon^{-1}$.

We want to discuss the Besov regularity of $[\![X^3]\!](t,x)$.

Let's start by reproving that $X\in C([0,T],\mathscr{C}^{-\kappa}(\Lambda_{\varepsilon,M}))$ for some small κ with $\mathscr{C}^{-\kappa} = B^{-\kappa}_{\infty,\infty}$. This is useful to have a blueprint for the general argument.

We will use the Besov embedding

$$B^{-\kappa}_{\infty,\infty} \subset B^{d/2p-\kappa}_{2p,2p}$$

and the Kolmogorov lemma to estimate the Hölder norm in time (another Besov embedding in disguise).

Lemma 4. *(Kolmogorov) Let $(X_t)_{t\in[0,T]}$ a continuous stochastic process with values in the Polish space $\mathscr{E}$ then*

$$\mathbb{E}\left[\left(\sup_{t>s\in[0,T]} \frac{\|X_t - X_s\|_{\mathscr{E}}}{|t-s|^{\alpha-\kappa}}\right)^p\right] \lesssim_\kappa \sup_{t>s\in[0,T]} \mathbb{E}\left[\left(\frac{\|X_t-X_s\|_{\mathscr{E}}}{|t-s|^\alpha}\right)^p\right] \tag{6}$$

for any small $\kappa>0$ and $p\geqslant 1$ and $\alpha\in(0,1)$.

Proof. Let us sketch how does it work. Decompose $[0,T]$ into dyadic intervals $[t_k^n, t_{k+1}^n]$ with $t_k^n = T2^{-n}k$ for $k=0,\ldots,2^n$ and consider the quantity

$$Q(X) := \sum_{n\geqslant 0} \frac{1}{2^{2n}} \sum_{k=0}^{2^n-1} \left[\frac{\|X_{t_{k+1}^n} - X_{t_k^n}\|_{\mathscr{E}}}{|t_{k+1}^n - t_k^n|^\alpha}\right]^p$$

for some $\alpha\in(0,1)$, $p\geqslant 2$. Reasoning with telescopic sums over dyadic generations one arrives to deduce that

$$\frac{\|X_t - X_s\|_{\mathscr{E}}}{|t-s|^\alpha} \leqslant \sum_{n:|t-s|\leqslant 2^{-n}} \frac{\|X_{t^n_{k_n+1}} - X_{t^n_{k_n}}\|_{\mathscr{E}}}{|t^n_{k_n} - t^n_{k_n}|^\alpha} + \frac{\|X_{t^n_{k'_n+1}} - X_{t^n_{k'_n}}\|_{\mathscr{E}}}{|t^n_{k'_n} - t^n_{k'_n}|^\alpha}$$

where t,s are dyadic numbers in $\mathbb{D} = \{t_k^n\}$ and $(k_n)_n$ and $(k'_n)_n$ are integers depending on t,s. Since each term in the r.h.s. is bounded by

$$\left\{\sum_{k=0}^{2^n-1} \left[\frac{\|X_{t_{k+1}^n} - X_{t_k^n}\|_{\mathscr{E}}}{|t_{k+1}^n - t_k^n|^\alpha}\right]^p\right\}^{1/p} \leqslant [2^{2n} Q(X)]^{1/p}$$

and they are at most $n \leqslant \log|t-s|$, we have

$$\left[\frac{\|X_t - X_s\|_{\mathscr{C}}}{|t-s|^{\alpha}}\right]^p \leqslant |t-s|^{\kappa p} Q(X)^{1/p}$$

for some $0<\kappa<\alpha$. So we deduce that

$$\left[\sup_{t>s\in\mathbb{D}} \frac{\|X_t - X_s\|_{\mathscr{C}}}{|t-s|^{\alpha-\kappa}}\right]^p \lesssim Q(X)$$

and by considering a continuous version of X we can extend this to all $t>s\in[0,T]$. Then taking expectations and using Fubini:

$$\mathbb{E}\left[\sup_{t>s\in[0,T]} \frac{\|X_t - X_s\|_{\mathscr{C}}}{|t-s|^{\alpha-\kappa}}\right]^p \lesssim \sum_{n\geqslant 0} \frac{1}{2^{2n}} \sum_{k=0}^{2^n-1} \mathbb{E}\left[\frac{\|X_{t^n_{k+1}} - X_{t^n_k}\|_{\mathscr{C}}}{|t^n_{k+1}-t^n_k|^{\alpha}}\right]^p$$

from which we obtain finally the basic estimate (6) for any small $\kappa>0$ and $p\geqslant 1$ and $\alpha\in(0,1)$. □

We need to estimate $\mathbb{E}\|X(t)-X(s)\|^{2p}_{\mathscr{C}^{-\kappa}}$ in terms of $|t-s|$.

$$\mathbb{E}\|X(t)-X(s)\|^{2p}_{\mathscr{C}^{-\kappa}} \lesssim \mathbb{E}\|X(t)-X(s)\|^{2p}_{B^{d/2p-\kappa}_{2p,2p}} \qquad \text{(Besov embedding)}$$

$$\lesssim \sum_{i\geqslant -1} 2^{(d-2p\kappa)i} \varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} \mathbb{E}|\Delta_i X(t,x) - \Delta_i X(s,x)|^{2p} \qquad \text{(Fubini)}$$

$$\lesssim_p \sum_{i\geqslant -1} 2^{(d-2p\kappa)i} \varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} \left[\mathbb{E}|\Delta_i X(t,x) - \Delta_i X(s,x)|^2\right]^p \qquad \text{(Hypercontractivity)}$$

$$\lesssim_p M^d \sum_{i\geqslant -1} 2^{(d-2p\kappa)i} \left[\mathbb{E}|\Delta_i X(t,0) - \Delta_i X(s,0)|^2\right]^p \qquad \text{(Stationarity)}$$

Therefore consider $\mathbb{I} := \mathbb{E}|\Delta_i X(t,0) - \Delta_i X(s,0)|^2$.

$$\Delta_i X(t,0) - \Delta_i X(s,0) = \frac{1}{M^{d/2}} \sum_{k\in\Lambda^*_{\varepsilon,M}} \exp(2\pi i k\cdot x) \int_s^t e^{-2(m^2+\hat{k}^2)(t-u)} \mathrm{d}\beta^{(k)}_s$$

$$+ \frac{1}{M^{d/2}} \sum_{k\in\Lambda^*_{\varepsilon,M}} \exp(2\pi i k\cdot x) \underbrace{\left(e^{-2(m^2+\hat{k}^2)(t-s)} - 1\right)}_{\lesssim |(m^2+\hat{k}^2)(t-s)|^{\alpha\in(0,1)}} \int_{-\infty}^s e^{-2(m^2+\hat{k}^2)(s-u)} \mathrm{d}\beta^{(k)}_s$$

Consider the first term: by Ito (for $t>s$)

$$\mathbb{I} = \frac{1}{M^d} \sum_{k\in\Lambda^*_{\varepsilon,M}} [\varphi(2^{-i}k)]^2 e^{-(m^2+\hat{k}^2)(t-s)} \int_s^t e^{-2(m^2+\hat{k}^2)(t-u)} \mathrm{d}u$$

$$\lesssim \frac{1}{M^d} \sum_{k\in\Lambda^*_{\varepsilon,M}} \frac{[\varphi(2^{-i}k)]^2}{(m^2+\hat{k}^2)} |(m^2+\hat{k}^2)(t-s)|^{\alpha\in(0,1)}$$

$$\lesssim \frac{1}{M^d} \sum_{k\in\Lambda^*_{\varepsilon,M}} \frac{[\varphi(2^{-i}k)]^2}{(m^2+k^2)} k^{2\alpha} (t-s)^{\alpha} \lesssim \int_{(-\varepsilon^{-1},\varepsilon^{-1})^d} \mathrm{d}k \frac{[\varphi(2^{-i}k)]^2}{(m^2+k^2)} \lesssim 2^{i(d-2+2\alpha)}(t-s)^{\alpha}$$

and similarly for the other term.

So

$$\mathbb{E}\|X(t)-X(s)\|_{\mathscr{C}^{-\kappa}}^{2p} \lesssim M^d \sum_{i\geqslant -1} 2^{i(d/2-1+d/p-\kappa+\alpha)2p}(t-s)^{\alpha p} \lesssim M^d (t-s)^{\alpha p}$$

when

$$\kappa > \frac{d-2}{2}+\frac{d}{p}+\alpha.$$

From this one deduces that

$$X \in C([0,T],\mathscr{C}^{-\kappa})$$

almost surely.

For $[\![X^3]\!]$ the computation is more laborious. Consider the fixed time moments.

$$\Delta_i[\![X^3]\!](t,x) = \frac{1}{M^{3d/2}} \sum_{k_1,k_2,k_3} \varphi(2^{-i}(k_1+k_2+k_3)) e^{2\pi i(k_1+k_2+k_3)\cdot x} [\![\hat{X}(t,k_1)\hat{X}(t,k_2)\hat{X}(t,k_3)]\!]$$

a computation with Wick's theorem gives (*the sunset diagram*)

$$\mathbb{E}|\Delta_i[\![X^3]\!](t,x)|^2 = \frac{1}{M^{3d}} \sum_{k_1,k_2,k_3} [\varphi(2^{-i}(k_1+k_2+k_3))]^2 \mathbb{E}|\hat{X}(t,k_1)|^2 \mathbb{E}|\hat{X}(t,k_2)|^2 \mathbb{E}|\hat{X}(t,k_3)|^2$$

$$\approx \int \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \frac{[\varphi(2^{-i}(k_1+k_2+k_3))]^2}{(m^2+k_1^2)(m^2+k_2^2)(m^2+k_3^2)}$$

$$\approx \sum_{a,b,c} \int \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \frac{[\varphi(2^{-i}(k_1+k_2+k_3))]^2}{(m^2+k_1^2)(m^2+k_2^2)(m^2+k_3^2)} \mathbb{I}_{k_1\approx 2^a, k_2\approx 2^b, k_3\approx 2^c}$$

$$\approx \sum_{a,b,c} 2^{-2a-2b-2c} \int \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 [\varphi(2^{-i}(k_1+k_2+k_3))]^2 \mathbb{I}_{k_1\approx 2^a, k_2\approx 2^b, k_3\approx 2^c}$$

Assume an order among $|k_1|, |k_2|, |k_3|$, since the integral is symmetric, it is enough to consider $a \gtrsim b \gtrsim c$. Then since $k_1+k_2+k_3 \approx 2^i$ we only have two possibilities. Either

a) $k_1 \approx k_2 \approx 2^a \gtrsim k_3 = 2^c \gtrsim 2^i$. In this case note that $k_2+k_1 \approx 2^c$, so the sum over k_2 gives a contribution of the order 2^{cd}, the sum over k_1 a contribution of order 2^{ad} and the sum over k_3 a contribution of order 2^{cd}. Together with the behaviour of the propagator this gives (for $d<4$)

$$\sum_{a\gtrsim c\gtrsim i} 2^{-4a-2c} 2^{2cd+ad} \lesssim \sum_{a\gtrsim c\gtrsim i} 2^{3cd-6c}$$

and which can be bounded, when $d<3$ as

$$\lesssim 2^{(3d-6)i} \approx 2^{0i}.$$

b) $k_1 \approx 2^a \approx 2^i \gtrsim k_2 \gtrsim k_3$. In this case note that, denoting $q = k_1 + k_2 + k_3 \approx 2^i$ we have $2^b \approx k_2 + k_3 \approx q - k_1$ so the sum over k_1 gives a contribution of the order 2^b, the sum over k_2 order 2^c and also the sum over k_3 is of order 2^c . Together with the behaviour of the propagator this gives (for $d \geqslant 2$)

$$\sum_{i \approx a \gtrsim b \gtrsim c} 2^{-2a-2b-2c} 2^{bd+2cd} \lesssim \sum_{i \approx a \gtrsim b} 2^{-2a-4b+3bd} \gtrsim \sum_{i \approx a} 2^{-6a+3ad} \approx 2^{(3d-6)i} \approx 2^{0i}.$$

Both cases are fine for $d = 2$ but diverge logarithmically in ε^{-1} for $d = 3$.

For $d = 2$, time regularity can be argued as above so we end up with

$$\llbracket X^3 \rrbracket \in C([0,T], \mathscr{C}^{-\kappa}),$$

for $\kappa > 0$ in $d = 2$.

(end of Monday's lecture)

In $d = 3$ we will need to consider instead a distributional norm in the time variable. Alternatively we can apply $\mathscr{L}^{-1}$ to $\llbracket X^3 \rrbracket$ and define the stationary tree

$$X^{\Psi} := \mathscr{L}^{-1} \llbracket X^3 \rrbracket = \int_{-\infty}^{t} e^{-(m^2-\Delta)(t-s)} \llbracket X^3 \rrbracket(s) \mathrm{d}s.$$

Now we have

$$\mathbb{E}|\Delta_i X^{\Psi}(t,x)|^2 = \frac{1}{M^{3d/2}} \sum_{k_1,k_2,k_3} [\varphi(2^{-i}(k_1+k_2+k_3))]^2 \times$$

$$\times \int_{-\infty}^{t} \int_{-\infty}^{s} \frac{e^{-\left(m^2 + \widehat{(k_1+k_2+k_3)}^2\right)(2t-s-s') - (m^2+\hat{k}_1^2)(s-s') - (m^2+\hat{k}_2^2)(s-s') - (m^2+\hat{k}_3^2)(s-s')}}{(m^2+\hat{k}_1^2)(m^2+\hat{k}_2^2)(m^2+\hat{k}_3^2)} \mathrm{d}s \mathrm{d}s'$$

$$= \frac{1}{M^{3d/2}} \sum_{k_1,k_2,k_3} [\varphi(2^{-i}(k_1+k_2+k_3))]^2 \times$$

$$\times \frac{1}{\left(m^2 + \widehat{(k_1+k_2+k_3)}^2\right)(m^2+\hat{k}_1^2)(m^2+\hat{k}_2^2)(m^2+\hat{k}_3^2)\left(4m^2 + \hat{k}_1^2 + \hat{k}_2^2 + \hat{k}_3^2 + \widehat{(k_1+k_2+k_3)}^2\right)}$$

So now the same argument as before gives (for $d < 4$)

$$\mathbb{E}|\Delta_i \llbracket X^3 \rrbracket(t,x)|^2 \lesssim 2^{(d-2)i} \sum_{c \gtrsim i} 2^{c(d-2)} \sum_{a \gtrsim c} 2^{ad-6a}$$

$$\lesssim 2^{(d-2)i} \sum_{c \gtrsim i} 2^{c(2d-8)} \lesssim \sum_{c \gtrsim i} 2^{c(3d-10)}$$

(othe other case can be treated similarly) and when $d = 3$ this implies that

$$X^{\Psi} \in C([0,T], \mathscr{C}^{1/2-\kappa})$$

for some small $\kappa > 0$.

Let us remark that, as $X^{\Psi} = \llbracket X^3 \rrbracket$, even X^{Ψ} is not well defined in $d = 4$.

These computation are not uniform as $M\to\infty$ due to the constant in the initial bounds. For this reason we have to use weighted spaces. For example:

$$\mathbb{E}\|X(t)\|^{2p}_{\mathscr{C}^{-\kappa}(\rho)} \lesssim \mathbb{E}\|X(t)\|^{2p}_{B^{d/2p-\kappa}_{2p,2p}(\rho)} \qquad \text{(Besov embedding)}$$

$$\lesssim \sum_{i\geqslant -1} 2^{(d-2p\kappa)i}\varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} \mathbb{E}|\rho(x)\Delta_i X(t,x)|^{2p} \qquad \text{(Fubini)}$$

$$\lesssim_p \sum_{i\geqslant -1} 2^{(d-2p\kappa)i}\varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} \rho(x)^{2p}\left[\mathbb{E}|\Delta_i X(t,x)|^2\right]^p \qquad \text{(Hypercontractivity)}$$

$$\lesssim_p \sum_{i\geqslant -1} 2^{(d-2p\kappa)i}\left[\mathbb{E}|\Delta_i X(t,0)|^2\right]^p \left(\sum_{x\in\Lambda_{\varepsilon,M}} \rho(x)^{2p}\right) \qquad \text{(Stationarity)}$$

and now since

$$\sum_{x\in\Lambda_{\varepsilon,M}} \rho(x)^{2p} < 1$$

for any algebraic decay exponent for ρ, provided p is large enough. This gives now the uniform bound.

These arguments can be used to prove that in $d=2$

$$[\![X^n_{\varepsilon,M}]\!] \in C([0,T];\mathscr{C}^{-\kappa}_{\varepsilon}(\rho))$$

almost surely with bounds uniform in ε, M in L^p spaces, i.e.

$$\sup_{\varepsilon,M}\mathbb{E}\left[\|[\![X^n_{\varepsilon,M}]\!]\|^p_{C([0,T];\mathscr{C}^{-\kappa}_{\varepsilon}(\rho))}\right] < +\infty.$$

This is enough for stochastic quantisation. With some more work one can show that the stochastic objects converge as $\varepsilon\to 0$, $M\to\infty$ (with suitable embeddings in $\mathscr{S}'(\mathbb{R}^d)$) in

$$C([0,T];\mathscr{C}^{-\kappa-\delta}(\rho^{1+\delta}))$$

for any $\delta>0$.

4 The construction of the Euclidean Φ^4_2 theory

In order the explain the main ideas of the contruction in a simpler setting, we restrict ourselves now to $d=2$. Our aim is find a decomposition of (4) where all the quantities can be controlled uniformly in M,ε. In the first step, we isolate the term coming from the noise as it is expected to be the most irregular in the limit $M\to\infty$, $\varepsilon\to 0$. Recall that in $d=2$, $(\xi_{M,\varepsilon})_{M,\varepsilon}$ only has uniform bounds in a weighted Besov space of regularity $-2-\kappa$ for every $\kappa>0$. This regularity becomes worse in 3 dimensions and we will see later that creates further difficulties and additional ideas are needed to complete the proof. To remove this irregularity, we solve the corresponding linear counterpart of (4) and let $X_{M,\varepsilon}$ be its stationary solution, that is,

$$\mathscr{L}_{\varepsilon}X_{M,\varepsilon} = \xi_{M,\varepsilon}. \qquad (7)$$

By Schauder estimates, $(X_{M,\varepsilon})_{M,\varepsilon}$ is bounded uniformly in the weighted function space $C\mathscr{C}^{-\kappa}(\rho^{\sigma})$ a.s. for $\kappa, \sigma > 0$ arbitrary and ρ as in (2).

Decomposing $\varphi_{M,\varepsilon} = X_{M,\varepsilon} + \eta_{M,\varepsilon}$ we expect $\eta_{M,\varepsilon}$ to be more regular. It satisfies

$$\mathscr{L}_{\varepsilon}\eta_{M,\varepsilon} + (X_{M,\varepsilon}^3 - 3c_{M,\varepsilon}X_{M,\varepsilon}) + 3(X_{M,\varepsilon}^2 - c_{M,\varepsilon})\eta_{M,\varepsilon} + 3X_{M,\varepsilon}\eta_{M,\varepsilon}^2 + \eta_{M,\varepsilon}^3 = 0. \tag{8}$$

The advantage is that choosing $c_{M,\varepsilon} := \mathbb{E}[X_{M,\varepsilon}^2(t)]$ the terms

$$[\![X_{M,\varepsilon}^2]\!] := X_{M,\varepsilon}^2 - c_{M,\varepsilon}, \qquad [\![X_{M,\varepsilon}^3]\!] := X_{M,\varepsilon}^3 - 3c_{M,\varepsilon}X_{M,\varepsilon}, \tag{9}$$

are bounded uniformly in M, ε in $C\mathscr{C}^{-\kappa}(\rho^{\sigma})$ a.s. In particular, the terms defined in (9) are the second and third Wick power of the Gaussian random variable $X_{M,\varepsilon}$. Then by Schauder estimates we expect $(\eta_{M,\varepsilon})_{M,\varepsilon}$ to be bounded uniformly in $C\mathscr{C}^{2-\kappa}(\rho^{\sigma})$, hence it is function valued and all the products in (8) are well-defined.

4.1 Weighted energy estimate

As the next step, we want to derive a weighted energy estimate for $\eta_{M,\varepsilon}$ and make it uniform in M, ε. To this end, we test (8) by $\rho^4\eta_{M,\varepsilon}$, or alternatively we apply the chain rule to calculate $\frac{1}{2}\partial_t\|\rho^2\eta_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2$. We obtain

$$\frac{1}{2}\partial_t\|\rho^2\eta_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2 + \|\rho\eta_{M,\varepsilon}\|_{L^{4,\varepsilon}}^4 + \|\rho^2\eta_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2 + \|\rho^2\nabla_{\varepsilon}\eta_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2$$

$$= -\langle \rho^4\eta_{M,\varepsilon}, [\![X_{M,\varepsilon}^3]\!] + 3[\![X_{M,\varepsilon}^2]\!]\eta_{M,\varepsilon} + 3X_{M,\varepsilon}\eta_{M,\varepsilon}^2\rangle_{\varepsilon}.$$

The $L^{4,\varepsilon}$-norm on the left hand side is obtained from the cubic term, the $L^{2,\varepsilon}$-norm from the massive term $-\eta_{M,\varepsilon}$, and finally the norm of the gradient comes after integration by parts from the term $-\Delta_{\varepsilon}\eta_{M,\varepsilon}$. Observe that by Lemma 1 and Lemma ?, the two last terms on the left hand side can be estimated from below by the $H^{1-\kappa}(\rho^2)$-norm of $\eta_{M,\varepsilon}$ for $\kappa > 0$ small so that the above rewrites as

$$\frac{1}{2}\partial_t\|\rho^2\eta_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2 + \|\rho\eta_{M,\varepsilon}\|_{L^{4,\varepsilon}}^4 + \|\rho^2\eta_{M,\varepsilon}\|_{H^{1-\kappa,\varepsilon}}^2 \tag{10}$$

$$\lesssim -\langle \rho^4\eta_{M,\varepsilon}, [\![X_{M,\varepsilon}^3]\!] + 3[\![X_{M,\varepsilon}^2]\!]\eta_{M,\varepsilon} + 3X_{M,\varepsilon}\eta_{M,\varepsilon}^2\rangle_{\varepsilon}. \tag{11}$$

All the terms on the left hand side will be crucially used to control the right hand side. More precisely, it is necessary to absorb into the left hand side the norms of $\eta_{M,\varepsilon}$ needed in the estimates of the right hand side. The final estimate can only depend in a polynomial way on the given data, namely, the uniform in M, ε estimates of

$$\mathbb{X}_{M,\varepsilon} := (X_{M,\varepsilon}, [\![X_{M,\varepsilon}^2]\!], [\![X_{M,\varepsilon}^3]\!]) \in [C_T\mathscr{C}^{-\kappa,\varepsilon}(\rho^{\sigma})]^3$$

for $\kappa, \sigma > 0$ small and an arbitrary $T \in (0, \infty)$. This is achieved by combining the basic results for discrete Besov space from Section 1 with weighted Young inequality in a suitable way. The key role is especially played by the L^4-norm because it permits to balance the loss of weight on the right hand side. This is due to the fact that the noise terms $\mathbb{X}_{M,\varepsilon}$ always require a part of the weight to be bounded uniformly in M, ε.

For the sequel, we fix the parameters $\kappa, \sigma > 0$ small as well as a weight ρ as in (2) and a small parameter $\iota \in (0,1)$ so that $\rho^{\iota} \in L^{4,0}$. This technical assumption will help us with several embeddings to close the desired energy estimate.

First, we apply Lemma 1, the duality (Lemma 2), the embedding $H^{1-\kappa,\varepsilon}(\rho^2) = B^{1-\kappa}_{2,2} \subset B^{\kappa}_{1,1}(\rho^{4-\sigma})$ which holds provided $\rho^{2-\sigma} \in L^{2,\varepsilon}$ and the weighted Young inequality, to bound for an arbitrary $\delta \in (0,1)$

$$|\langle \rho^4 \eta_{M,\varepsilon}, [\![X^3_{M,\varepsilon}]\!]\rangle_{\varepsilon}| \lesssim \|\rho^{\sigma}[\![X^3_{M,\varepsilon}]\!]\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho^{4-\sigma}\eta_{M,\varepsilon}\|_{B^{\kappa,\varepsilon}_{1,1}}$$

$$\lesssim \|\rho^{\sigma}[\![X^3_{M,\varepsilon}]\!]\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho^2 \eta_{M,\varepsilon}\|_{H^{1-\kappa,\varepsilon}} \leqslant Q(\mathbb{X}_{M,\varepsilon}) + \delta \|\rho^2 \eta_{M,\varepsilon}\|^2_{H^{1-\kappa,\varepsilon}}. \tag{12}$$

Here and in the sequel, $Q(\mathbb{X}_{M,\varepsilon})$ always denotes a polynomial in the uniform norms of $\mathbb{X}_{M,\varepsilon}$ and it may change from line to line. Due to the small constant δ, the last term on the right hand side of (12) can be absorbed into (10).

Starting similarly for the second term in (11) and using the estimate for powers from Lemma ? as well as the interpolation from Lemma 3 with $\theta = \frac{1-3\kappa}{1-\kappa}$, the embedding from Lemma ?, we obtain

$$|\langle \rho^4 \eta_{M,\varepsilon}, 3[\![X^2_{M,\varepsilon}]\!]\eta_{M,\varepsilon}\rangle_{\varepsilon}| \lesssim \|\rho^{\sigma}[\![X^2_{M,\varepsilon}]\!]\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho^{4-\sigma}\eta^2_{M,\varepsilon}\|_{B^{\kappa,\varepsilon}_{1,1}}$$

$$\lesssim \|\rho^{\sigma}[\![X^2_{M,\varepsilon}]\!]\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho^{1+\iota}\eta_{M,\varepsilon}\|_{L^{2,\varepsilon}} \|\rho^{3-\iota-\sigma}\eta_{M,\varepsilon}\|_{H^{2\kappa,\varepsilon}}$$

$$\lesssim \|\rho^{\sigma}[\![X^3_{M,\varepsilon}]\!]\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho^{1+\iota}\eta_{M,\varepsilon}\|^{1+\theta}_{L^{2,\varepsilon}} \|\rho^2 \eta_{M,\varepsilon}\|^{1-\theta}_{H^{1-\kappa,\varepsilon}}$$

$$\lesssim \|\rho^{\sigma}[\![X^3_{M,\varepsilon}]\!]\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho \eta_{M,\varepsilon}\|^{1+\theta}_{L^{4,\varepsilon}} \|\rho^2 \eta_{M,\varepsilon}\|^{1-\theta}_{H^{1-\kappa,\varepsilon}}$$

$$\leqslant Q(\mathbb{X}_{M,\varepsilon}) + \delta(\|\rho \eta_{M,\varepsilon}\|^4_{L^{4,\varepsilon}} + \|\rho^2 \eta_{M,\varepsilon}\|^2_{H^{1-\kappa,\varepsilon}}).$$

In the same spirit, the last term is bounded as

$$|\langle \rho^4 \eta_{M,\varepsilon}, 3X_{M,\varepsilon}\eta^2_{M,\varepsilon}\rangle_{\varepsilon}| \lesssim \|\rho^{\sigma} X_{M,\varepsilon}\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho^{4-\sigma}\eta^3_{M,\varepsilon}\|_{B^{\kappa}_{1,1}}$$

$$\lesssim \|\rho^{\sigma} X_{M,\varepsilon}\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho \eta_{M,\varepsilon}\|^2_{L^{4,\varepsilon}} \|\rho^{2-\sigma}\eta_{M,\varepsilon}\|_{H^{2\kappa,\varepsilon}}$$

$$\lesssim \|\rho^{\sigma} X_{M,\varepsilon}\|_{\mathscr{C}^{-\kappa,\varepsilon}} \|\rho \eta_{M,\varepsilon}\|^2_{L^{4,\varepsilon}} \|\rho^{1+\iota}\eta_{M,\varepsilon}\|^{\theta}_{L^{2,\varepsilon}} \|\rho^2 \eta_{M,\varepsilon}\|^{1-\theta}_{H^{1-\kappa,\varepsilon}}$$

$$\leqslant Q(\mathbb{X}_{M,\varepsilon}) + \delta(\|\rho \eta_{M,\varepsilon}\|^4_{L^{4,\varepsilon}} + \|\rho^2 \eta_{M,\varepsilon}\|^2_{H^{1-\kappa,\varepsilon}}).$$

Finally, this brings us to the estimate

$$\frac{1}{2}\partial_t\|\rho^2\eta_{M,\varepsilon}\|^2_{L^{2,\varepsilon}}+\|\rho\eta_{M,\varepsilon}\|^4_{L^{4,\varepsilon}}+\|\rho^2\eta_{M,\varepsilon}\|^2_{H^{1-\kappa,\varepsilon}}\leqslant Q(\mathbb{X}_{M,\varepsilon}) \tag{13}$$

which we exploit further in the next section in order to deduce tightness of the approximate invariant measures (5).

4.2 Extension operators

In order to construct the Euclidean quantum field theory as a limit of lattice approximations, we need a suitable extension operator that allows to extend distributions defined on the lattice Λ_ε to the full space $\mathbb{R}^d$. To this end, we fix a smooth, compactly supported and radially symmetric nonnegative function $w\in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} w\subset B_{1/2}$ where $B_{1/2}\subset\mathbb{R}^d$ is the ball centered at 0 with radius $1/2$ and $\int_{\mathbb{R}^d}w(x)\,\mathrm{d}x=1$. Let $w^\varepsilon(\cdot)\coloneqq\varepsilon^{-d}w(\varepsilon^{-1}\cdot)$ and define the extension operator $\mathscr{E}^\varepsilon$ by

$$\mathscr{E}^\varepsilon f\coloneqq w^\varepsilon *_\varepsilon f,\qquad f\in\mathscr{S}'(\Lambda_\varepsilon).$$

With a slight abuse of notation we used the same notation $*_\varepsilon$ as for the convolution on the lattice Λ_ε to denote the operation

$$(w^\varepsilon *_\varepsilon f)(x)\coloneqq\varepsilon^d\sum_{y\in\Lambda_\varepsilon}w^\varepsilon(x-y)\,f(y),\qquad x\in\mathbb{R}^d,$$

which defines a function on the full space $\mathbb{R}^d$. The following result is proved in Section A.4 in [1].

Lemma 5. *Let $\alpha\in\mathbb{R}$, $p,q\in[1,\infty]$ and let ρ be a weight. Then the operators*

$$\mathscr{E}^\varepsilon\colon B^{\alpha,\varepsilon}_{p,q}(\rho)\to B^\alpha_{p,q}(\rho)$$

are bounded uniformly in ε.

4.3 Tightness

Let $\varphi_{M,\varepsilon}$ and $X_{M,\varepsilon}$ be stationary solutions to (4) and (7), respectively, defined on a common probability space $(\Omega,\mathscr{F},\mathbb{P})$. These solutions can be obtained by solving (4) and (7) with random initial conditions sampled from the probability measures (5) and

$$\frac{1}{Z_{M,\varepsilon}}\exp\left\{-2\varepsilon^d\sum_{x\in\Lambda_{M,\varepsilon}}\left[\frac{-3c_{M,\varepsilon}+1}{2}|\varphi_x|^2+\frac{1}{2}|\nabla_\varepsilon\varphi_x|^2\right]\right\}\prod_{x\in\Lambda_{M,\varepsilon}}\mathrm{d}\varphi_x,$$

respectively, where $Z_{M,\varepsilon}$ denotes a normalizing constant. As above, we define $\eta_{M,\varepsilon}\coloneqq\varphi_{M,\varepsilon}-X_{M,\varepsilon}$ and observe that it is a stationary solution to (8) so that the weighted energy estimate (13) holds true. Taking expectation and using stationarity therefore implies

$$\mathbb{E}\|\rho\eta_{M,\varepsilon}\|^4_{L^{4,\varepsilon}}+\mathbb{E}\|\rho^2\eta_{M,\varepsilon}\|^2_{H^{1-\kappa,\varepsilon}}\leqslant\mathbb{E}Q(\mathbb{X}_{M,\varepsilon})\lesssim 1, \tag{14}$$

where the implicit constant is independent of M, ε. In particular, it is important to note that the form of the polynomial Q does not depend on M, ε: the only dependence of $Q(\mathbb{X}_{M,\varepsilon})$ on these parameters comes through the norms of $\mathbb{X}_{M,\varepsilon}$ and the qunatity is uniformly bounded in expectation.

All the involved objects $\varphi_{M,\varepsilon}, \mathbb{X}_{M,\varepsilon}, \eta_{M,\varepsilon}$ are extended periodically to the full lattice Λ_ε. In order to establish tightness of the measures (5), that is, the laws of $\varphi_{M,\varepsilon}$ at an arbitrary time, we make use of the extension operator $\mathscr{E}^\varepsilon$ defined in Appendix 4.2, which further permits to extend $\varphi_{M,\varepsilon}$ as a distribution on Λ_ε to a distribution on the full space $\mathbb{R}^2$.

Theorem 6. *The family of laws of* $(\mathscr{E}^\varepsilon \varphi_{M,\varepsilon}(0))_{M,\varepsilon}$ *is tight on* $\mathscr{C}^{-2\kappa}(\rho^{2+\kappa})$. *Every accumulation point* ν *is the candidate Euclidean* Φ^4_2 *quantum field theory.*

Proof. Lemma 5 below and (14) lead to

$$\mathbb{E}\|\mathscr{E}^\varepsilon X_{M,\varepsilon}(0)\|^2_{\mathscr{C}^{-\kappa}(\rho^\sigma)} \lesssim 1, \qquad \mathbb{E}\|\mathscr{E}^\varepsilon \eta_{M,\varepsilon}(0)\|^2_{H^{1-\kappa,\varepsilon}(\rho^2)} \lesssim 1$$

and consequently

$$\mathbb{E}\|\mathscr{E}^\varepsilon \varphi_{M,\varepsilon}(0)\|^2_{\mathscr{C}^{-\kappa}(\rho^2)} \lesssim \mathbb{E}\|\mathscr{E}^\varepsilon X_{M,\varepsilon}(0)\|^2_{\mathscr{C}^{-\kappa}(\rho^\sigma)} + \mathbb{E}\|\mathscr{E}^\varepsilon \eta_{M,\varepsilon}(0)\|^2_{H^{1-\kappa,\varepsilon}(\rho^2)} \lesssim 1.$$

Since the embedding $\mathscr{C}^{-\kappa}(\rho^2) \subset \mathscr{C}^{-2\kappa}(\rho^{2+\kappa})$ is compact, the result follows. □

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