Lecture 8: Small scale limit of ϕ^4_2

Recall: $X = Y + Z$. $\partial_t X = (\Delta_\varepsilon - m^2) X - \lambda X^3 + \xi_\varepsilon$

$\left(\xi_\varepsilon \stackrel{\text{law}}{=} \frac{dB_t}{dt} \cdot \varepsilon^{-1} \right)$

$$\begin{cases} \partial_t Y_t = (\Delta_\varepsilon - m^2) Y_t + \xi_\varepsilon \\ \partial_t Z_t = (\Delta_\varepsilon - m^2) Z_t \\ \qquad - \lambda (Y^3 + 3Y^2 Z + 3 Y Z^2 + Z^3) \end{cases}$$

$\boxed{\lambda > 0 \text{ important}}$

- Recall: $Y^\varepsilon \in C(\mathbb{R}; C^{-\frac{d-2}{2} - \kappa})$ a.s. uniformly in ε

$\lim_{\varepsilon \to 0} Y_\varepsilon^2 - C_\varepsilon$

- recall $\stackrel{\shortparallel}{Y}^2 \in C(\mathbb{R}; C^{2\alpha})$ $\quad \alpha = -\frac{d-2}{2} - \kappa < 0$, $d \geq 2$

$$\lim_{\varepsilon \to 0} Y_\varepsilon^3 - 3C_\varepsilon Y_\varepsilon = Y^3 \in \begin{cases} C(\mathbb{R}; C^{-\kappa}) & d = 2 \\ C^{-\kappa}(\mathbb{R}; C^{3\alpha}) & d = 3 \end{cases}$$

We focus on $d = 2$ in this lecture.

Step 1

Renormalized equ: $\partial_t X = (\Delta_\varepsilon - m^2) X - \lambda (X^3 - 3C_\varepsilon X) + \xi_\varepsilon$

Rmk: "$e^{-\int \frac{(\nabla\phi)^2}{2} + \frac{m^2 \phi^2}{2} + \frac{\lambda \phi^4}{4} - \frac{3\lambda C_\varepsilon}{2} \phi^2 \, dx} d\phi$"

$$\lambda (\underbrace{Y^3} + 3Y^2 Z + 3 Y Z^2 + Z^3) - \lambda \cdot \underbrace{3 C_\varepsilon (Y + Z)}$$

$$= \lambda (\underbrace{Y^3 - 3C_\varepsilon Y} + \underline{3(Y^2 - C_\varepsilon) Z} + 3 Y Z^2 + Z^3)$$

$$= \lambda (Y^3 + 3 Y^2 Z + 3 Y Z^2 + Z^3)$$

Da Prato – Debussche AoP 2003

"magic" is that one constant C_ε works for all terms.

Rmk: (1) $H_n(y+z) = \sum_{k=0}^{n} \binom{n}{k} H_k(y) \cdot z^{n-k}$

(2) Deeper reasons: subcriticality etc.

Hairer et al. "BPHZ ... " "Renormalizing SPDE ... "

Step 2 Energy estimate:

Energy identity: $\int_{\mathbb{T}_\varepsilon^d} = \varepsilon^d \sum_{x \in \mathbb{T}_\varepsilon^d}$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2\,dx + \int_{\mathbb{T}_\varepsilon^d} |\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\,dx$$

$$= -\frac{1}{2}\int_{\mathbb{T}_\varepsilon^d} \lambda\left(\Psi^3 Z + 3\Psi^2 Z^2 + 3\Psi Z^3\right) dx$$

goal: bound RHS by ① norms of Ψ, Ψ^2, Ψ^3

② small const $\times$ $\|Z\|_{L^2}^2$, $\|\nabla Z\|_{L^2}^2$, $\|Z\|_{L^4}^4$

and absorb them to LHS.

The following can be found in [GH] (TA session)

Lemma (Besov duality): $\alpha \in \mathbb{R}$, $p, p', q, q' \in [1,\infty]$.

$$\frac{1}{p} + \frac{1}{p'} = 1 \qquad \frac{1}{q} + \frac{1}{q'} = 1$$

Then $\langle f, g\rangle_\varepsilon \lesssim \|f\|_{B^{\alpha,\varepsilon}_{p,q}} \|g\|_{B^{-\alpha,\varepsilon}_{p',q'}}$

Lemma (Interpolate): $\|f\|_{B^\alpha_{p,q}} \le \|f\|^{\theta}_{B^{\alpha_0}_{p_0 q_0}} \|f\|^{1-\theta}_{B^{\alpha_1}_{p_1 q_1}}$

where $\alpha = \theta\alpha_0 + (1-\theta)\alpha_1$.
$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$$
$$\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{1-q_1}$$

Facts: $L^2 = B^0_{2,2}$, $C^\alpha = B^\alpha_{\infty\infty}$. $H^\alpha = B^\alpha_{2,2}$.

Besov duality

$\alpha = -\kappa$

$$\left|\int_{\mathbb{T}_\varepsilon^d} \Psi^3 Z \, dx\right| \overset{\text{Besov duality}}{\leq} \|\Psi^3\|_{B^{3\alpha}_{\infty\infty}} \underbrace{\|Z\|_{B^{4\kappa}_{1,1}}}$$

Want: $L^2 = B^0_{2,2}$
and $H^1 = B^1_{2,2}$.

Besov embedding

$$\|Z\|_{B^{4\kappa}_{1,1}} \leq \|Z\|_{B^{4\kappa}_{2,2}}$$

Interpolate

$$\leq \|Z\|^{\theta}_{B^0_{22}} \|Z\|^{1-\theta}_{B^1_{22}}$$

$4\kappa = \theta \cdot 0 + (1-\theta)\cdot 1$

$\theta = 1 - 4\kappa$

$$= \|Z\|^{1-4\kappa}_{L^2} \|Z\|^{4\kappa}_{H^1}$$

Then use Young to write $a \cdot b^{1-4\kappa} \cdot c^{4\kappa}$ into a sum.

$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \qquad p_2 = \frac{2}{1-4\kappa} \qquad p_3 = \frac{2}{4\kappa}$

So $\quad \frac{1}{p_1} + \frac{1-4\kappa}{2} + \frac{4\kappa}{2} = 1 \quad \Rightarrow \quad p_1 = 2$

Therefore,

$$\left|\int_{\mathbb{T}_\varepsilon^d} \Psi^3 Z \, dx\right| \leq C_\delta \|\Psi^3\|^2_{C^{3\alpha}} + \delta \|\nabla_\varepsilon Z\|^2_{L^2} + \delta \|Z\|^2_{L^2}$$

Similarly:

If time permits we do one more. or, TA session

$$\left|\int_{\mathbb{T}_\varepsilon^d} \Psi^2 Z^2 dx\right| \leq \|\Psi^2\|_{C^{2\alpha}} \|Z^2\|_{B^{3\kappa}_{1,1}}$$

$$\overset{\dots}{\leq} C_\delta \|\Psi^2\|_{C^{2\alpha}} + \delta \|\nabla Z\|^2_{L^2} + \delta \|Z\|^4_{L^4}$$

$$\left|\int_{\mathbb{T}_\varepsilon^d} \Psi^1 Z^3 dx\right| \leq \|\Psi^1\|_{C^\alpha} \|Z^3\|_{B^{2\kappa}_{1,1}}$$

$$\overset{\dots}{\leq} C_\delta \|\Psi^2\|_{C^\alpha} + \delta \|\nabla Z\|^2_{L^2} + \delta \|Z\|^4_{L^4}$$

Therefore,

$$\frac{1}{2}\partial_t \int_{\mathbb{T}_\varepsilon^d} Z_t^2\,dx + (1-\delta)\int_{\mathbb{T}_\varepsilon^d} |\nabla_\varepsilon Z_t|^2 + m^2|Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\,dx$$

$$\leq Q_t$$

$$Q_t = 1 + C\left(\|Y_t\|_{C^\alpha}^K + \|Y_t^2\|_{C^{2\alpha}}^K + \|Y_t^3\|_{C^{3\alpha}}^K\right)$$

for some large power K.

Step 3: Tightness.

Now, we can study what happens when $\varepsilon \to 0$:

recall stationary coupling $\mathbb{P}^\varepsilon$.

under $\mathbb{P}^\varepsilon$, Y, Z are stationary, $X = Y + Z$ is stationary

$X_t \sim \nu^\varepsilon$

Take $\mathbb{E}$ ∂_t term vanishes by stationarity.

$$\mathbb{E}\int_{\mathbb{T}_\varepsilon^d} |\nabla_\varepsilon Z_0|^2 + m^2|Z_0|^2 + \frac{\lambda}{2}|Z_0|^4\,dx \leq \mathbb{E}Q_t = \mathbb{E}Q_0 < \infty$$

uniformly in ε

$\boxed{\text{Claim}}$: above bound $\Longrightarrow$ tightness of $(\nu^\varepsilon)_{\varepsilon>0}$

Rmk: ν^ε lives on lattices with different ε. To compare them, we should extend fields ϕ on Λ_ε to continuum.

e.g. $(\mathcal{E}^\varepsilon \phi)(x) = \varepsilon^d \sum_{y\in\Lambda_\varepsilon} \delta^\varepsilon(x-y) f(y)$ $\quad \delta^\varepsilon \to \delta$

But we ignore this technical detail.

Indeed, write $\gamma^\varepsilon = \text{Law of } (Y_0, Z_0)$

Could s

$$\sup_\varepsilon \int \left(\|\psi\|^2_{C^{-\alpha}} + \|\nabla \zeta\|^2_{L^2} + \|\zeta\|^2_{L^2} + \|\zeta\|^4_{L^4}\right) \gamma^\varepsilon(d\psi \times d\zeta) \Bigg\} \overset{T_1}{\leftarrow}$$

def of γ

$$= \sup_\varepsilon \mathbb{E}_{P^\varepsilon}\left[\|Y_0\|^2_{C^{-\alpha}} + \|\nabla Z_0\|^2_{L^2} + \|Z_0\|^2_{L^2} + \|Z_0\|^4_{L^4}\right]$$

$$\leq \sup_\varepsilon \mathbb{E}\, Q_0 < \infty$$

above bound and trivial bound on Y_0 by Q_0

This gives tightness of $(\gamma^\varepsilon)_{\varepsilon>0}$ on $C^{-2\alpha} \times H^{1-\kappa}$

since $C^{-\alpha} \hookrightarrow C^{-2\alpha}$, $H^1 \hookrightarrow H^{1-\kappa}$ compact embedding.

But what we care about is ν^ε

which is projection of γ^ε (by summing two factors)

$$\int \|\varphi\|^2_{B^{-\alpha}_{22}} \nu^\varepsilon(d\varphi) = \int \|\psi + \zeta\|^2_{B^{-\alpha}_{2,2}} \gamma^\varepsilon(d\psi \times d\zeta)$$

$$\leq 2\int \left(\|\psi\|^2_{B^{-\alpha}_{22}} + \|\zeta\|^2_{B^{-\alpha}_{22}}\right) \gamma^\varepsilon(d\psi \times d\zeta)$$

$$\leq 2\int \left(\|\psi\|^2_{C^{-\alpha}} + \|\zeta\|^2_{H^1}\right) \gamma^\varepsilon(d\psi \times d\zeta) \lesssim 1 \quad \text{unif in } \varepsilon$$

Basically we're saying

$$\|X\|^2_{B^{-\alpha}_{22}} \leq \|Y\|^2_{B^{-\alpha}_{22}} + \|Z\|^2_{B^{-\alpha}_{22}}$$

$$C^{-\alpha} = B^{-\alpha}_{\infty\infty} \hookrightarrow B^{-\alpha}_{2,2}$$

$$H^1 = B^1_{22} \hookrightarrow B^{-\alpha}_{22}$$

This gives tightness of ν^ε on $B^{-2\alpha}_{22}(\mathbb{T}^2)$

Since $B^{-\alpha}_{22} \hookrightarrow B^{-2\alpha}_{22}$ is compact embedding.

So $\exists$ subseq $\longrightarrow$ limit ν on $B^{-2\alpha}_{22}(\mathbb{T}^2) = H^{-2\alpha}(\mathbb{T}^2)$

Remarks:

① Combined with the infinite volume limit argument <u>with weights</u>
One can construct $\underbrace{\Phi^4 \text{ on whole } \mathbb{R}^2}$.

② This limiting measure is $\underbrace{\underline{\text{not}} \text{ Gaussian}}$
It follows by an interesting argument, but we postpone it to 3D.

③ IBP formula/Dyson-Schwinger extends to continuum:
assume F is cylinder functional on $S'(\mathbb{R}^2)$

i.e. $F(\varphi) = \mathrm{poly}(\varphi(f_1), \cdots, \varphi(f_n))$ $\quad f_i \in S(\mathbb{R}^2)$

Compute like Wednesday but now with renormalization C_ε

$$\int DF(\varphi)(x)\, \nu(d\varphi) = \int F(\varphi)\,(m^2 - \Delta)\,\varphi(x)\,\nu(d\varphi)$$
$$+ \lambda \int F(\varphi) \underbrace{\left(\varphi(x)^3 - 3C\varphi(x)\right)}_{=:\,[\varphi^3]} \nu(d\varphi)$$

meaningful as distributions in x
(f_i smooth)

$$\varphi = Y_0 + Z_0$$

$$(Y_0^3 - 3cY_0) + 3(Y_0^2 - c)Z_0 + 3Y_0 Z_0^2 + Z_0^3$$

Difficulty in 3D:

$\mathbf{Y}^2 \in C^{-1-\kappa}$.

① For duality, $\langle \mathbf{Y}^2, Z^2 \rangle \leq \|\mathbf{Y}^2\|_{B^{-1-\kappa}_{\cdot\cdot}} \|Z^2\|_{B^{1+\kappa}_{\cdot\cdot}}$

But LHS only has $\|\nabla Z\|_{L^2} \approx \|Z\|_{H^1}$

There's no way to control $B^{1+\kappa}$ norm of Z.

TA session

If time permits:

② The problem is even "before energy estimate".

Classical result for product (Young theorem)

$$\|fg\|_{C^{\min(\alpha,\beta)}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} \quad \text{if } \alpha+\beta>0$$

So for $\Psi^{:2:} Z$ in the Z equation, Z must be $C^{1+\kappa}$.

There's no way to control $C^{1+\kappa}$ norm of Z

(could mention Schauder)

③ One more renormalization beyond Wick is necessary

$$Z \approx (\partial_t - \Delta)^{-1} (\Psi^3 + 3\Psi^2)$$

$$E[\Psi^2 Z] \approx E[\Psi^2 (\partial_t - \Delta)^{-1} (\Psi^3 + 3\Psi^2)]$$

both terms diverge

"$-1+2-1 \approx \log$"

Comments about Φ^4_3 (if time permits):

- local solution: Hairer 2013/3 $\xi_\varepsilon = \xi * \varphi_\varepsilon$ "smooth regularization"
  Using Reg. Stru.
  Catellier - Chouk 2013/10.
  using Paracontrol distribution (Gubinelli Imkeller Perkowski)
- Global solution: Mourrat - Weber 2016. on $\mathbb{T}^3$. $\to$ meas on $\mathbb{T}^3$.

Gubinelli - Hofmanova 2018.4
Moinat - Weber 2018.11
global bound over space-time

Gubinelli - Hofmanova 2018: PDE construction of Φ^4 on $\mathbb{R}^3$ (lattice approx) & axioms.