LECTURE 6

Let $T \in \mathcal{D}'(\mathbb{R}^d)$ (or $\mathcal{S}'(\mathbb{R}^d)$).

Then $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$ we can

define canonically $T \cdot \varphi \in \mathcal{D}'(\mathbb{R}^d)$:

$$T \cdot \varphi\,(\psi) := T(\varphi\psi), \quad \forall \psi \in C_c^\infty(\mathbb{R}^d).$$

Let now $T \in \mathcal{E}'(\mathbb{R}^d)$ and

$\varphi \in C_c(\mathbb{R}^d) \setminus C_c^\infty(\mathbb{R}^d)$. Then

what is $T \cdot \varphi\,(\psi) = ?$ Since

in general $\varphi\psi \notin C_c^\infty(\mathbb{R}^d)$, we can

NOT compute $T(\varphi\psi)$.

EVEN WORSE: if $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^d)$,

there is NO CANONICAL WAY of

defining $T_1 T_2(\varphi)$, for $\varphi \in C_c^\infty(\mathbb{R}^d)$.

In particular: $T^2, T^3, T^4 \ldots$

are in general ill-defined.

THIS IS WHAT WE MEAN when we

say that $\int_{\mathbb{R}^d} V(X(\delta_x))\,dx$ is

ill-defined for the GFF_m

$x \longrightarrow X(\delta_x)$ is NOT a function

if $d \geq 2$ and for generic

(even polynomial) $V : \mathbb{R} \to \mathbb{R}$,

$x \longrightarrow V(X(\delta_x))$ is ill-defined.

NOTE that "ill-defined" does NOT mean "impossible to define" but rather "impossible to define canonically".

We shall see that Y^2 and Y^3 can and must be defined for a specific distribution, but this will be a choice.

EXAMPLE

If B is a two-sided Brownian motion, $B'_t = \frac{dB_t}{dt} \in \mathcal{D}'(\mathbb{R})$

and $B \in C(\mathbb{R})$.

What is the product

$$B \cdot B' \quad ?$$

Recall Itô's formula:

$$B_t^2 = 2\int_0^t B_s \, dB_s + t, \quad t \geq 0.$$

In Stratonovich form:

$$B'_t = 2\int_0^t B_s \circ dB_s$$

(Recall: $\forall\, t \geq 0$

$$\int_0^t B_s \, dB_s = \lim_{n \to \infty} \sum_{\frac{i}{n} \leq t} B_{\frac{i}{n}} \left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}} \right)$$

$$\int_0^t B_s \circ dB_s =$$

$$= \lim_{n \to \infty} \sum_{\frac{i+1}{n} \le t} \frac{B_{\frac{i}{n}} + B_{\frac{i+1}{n}}}{2} \left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}} \right)$$

(limits in $L^2(\Omega)$).

Then we have two possible definitions:

$$B_t B'_t := \frac{d}{dt} \frac{B_t^2 - t}{2} \quad \text{(Itô)}$$

$$B_t \circ B'_t := \frac{d}{dt} \frac{B_t^2}{2} \quad \text{(Stratonovich)}$$

and there many more!

Neither of these are canonical.
One has to choose, and then work with this choice.

EXAMPLE: in Lecture 4 we saw that

$$\dot{Z} = (m - \Delta_\varepsilon) Z - \lambda (Z^3 + 3ZY^2 + 3Z^2Y + Y^3)$$

We're going to see that as $\varepsilon \downarrow 0$,

$$Y = Y^{(\varepsilon)} \longrightarrow \overline{Y}$$

where $\overline{Y}$ is a distribution.

So, what about Y^2 and Y^3?

Reversibility of gradient-type SDEs

Let $U : \mathbb{R}^d \longrightarrow \mathbb{R}$ of class C^2 s.t. U, DU are Lipschitz:

$$|U(x) - U(y)| \le L|x-y|$$

$$|DU(x) - DU(y)| \le L|x-y|$$

Let $B = (B^1, \dots, B^d)$ be a $\mathbb{R}^d$-valued Brownian Motion (namely d independent BMs).

NOTE: Has used opposite sign

We study the SDEs

$$X_t(x) = x - \int_0^t DU(X_s)\,ds + B_t, \quad t \ge 0.$$

Note: as in Lecture 4, if

$$Z_t := X_t - B_t, \quad \text{then}$$

$$Z_t = x - \int_0^t DU(Z_s + B_s)\,ds,$$

a Random ODE rather than a SDE.

We want to prove that

$$\int_{\mathbb{R}^d} f(x)\, \mathbb{E}\big(g(X_t(x))\big) e^{-2V(x)}\, dx =$$

$$= \int_{\mathbb{R}^d} g(x)\, \mathbb{E}\big(f(X_t(x))\big) e^{-2V(x)}\, dx$$

for all $f, g : \mathbb{R}^d \to \mathbb{R}$ measurable bounded.

We define

$$L_t := \int_0^t \langle \nabla V(X_s), dB_s \rangle$$

$$= \int_0^t \sum_{i=1}^d \frac{\partial V}{\partial x_i}(X_s)\, dB^i_s$$

and $M_t = \mathcal{E}(L)_t$, the associated exponential (local) martingale, i.e.

$$M_t = \exp\left(\int_0^t \langle \nabla U(X_s), dB_s \rangle - \frac{1}{2} \int_0^t \|\nabla U(X_s)\|^2 ds \right)$$

Since $\|\nabla U\|$ is bounded (U is Lipschitz), by Novikov's condition $(M_t)_{t \geq 0}$ is a martingale, $\mathbb{E}(M_t) = 1$ and $M_t \, d\mathbb{P} =: \mathbb{Q}_t$ is a new probability measure on Ω. By Girsanov's theorem

under $\mathbb{Q}_t$

$$X_s(x) - x = B_s - \langle L, B \rangle_s \quad , \quad s \in [0,t]$$

is a BM in $\mathbb{R}^d$.

By Itô's formula

$$U(X_t(x)) = U(x) + \int_0^t \langle \nabla U(X_s), dX_s \rangle + \frac{1}{2}\int_0^t \mathrm{Tr}\left[D^2 U(X_s)\right] ds$$

$$= U(x) + \int_0^t \left(-\|\nabla U(X_s)\|^2 + \frac{1}{2}\mathrm{Tr}\left[D^2 U(X_s)\right]\right) ds + \int_0^t \langle \nabla U(X_s), dB_s \rangle .$$

Then:

$$\int_0^t \langle \nabla U(X_s), dB_s \rangle = $$

$$= U(X_t) - U(x) + \int_0^t \left(\|\nabla U(X_s)\|^2 - \frac{1}{2}\mathrm{Tr}\left[D^2 U(X_s)\right]\right) ds$$

and

$$M_t = \exp\left(U(X_t) - U(x) + \int_0^t v(X_s)\, ds\right)$$

with $v(y) := \frac{1}{2}\left(\|\nabla U(y)\|^2 - \mathrm{Tr}\left[D^2 U(y)\right]\right)$

NOW:

$$\int f(x)\, \mathbb{E}\left[g(X_t(x))\right] e^{-2V(x)}\, dx =$$

$$= \int f(x)\, \mathbb{E}_{\mathbb{Q}_t}\left[g(X_t(x))\, e^{-2V(x)} \frac{1}{M_t}\right] dx$$

$$= \int \mathbb{E}_{\mathbb{Q}_t}\left[f(x)\, g(X_t(x))\, e^{-V(X_t)-V(x)-\int_0^t v(X_s)\,ds}\right] dx$$

Girsanov

$$\downarrow$$

$$= \int \mathbb{E}\left[f(x)\, g(x+B_t)\, e^{-V(x+B_t)-V(x)-\int_0^t v(x+B_s)\,ds}\right] dx$$

$[y = x + B_t]$

$$\downarrow$$

$$= \int \mathbb{E}\left[f(y-B_t)\, g(y)\, e^{-V(y)-V(y-B_t)-\underbrace{\int_0^t v(y-B_t+B_s)\,ds}}\right] dy$$

$$\underbrace{\int_0^t v(y-B_t+B_s)\,ds}_{\int_0^t v(y+B_{t-s}-B_t)\,ds}$$

$$(B_{t-s}-B_t)_{s\in[0,t]} \overset{(d)}{=} (B_s)_{s\in[0,t]}$$

$$\downarrow$$

$$= \int \mathbb{E}\left[f(y+B_t)\, g(y)\, e^{-V(y)-V(y+B_t)-\int_0^t v(y+B_s)\,ds}\right] dy$$

$$= \int g(x)\, \mathbb{E}\big(f(X_t(x))\big)\, e^{-2U(x)}\, dx$$

Exercise: prove in the same way that

$\forall\, \Phi : C([0,t];\mathbb{R}^d) \to \mathbb{R}$ measurable bounded:

$$\int \mathbb{E}\left[\Phi\left(X_{t-s}(x),\, s\in[0,t]\right)\right] e^{-2U(x)}\, dx$$

$$= \int \mathbb{E}\left[\Phi\left(X_s(x),\, s\in[0,t]\right)\right] e^{-2U(x)}\, dx$$

We have obtained that the transition semigroup $P_t f(x) = \mathbb{E}\big(f(X_t(x))\big)$ is symmetric in $L^2(\mathbb{R}^d, e^{-2U(x)}dx)$

Moreover: if $g \equiv 1$

$$\int (P_t f)\, e^{-2U} dx = \int f\, e^{-2U} dx$$

namely $e^{-2U}dx$ is an invariant measure for $(X_t(x))_{t\geq 0, x\in\mathbb{R}^d}$.

If $\int e^{-2U}dx < +\infty$ then

$$\nu := \frac{1}{\int e^{-2U}dx} \cdot e^{-2U(x)}dx$$ is

a probability measure and if

$X_0 \sim \nu$, $(X_t)_{t\geq 0}$ is stationary.

$X_t \sim \nu \quad \forall t \geq 0.$

NOTE: if ∇U is not bounded, then for the Novikov criterion we need additional information.

FOR EXAMPLE: if $d=1$ and

$U(x) = \lambda \frac{x^4}{4}$, $\quad U' = \lambda x^3$,

then everything works if $\lambda \geq 0$, but not if $\lambda < 0$.

In the latter case, the solution to the SDE even explodes in finite time.