Coming down from infinity

Ref: See Moinat-Weber "Space-time localization..." for an alternative proof

For Φ^4_2 in unit volume, recall the L^p estimate from Session 3:

$$\frac{1}{2p}\partial_t \|\varphi\|_{L^{2p}}^{2p} + m\|\varphi\|_{L^{2p}}^{2p} + \|\varphi^{p-1}\nabla\varphi\|_{L^2}^2 + \frac{1}{4}\|\varphi\|_{L^{2(p+1)}}^{2(p+1)}$$

$$\leq C_p\left(\|Z\|_{B^{-\kappa}_{\infty,\infty}}^{4(p+1)} + \|:Z^2:\|_{B^{-\kappa}_{\infty,\infty}}^{\frac{4(p+1)}{3}} + \|:Z^3:\|_{B^{-\kappa}_{\infty,\infty}}^{\frac{4(p+1)}{5}}\right).$$

For simplicity, we take $\lambda = 1$.
Using this, we will show that

$$\|Y(t)\|_{L^p}$$

satisfies a bound which is completely independent of $\|Y(0)\|_{L^p}$. Estimates of this type are known as "coming down from infinity".

We will see that the essence of the argument reduces to understanding the behavior of the ODE:

$$\dot{y}(t) \leq -\alpha y(t)^3 .$$

Here $\alpha > 0$ is some parameter.
We have that

$$-\frac{1}{2}\frac{d}{dt}y(t)^{-2} = \frac{\dot{y}(t)}{y(t)^3} \leq -\alpha,$$

which implies

$$\frac{d}{dt} y(t)^{-2} \geq 2\alpha,$$

and thus

$$y(t)^{-2} \geq y(0)^{-2} + 2\alpha t,$$

$$y(t) \leq \left(\frac{1}{y(0)^{-2} + 2\alpha t} \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{2\alpha t} \right)^{\frac{1}{2}}.$$

Note that the bound is independent of $y(0)$.

More generally, we have the following lemma.

Lemma. Let $y : [0,1] \to [0,\infty)$ be

C^{-} and st for some constants $\alpha, \beta > 0$,

$$\dot{y}(t) \leq -\alpha y(t)^3 + \beta.$$

Then there is some constant C st $\forall\ t \in [0,1]$, we have

$$y(t) \leq C \max\left((\alpha t)^{-\frac{1}{2}}, \left(\frac{\beta}{\alpha}\right)^{\frac{1}{3}}\right).$$

Pf. First, suppose that

$$\frac{1}{2} \alpha\, y(0)^3 \leq \beta,$$

i.e. $$y(0) \leq \left(\frac{2\beta}{\alpha}\right)^{\frac{1}{3}}.$$

Then we claim that

$$\sup_{t \in [0,1]} y(t) \leq \left(\frac{2\beta}{\alpha}\right)^{\frac{1}{3}}.$$

Indeed, if the sup is attained at some $t_0 \in [0,1]$, then the left derivative of y at t_0 is ≥ 0, and thus

$$0 \leq \dot{y}(t_0) \leq -\alpha y(t_0)^3 + \beta,$$

i.e.

$$y(t_0) \leq \left(\frac{\beta}{\alpha}\right)^{\frac{1}{3}}.$$

Now, suppose that $y(0) > \left(\frac{2\beta}{\alpha}\right)^{\frac{1}{3}}$. Let

$$\tau = \inf\left\{ t \in [0,1] : y(t) = \left(\frac{2\beta}{\alpha}\right)^{\frac{1}{3}} \right\}$$

If no such t exists, set $\tau = 1$. Then for all $t \in [0,\tau]$, we have that

$$\dot{y}(t) \leq -\alpha y(t)^3 + \beta$$

$$\leq -\frac{1}{2}\alpha y(t)^3 .$$

Using our previous bound ,
we obtain that for all $t \in [0,\tau]$,

$$y(t) \leq (\alpha t)^{-\frac{1}{2}} .$$

For $t \in [\tau, 1]$, by the previous argument we have that

$$y(t) \leq \left(\frac{2B}{\alpha}\right)^{\frac{1}{3}} .$$

The desired result now follows. $\square$

We now apply this to Φ^4_2. Let

$$R_p = \|Z\|^{4(p+1)}_{L^\infty_4 B^{-\kappa}_{\infty,\infty}} + \|{:}Z^2{:}\|^{\frac{4(p+1)}{3}}_{L^\infty_4 B^{-\kappa}_{\infty,\infty}} + \|{:}Z^3{:}\|^{\frac{4(p+1)}{5}}_{L^\infty_4 B^{-\kappa}_{\infty,\infty}} + 1 .$$

Here, L_t^∞ is taken over the unit time interval.

By the L^p estimate, we have that

$$\frac{1}{2p}\partial_t\left(\|Y\|_{L^{2p}}^{2p}\right) \le -\frac{1}{4}\|Y\|_{L^{2(p+1)}}^{2(p+1)} + C_p R_p .$$

Thus,

$$\partial_t\left(\|Y\|_{L^{2p}}\right) = \partial_t\left(\left(\|Y\|_{L^{2p}}^{2p}\right)^{\frac{1}{2p}}\right)$$

$$= \frac{1}{2p}\left(\|Y\|_{L^{2p}}^{2p}\right)^{\frac{1}{2p}-1}\partial_t\left(\|Y\|_{L^{2p}}^{2p}\right)$$

$$\le \|Y\|_{L^{2p}}^{1-2p}\left(-\frac{1}{4}\|Y\|_{L^{2(p+1)}}^{2(p+1)} + R_p\right)$$

$$\le -\frac{1}{4}\|\psi\|_{L^{2p}}^{3} + \frac{R_p}{\|\psi\|_{L^{2p}}^{2p-1}}.$$

(Here we use that in unit volume,

$$\|\psi\|_{L^{2p}} \le \|\psi\|_{L^{2(p+1)}}.)$$

This implies

$$-\frac{1}{2}\partial_t\left(\|\psi\|_{L^{2p}}^{-2}\right) = \|\psi\|_{L^{2p}}^{-3}\,\partial_t\left(\|\psi\|_{L^{2p}}\right)$$

$$\le -\frac{1}{4} + \frac{C_p R_p}{\|\psi\|_{L^{2p}}^{2(p+1)}}$$

Now, if

$$\|\psi\|_{L^{2p}} > 10\left(C_p R_p\right)^{\frac{1}{2(p+1)}}$$

for all $t \in [0,1]$, then obtain

$$-\frac{1}{2}\partial_t\left(\|Y\|_{L^{2p}}^{-2}\right) \leq -\frac{1}{10}$$

$$\Rightarrow \quad \|Y(t)\|_{L^{2p}} \lesssim t^{-\frac{1}{2}}$$

for all $t \in [0,1]$. Otherwise, let τ be the first time st

$$\|Y(t)\|_{L^{2p}} \leq 10\left(C_p R_p\right)^{\frac{1}{2(p+1)}}.$$

Then for all $t \in [0,\tau]$, have

$$\|Y(t)\| \lesssim t^{-\frac{1}{2}},$$

and for all $t \in [\tau, 1]$, have

$$\|Y(t)\|_{L^{2p}} \leq 10\left(C_p R_p\right)^{\frac{1}{2(p+1)}}.$$

Thus, obtain

$$\|Y(t)\|_{L^{2p}} \lesssim \max\left(t^{-\frac{1}{2}}, (C_p R_p)^{\frac{1}{2(p+1)}}\right)$$

for all $t \in [0,1]$.

Using a similar argument, can also show the following result.

Lemma. (Cf Lemma 2.7 Mourrat–Weber, Lemma 6.1 Hairer–Steele).
Let $u : [0,1] \times \mathbb{T}^d \to \mathbb{R}$ satisfy

$$(\partial_t - \Delta) u \leq -u^3 + C_0.$$

Then for all $t \in [0,1]$,

$$\|u(t)\|_{L^\infty} \lesssim \max\left(t^{-\frac{1}{2}}, C_0^{\frac{1}{3}}\right).$$

Pf. Have

$$\frac{1}{2p}\partial_t\left(\|u\|_{L^{2p}}^{2p}\right) \leq -\|u\|_{L^{2(p+1)}}^{2(p+1)} + C_0\|u\|_{L^{2p-1}}^{2p-1}$$

$$\leq -\frac{1}{2}\|u\|^{2(p+1)} + C\,C_0^{\frac{2(p+1)}{3}}.$$

Thus

$$\partial_t\left(\|u\|_{L^{2p}}\right) = \partial_t\left(\left(\|u\|^{2p}\right)^{\frac{1}{2p}}\right)$$

$$= \|u\|^{2p\left(\frac{1}{2p}-1\right)}\frac{1}{2p}\partial_t\left(\|u\|_{L^{2p}}^{2p}\right)$$

$$\leq -\frac{1}{2}\|u\|^{3} + \frac{C\,C_0^{\frac{1}{3}2(p+1)}}{\|u\|^{2p-1}}.$$

Thus

$$-\frac{1}{2}\partial_t\left(\|u\|_{L^{2p}}^{-2}\right) = \|u\|_{L^{2p}}^{-3}\,\partial_t\left(\|u\|_{L^{2p}}\right)$$

$$\leq -\frac{1}{2} + \frac{C\, C_0^{\frac{1}{3}2(p+1)}}{\|u\|_{L^{2p}}^{2(p+1)}}.$$

If

$$\|u\|_{L^{2p}} > 10\, C\, C_0^{\frac{1}{3}}$$

for all $t \in [0,1]$, then obtain bound like

$$\|u(t)\|_{L^{2p}} \lesssim t^{-\frac{1}{2}}$$

as before. Can handle other case similar to before. $\square$