Schauder estimates.

Ref: Bahouri, Chemin, Danchin "Fourier analysis and nonlinear PDE"

Simplification: work on $\mathbb{T}^d$, since this avoids weights (which are needed in infinite volume) and lattice effects.

Recall given $f: \mathbb{T}^d \to \mathbb{C}$,

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(x) \underbrace{e^{-i 2\pi n \cdot x}}_{\overline{e_n}} dx,$$

$$f = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e_n$$

Since $\Delta e_n = -4\pi^2 |n|^2 e_n$, have that

$$(\Delta - m) f = -\sum_n \hat{f}(n) (4\pi^2 |n|^2 + m) e_n$$

and

$$e^{t(\Delta - m)} f = \sum_n e^{-t(4\pi^2|n|^2+m)} \times \hat{f}(n) e_n .$$

Recall also Besov norm

$$\|f\|_{B^\alpha_{p,q}} = \left(\sum_{j \geq -1} \left(2^{j\alpha} \|\Delta_j f\|_{L^p} \right)^q \right)^{\frac{1}{q}} .$$

Prop. Let $A \subseteq \mathbb{R}^d$ be an annulus. For $\lambda > 0$, have that if

$$\operatorname{supp}(\hat{f}) \subseteq \lambda A ,$$

then for $1 \leq p \leq \infty$,

$$\|e^{t\Delta} f\|_{L^p} \leq C e^{-c\lambda^2 t} \|f\|_{L^p},$$

where C, c do not depend on λ, t, p.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^n \setminus 3\mathbb{D}3)$ be supported on an annulus containing A, and moreover $\psi \equiv 1$ on A. Then if $\operatorname{supp}(\hat{f}) \subseteq \lambda A$, we have that

$$e^{t\Delta} f = \sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} \psi(\lambda^{-1} n) \hat{f}(n) e_n$$

$$= (e^{t\Delta} \phi_\lambda) * f$$

where

$$\phi_\lambda = \sum_n \psi(\lambda^{-1} n) e_n.$$

By Young's convolution inequality, we have that

$$\|(e^{t\Delta}\phi_\lambda) * f\|_{L^p} \leq \|e^{t\Delta}\phi_\lambda\|_{L^1}\|f\|_{L^p},$$

and so it remains to show that

$$\|e^{t\Delta}\phi_\lambda\|_{L^1} \leq Ce^{-\lambda^2 t}.$$

Towards this end, let $\eta \in S(\mathbb{R}^d)$ be s.t.

$$\hat{\eta} = \psi.$$

Let $\eta_\lambda(x) = \lambda^d \eta(\lambda x)$.

Then claim that

$$\phi_\lambda(x) = \sum_{n \in \mathbb{Z}^d} \eta_\lambda(x + 2\pi n).$$

To see this, compute Fourier transform

$$\hat{\phi}_\lambda(n) = \int_{[0,1]^d} \phi_\lambda(x)\, e^{-i2\pi n\cdot x}\, dx$$

$$= \sum_{m\in\mathbb{Z}^d} \int_{[0,1]^d} \eta_\lambda(x+2\pi m)\, e^{-i2\pi n\cdot x}\, dx$$

$$= \int_{\mathbb{R}^d} \eta_\lambda(x)\, e^{-i2\pi n\cdot x}\, dx$$

$$= \hat{\eta}_\lambda(n) = \hat{\eta}(\tilde{\lambda}^{-1} n)$$

$$= \psi(\tilde{\lambda}^{-1} n).$$

Since ϕ_λ was defined by

$$\hat{\phi}_\lambda(n) = \psi(\tilde{\lambda}^{-1} n),$$

this shows the claim. Now, let

$$p_t^{euc}(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4t}\right)$$

the heat kernel on $\mathbb{R}^d$. Claim

$$p_t = \sum_n p_t^{euc}(x + 2\pi n) .$$

Proof is similar to previous claim. Now, have

$$e^{t\Delta} \phi_\lambda(x) = p_t * \phi_\lambda(x)$$

$$= \int_{[0,1]^d} dy \, p_t(x-y) \, \phi_\lambda(y)$$

$$= \sum_n \int_{[0,1]^d} dy \, p_t^{euc}(x + 2\pi n - y) \, \phi_\lambda(y)$$

$$= \int_{\mathbb{R}^d} dy\, p_t^{euc}(x-y)\, \varphi_\lambda(y)$$

(think of φ_λ as a periodic fn on $\mathbb{R}^d$). Using first claim, get

$$\|e^{t\Delta}\varphi_\lambda\|_{L^1([0,1]^d)}$$

$$= \int_{[0,1]^d} dx \left| \int_{\mathbb{R}^d} dy\, p_t^{euc}(x-y) \sum_n \varphi_\lambda(y+2\pi n) \right|$$

$$\leq \sum_n \int_{[0,1]^d} dx \left| \int_{\mathbb{R}^d} dy\, p_t^{euc}(x-y)\, \varphi_\lambda(y+2\pi n) \right|$$

$$= \sum_n \int_{[0,1]^d} dx \left| \int_{\mathbb{R}^d} dy \; p_t^{euc}(x+2\pi(n-y)) \, \eta_\lambda(y) \right|$$

$$= \int_{\mathbb{R}^d} dx \left| \int_{\mathbb{R}^d} dy \; p_t^{euc}(x-y) \, \eta_\lambda(y) \right|$$

$$= \| p_t^{euc} * \eta_\lambda \|_{L^1(\mathbb{R}^d)}.$$

In summary, have gone from torus to $\mathbb{R}^d$!

By scaling, we have that

$$\| p_t^{euc} * \eta_\lambda \|_{L^1(\mathbb{R}^d)} = \| p_{\lambda^2 t}^{euc} * \eta \|_{L^1(\mathbb{R}^d)}.$$

To see this, note

$$\lambda^{-d}(p_t^{euc} * \eta_\lambda)(\lambda^{-1}x)$$

$$= \lambda^{-d} \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} dy \, \exp\left(-\frac{|\lambda^{-1}x - y|^2}{4t}\right) \lambda^d \eta(\lambda y)$$

$$\overset{[z = \lambda y]}{=} \frac{1}{(4\pi\lambda^2 t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} dy \, \exp\left(-\frac{|x-z|^2}{4\lambda^2 t}\right) \eta(z)$$

$$= (p_{\lambda^2 t}^{euc} * \eta)(x).$$

Thus

$$\|p_t^{euc} * \eta_\lambda\|_{L^1} = \|\lambda^{-d} p_t^{euc} * \eta(\lambda^{-1}\cdot)\|_{L^1}$$

$$= \| p^{euc}_{\lambda^2 t} * \eta \|_{L^1} .$$

Thus, suffice to take $\lambda = 1$, i.e. show that

$$\| p^{euc}_t * \eta \|_{L^1} \leq C e^{-ct} .$$

To see this, first write

$$(p^{euc}_t * \eta)(x) = \int dp \, e^{i 2\pi p \cdot x} e^{-4\pi^2 |p|^2 t} \psi(p).$$

(recall $\hat{\eta} = \psi$)

Note by properties of ψ, have that

$$\| p^{euc}_t * \eta \|_{L^\infty} \leq C e^{-ct} .$$

However, need to show L^1 bound. Trick: multiply and divide by a polynomial which ensures integrability:

$$(1+4\pi^2|x|^2)^{-d}(1+4\pi^2|x|^2)^{d}\int dp\, e^{i2\pi p\cdot x} e^{-4\pi^2|p|^2 t}\psi(p)$$

Enough to show $\square$ has the same L^∞ bound. Note

$$\square = \int dp\,(1-\Delta_p)^{d} e^{i2\pi p\cdot x} e^{-4\pi^2|p|^2 t}\psi(p)$$

$$\overset{(IBP)}{=} \int dp\, e^{i2\pi p\cdot x}\{(1-\Delta_p)^{d}\{e^{-4\pi^2|p|^2 t}\psi(p)\}\}$$

Have that [redacted] is a sum of terms of the form

$$\partial_\alpha \left(e^{-4\pi^2 |p|^2 t} \right) \partial_\beta \psi(p)$$

where $|\alpha| + |\beta| \leq 2d$. Can bound each on the support of ψ by

$$C (1 + Ct)^d \, e^{-ct}.$$

Have that

$$\left\| \int dp \, e^{i 2\pi p \cdot x} \psi(p) \, x \right.$$

$$\left. C (1 + Ct)^d \, e^{-ct} \right\|_{L^\infty_x} \leq C e^{-ct},$$

and this shows the desired estimate for [redacted]. □

Given the proposition, we may prove the following Schauder estimate.

Prop. For $t \in (0,1]$, $\beta \geq \alpha$, $p, q \in [1, \infty]$, we have that

$$\|e^{t\Delta} f\|_{B^{\beta}_{p,q}} \lesssim (\beta-\alpha)^{\frac{\beta-\alpha}{2}} t^{-\frac{(\beta-\alpha)}{2}} \|f\|_{B^{\alpha}_{p,q}}.$$

Pf. For $j \geq 0$, we have by the Proposition that

$$\|\Delta_j e^{t\Delta} f\|_{L^p} = \|e^{t\Delta} \Delta_j f\|_{L^p}$$

$$\leq C e^{-c 2^{2j} t} \|\Delta_j f\|_{L^p}.$$

Thus,

$$2^{j\beta}\|\Delta_j e^{t\Delta} f\|_{L^p} \lesssim 2^{j\alpha}\|\Delta_j f\|_{L^p} \times$$

$$2^{j(\beta-\alpha)} e^{-c2^{2j}t}$$

$$\lesssim (\beta-\alpha)^{-\frac{(\beta-\alpha)}{2}} t^{-\frac{(\beta-\alpha)}{2}} 2^{j\alpha}\|\Delta_j f\|_{L^p},$$

where we used that

$$\sup_{x\geq 1} x^{\beta-\alpha} e^{-cx^2 t}$$

$$\lesssim (\beta-\alpha)^{\frac{\beta-\alpha}{2}} t^{-\frac{(\beta-\alpha)}{2}}.$$

The desired result now follows. □

Now, given $f:[0,T]\to B^{\alpha}_{p,q}$, define

$$\|f\|_{L^r_t B^{\alpha}_{p,q}} = \left(\int_0^T dt\, \|f(t)\|^r_{B^{\alpha}_{p,q}}\right)^{\frac{1}{r}}.$$

Define

$$Duh(f)(t) := \int_0^t e^{(t-s)(\Delta - m)} f(s)\,ds.$$

Then we have the following Schauder estimate.

Prop. Let $0 \le \beta < 2$, $p, q \in [1, \infty]$. Then

$$\|Duh(f)\|_{L^\infty_t B^\beta_{p,q}} \lesssim_{\beta-\alpha} T^{1-\frac{(\beta-\alpha)}{2}} \|f\|_{L^\infty_t B^\alpha_{p,q}}.$$

Pf. For $t \in [0, T]$, we bound

$$\|Duh(f)(t)\|_{B^\beta_{p,q}}$$

$$\leq \int_0^t \left\| e^{(t-s)(\Delta-m)} f(s) \right\|_{B^{\beta}_{p,q}} ds$$

$$\lesssim \int_0^t (t-s)^{-\frac{(\beta-\alpha)}{2}} \|f(s)\|_{B^{\alpha}_{p,q}} ds$$

$$\lesssim t^{1-\frac{(\beta-\alpha)}{2}} \|f\|_{L^{\infty}_t B^{\alpha}_{p,q}}$$

where we used that $\beta < \alpha+2$ to integrate the singularity. Taking sup over $t \in [0,T]$, the desired result now follows. □