Hermite polynomials.

Ref: Max's notes, Hairer's notes on Malliavin calculus.

Let $(H_n(x))_{n\geq 0}$ be the family of polynomials characterized by:

$$e^{i\lambda x} e^{\frac{\lambda^2}{2}} = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} H_n(x)$$

Ex. $H_0(x) = 1$

$H_1(x) = x$

$H_2(x) = x^2 - 1$.

Let $\rho(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

be the standard Gaussian density.

By differentiating in x, obtain

$$\partial_x \left(e^{i\lambda x} e^{\frac{\lambda^2}{2}}\right) = i\lambda\, e^{i\lambda x} e^{\frac{\lambda^2}{2}},$$

which implies

$$\sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} H_n' = \sum_{n=0}^{\infty} \frac{(i\lambda)^{n+1}}{n!} H_n$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(i\lambda)^{n+1}}{(n+1)!} H_{n+1}' = \sum_{n=0}^{\infty} \frac{(i\lambda)^{n+1}}{n!} H_n$$

$$\Rightarrow \frac{1}{n+1} H_{n+1}' = H_n$$

$$H_{n+1}' = (n+1) H_n \, .$$

Similarly, by differentiating in λ, can obtain

$$H_{n+1} = xH_n - H_n'$$

Indeed,

$$\partial_\lambda \left(e^{i\lambda x} e^{\frac{\lambda^2}{2}} \right) = (ix+\lambda) e^{i\lambda x} e^{\frac{\lambda^2}{2}}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{i^n \lambda^{n-1}}{(n-1)!} H_n = \sum_{n=0}^{\infty} \frac{i^{n+1} \lambda^n}{n!} xH_n$$

$$+ \frac{i^n \lambda^{n+1}}{n!} H_n,$$

or i.e.

$$\sum_{n=0}^{\infty} i \frac{(i\lambda)^n}{n!} H_n = \sum_{n=0}^{\infty} i \frac{(i\lambda)^n}{n!} xH_n$$

$$+\sum_{n=1}^{\infty} i\,\frac{(i\lambda)^n}{n!}\, i^{-2}\, n H_{n-1}$$

Obtain

$$H_n = xH_n - nH_{n-1},$$

and thus also

$$H_n = xH_n - H_n'.$$

Lemma. For all $n \geq 1$,

$$\int H_n(x)\, e(x)\,dx = 0.$$

Pf. Note that

$$1 = \int e^{i\lambda x} e^{\frac{\lambda^2}{2}} \varrho(x)\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \int H_n(x) \varrho(x)\, dx.$$

$\square$

Prop. $(H_n)_{n \geq 0}$ is an orthogonal basis of $L^2(\mathbb{R}, \varrho)$.

Have that

$$\int H_n(x) H_m(x) \varrho(x)\, dx$$

$$= \frac{1}{n+1} \int H_{n+1}'(x) H_m(x) \varrho(x)\, dx$$

$$\overset{(IBP)}{=} \frac{1}{n+1} \int H_{n+1}(x) \left(x H_m(x) - H_m'(x) \right) e(x)\, dx$$

$$= \frac{1}{n+1} \int H_{n+1}(x) H_{m+1}(x) e(x) dx$$

Combining w/ the lemma, obtain

$$\int H_n(x) H_m(x) e(x) dx = \delta_{n,m} n! .$$

This shows that the $(H_n)_{n \geq 0}$ are orthogonal. To show that this is a basis, assume that $f \in L^2(\mathbb{R}, e)$ is such that

$$\int f(x)\, H_n(x)\, \rho(x)\, dx = 0 \quad \forall n.$$

Then $\forall \lambda$,

$$\int f(x)\, e^{i\lambda x}\, e^{\frac{\lambda^2}{2}}\, \rho(x)\, dx$$

$$= \sum_{n \geq 0}^{\infty} \frac{(i\lambda)^n}{n!} \int f(x)\, H_n(x)\, \rho(x)\, dx$$

(exercise: justify exchange of sum and integral here)

$$= 0.$$

This implies $\widehat{f\rho}(\lambda) = 0 \;\; \forall \lambda$, which implies that

$$fe = 0 \quad \text{a.e.}$$

$$\Rightarrow \quad f = 0 \quad \text{a.e.} \qquad \square$$

Define creation and annihilation operators

$$C = x - \partial_x \,, \quad A = \partial_x$$

Have that

$$CH_n = H_{n+1} \,, \quad AH_n = nH_{n-1}$$

Thus, their action is very simple on $(H_n)_{n \geq 0}$.

Also,

$$CAH_n = nCH_{n-1} = nH_n,$$

$$[A, C] = \partial_x(x-\partial_x) - (x-\partial_x)\partial_x$$

$$= 1.$$

Now, define

$$P = i(x - 2\partial_x)$$

$$= i(C - A)$$

$$Q = x = C + A.$$

Then

$$[P, Q] = [C+A, i(C-A)]$$

$$= i\left([A,C] - [C,A]\right)$$

$$= 2i \quad ,$$

$$Q^2 = C^2 + CA + AC + A^2$$

$$P^2 = -\left(C^2 - CA - AC + A^2\right)$$

Thus,

$$P^2 + Q^2 = 2(CA + AC)$$

$$= 2([A,C] + 2CA)$$

$$= 2(1 + 2CA).$$

Have that

$$E := \frac{1}{4}(P^2 + Q^2 - 2) = CA,$$

and

$$E H_n = n H_n .$$

Note that

$$E = CA = x\partial_x - \partial_{xx}$$

$$= x\partial_x - \Delta$$

is the OU operator!

Alternative representation of quantum harmonic oscillator.

The Hermite functions

$$\tilde{H}_n = H_n e^{-\frac{x^2}{4}}$$

form an orthogonal basis of $L^2(\mathbb{R})$.

Have that

$$-i\partial_x \tilde{H}_n = -i\partial_x H_n e^{-\frac{x^2}{4}} + i\frac{x}{2} H_n e^{-\frac{x^2}{4}}$$

$$= i\left(\left(\frac{x}{2} - \partial_x\right)H_n\right)e^{-\frac{x^2}{4}}$$

$$= \left(\frac{P}{2} H_n\right) e^{-\frac{x^2}{4}}.$$

Have that

$$-\partial_x \partial_x \tilde{H}_n = -i\partial_x\left(-i\partial_x \tilde{H}_n\right)$$

$$= -i\partial_x\left(\left(\frac{p}{2}H_n\right)e^{-\frac{x^2}{4}}\right)$$

$$= -i\left(\partial_x\left(\frac{p}{2}H_n\right)e^{-\frac{x^2}{4}} - \frac{x}{2}\left(\frac{p}{2}H_n\right)e^{-\frac{x^2}{4}}\right)$$

$$= \left(\left(\frac{p}{2}\right)^2 H_n\right)e^{-\frac{x^2}{4}}.$$

$$\frac{x^2}{4}\tilde{H}_n = \left(\frac{x^2}{4}H_n\right)e^{-\frac{x^2}{4}}.$$

Thus if $\tilde{p} = -i\partial_x$

$$\tilde{Q} = \frac{x}{2},$$

then

$$\left(\tilde{P}^2 + \tilde{Q}^2\right)\tilde{H}_n = \left(\frac{1}{4}(P^2+Q^2)H_n\right) e^{-\frac{x^2}{4}}$$

$$= \left(E+\frac{1}{2}\right)H_n e^{-\frac{x^2}{4}}$$

$$= \left(n+\frac{1}{2}\right)\tilde{H}_n .$$

Thus

$$\left(\tilde{P}^2 + \tilde{Q}^2 - \frac{1}{2}\right)\tilde{H}_n = n\tilde{H}_n,$$

$$\tilde{P}^2 + \tilde{Q}^2 - \frac{1}{2} = \frac{x^2}{4} - \partial_x^2 - \frac{1}{2}.$$