

Distributions

Ref: Wikipedia, Max's notes

Formalize the notion of an object which is pointwise ill-defined, yet "averages" are well-defined.

Notation. Given multi-index

$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d,$$

let

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}.$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

Natural space of test functions:
Schwartz space

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d, \mathbb{C}) :$$

$$\forall \alpha, \beta \in \mathbb{N}^d,$$

$$\|f\|_{\alpha, \beta} < \infty \},$$

where

$$\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)|.$$

In words, f and all of its derivatives decreases faster than any polynomial.

Examples: $C_c^\infty(\mathbb{R}^d), x^\alpha e^{-|x|^2}$.

Topology on $\mathcal{S}$ characterized by:

$$f_k \rightarrow f \text{ if}$$

$$\|f_k - f\|_{\alpha, \beta} \rightarrow 0 \quad \forall \alpha, \beta.$$

This makes S into a Frechét space (a complete metrizable TVS whose topology comes from countable family of seminorms).

Def. A tempered distribution is a continuous linear functional

$$\phi : S \rightarrow \mathbb{C}.$$

The TVS of tempered distributions is denoted $S' = S'(\mathbb{R}^d)$.

Ex. Dirac delta $\delta_0(f) = f(0)$.
Continuous because if

$$f_k \rightarrow f \text{ in } S,$$

then

$$\delta_0'(f_k) = f_k(0) \rightarrow f(0) = \delta_0(f).$$

Derivative of BM on $\mathbb{R}$. Let B be a standard BM,

$$3'' = \frac{dB}{dt},$$

which means

$$3(f) = \int f dB$$

One can then show that a.s., this is continuous, and thus is a tempered distribution.

Fourier transform

For $f \in L^1(\mathbb{R}^d, \mathbb{C})$, define

$$\mathcal{F}f(p) := \int_{\mathbb{R}^d} dx \, f(x) e^{-2\pi i p \cdot x},$$

$$p \in \mathbb{R}^d.$$

Define also inverse Fourier transform

$$\mathcal{F}^{-1}f(x) := \int_{\mathbb{R}^n} dp \, f(p) e^{2\pi i p \cdot x}$$

Lemma. If $f \in \mathcal{S}$, then $\mathcal{F}f, \mathcal{F}^{-1}f \in \mathcal{S}$,
and

$$\mathcal{F}\mathcal{F}^{-1}f = \mathcal{F}^{-1}\mathcal{F}f = f.$$

$$\text{Lemma. } \mathcal{F}\partial^\alpha f(p) = (2\pi i)^{|\alpha|} p^\alpha \mathcal{F}f(p).$$

Corollary. $\mathcal{F} \Delta f(p) = -4\pi^2 |p|^2 \mathcal{F} f(p).$

Plancherel: for $f, g \in S$,

$$(f, g) = (\mathcal{F} f, \mathcal{F} g),$$

where

$$(f, g) = \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx.$$

(Since $S \subseteq L^2$ is dense, this allows to extend $\mathcal{F}$ to L^2 .)

Lemma. $\widehat{f * g} = \hat{f} \hat{g}.$

For $f \in L^1(\mathbb{T}^d, \mathbb{C})$, define

$$\hat{f}(n) := \hat{f}(n) := \int_{\mathbb{T}^d} dx \, f(x) e^{-2\pi i n \cdot x},$$

$$n \in \mathbb{Z}^d.$$

Then (Fourier inversion)

$$f = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e_n,$$

where $e_n(x) = e^{2\pi i n \cdot x}$.

If $f \in C^\infty(\mathbb{T}^d, \mathbb{C})$, then series converges ptwise unif., if $f \in L^2(\mathbb{T}^d, \mathbb{C})$, then series converges in L^2 .

Plancherel:

$$\int_{\mathbb{T}^d} dx f(x) \overline{g(x)} = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) \overline{\hat{g}(n)}.$$

Fourier transform on the lattice
As in Max's notes,

$$\mathcal{L} = \mathbb{Z}^N, \quad M = \mathbb{Z}^{N'},$$

$$\Lambda_{\mathcal{L}} = (\mathcal{L}\mathbb{Z})^d$$

$$\Lambda_{\mathcal{L}, M} = \Lambda_{\mathcal{L}} \cap \mathbb{T}_M^d$$

$$= (\mathcal{L}\mathbb{Z})^d \cap \left[-\frac{M}{2}, \frac{M}{2}\right)^d$$

↑
periodic boundary

Let $\hat{\Delta}_\varepsilon = (\bar{\varepsilon}' [-\frac{1}{2}, \frac{1}{2}])^d$ be the dual of Δ_ε .

For $f: \Delta_\varepsilon \rightarrow \mathbb{C}$, $g: \hat{\Delta}_\varepsilon \rightarrow \mathbb{C}$,

$$\mathcal{F}_\varepsilon f(k) = \varepsilon^d \sum_{x \in \Delta_\varepsilon} f(x) e^{-2\pi i k \cdot x}$$

$$\mathcal{F}_\varepsilon^{-1} g(x) = \int_{\hat{\Delta}_\varepsilon} g(k) e^{2\pi i k \cdot x} dk,$$

Remark. The $\bar{\varepsilon}'$ factor in $\hat{\Delta}_\varepsilon$ arises because if

$$k, k' \in \mathbb{R}^d, \quad k - k' \in \bar{\varepsilon}' \mathbb{Z}^d,$$

then

$$x \mapsto e^{2\pi i k \cdot x}, \quad x \mapsto e^{2\pi i k' \cdot x}$$

define the same function.

let $\hat{\Delta}_{\varepsilon, M} = ((M^{-1}\mathbb{Z}) \cap \varepsilon'[-\frac{1}{2}, \frac{1}{2}))^d$.

Remark. Note that if $k \notin M^{-1}\mathbb{Z}$, then

$$x \mapsto e^{2\pi i k \cdot x}$$

is not periodic in x .

For $f: \Delta_{\varepsilon, M} \rightarrow \mathbb{C}$, $g: \hat{\Delta}_{\varepsilon, M} \rightarrow \mathbb{C}$,
define

$$F_{\varepsilon, M} f(k) = \varepsilon^d \sum_{x \in \Delta_{\varepsilon, M}} f(x) e^{-2\pi i k \cdot x}$$

$$F_{\varepsilon, M}^{-1} g(x) = \frac{1}{M^d} \sum_{k \in \hat{\Delta}_{\varepsilon, M}} g(k) e^{2\pi i k \cdot x}$$

Now, for $f: \Delta_\varepsilon \rightarrow \mathbb{C}$, define

$$\nabla_\varepsilon^i f(x) = \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon},$$

$$\nabla_\varepsilon^{-i} f(x) = \frac{f(x) - f(x - \varepsilon e_i)}{\varepsilon},$$

$$\begin{aligned} \Delta_\varepsilon f(x) &= \sum_{i=1}^d \nabla_\varepsilon^{-i} \nabla_\varepsilon^i f(x) \\ &= \varepsilon^{-2} \sum_{i=1}^d (f(x + \varepsilon e_i) + f(x - \varepsilon e_i) - 2f(x)). \end{aligned}$$

Have that

$$\nabla_\varepsilon^j f(x) = \int_{\Delta_\varepsilon} dk \, \hat{f}_\varepsilon(k) \frac{e^{2\pi i \varepsilon k_j - 1}}{\varepsilon} e^{2\pi i k \cdot x},$$

$$\Delta_\varepsilon f(x) = - \int_{\Lambda_\varepsilon} dk \, Ff(k) \sum_{j=1}^d (2\bar{\varepsilon}^j \sin(\pi \varepsilon k_j))^2 e^{2\pi i k \cdot x}$$

For $m^2 > 0$, have that

$$\begin{aligned} & (m^2 - \Delta_\varepsilon^{-1})(x, y) \\ &= \frac{1}{4^d} \sum_{k \in (M^{-1}\mathbb{Z} \cap \bar{\varepsilon}^1(-\frac{1}{2}, \frac{1}{2}))^d} e^{ik \cdot (x-y)} \times \\ & \quad \frac{1}{m^2 + \sum_{i=1}^d (2\bar{\varepsilon}^i \sin(\pi \varepsilon k_i))^2} \end{aligned}$$

Appendix.

How to remember the formal identity

$$\int_{\mathbb{R}^d} dp \, e^{i2\pi p \cdot x} = \delta_0(x) \, ?$$

Or, equivalently,

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp \, e^{ip \cdot x} = \delta_0(x).$$

Let K_t be the heat kernel on $\mathbb{R}^d$, i.e.

$$K_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{2t}\right).$$

We have that

$$K_t \rightarrow \Delta_0 \quad \text{as } t \rightarrow 0.$$

Want to find $g_t(p)$ st

$$K_t(x) = \int_{\mathbb{R}^d} dp \, g_t(p) e^{ip \cdot x}.$$

Then take $\lim_{t \downarrow 0} g_t$ to obtain

formal identity for Δ_0 . We compute

$$\begin{aligned} \left(\frac{t}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} dp \exp\left(-\frac{t}{2}|p|^2\right) e^{ip \cdot x} \\ = \exp\left(-\frac{|x|^2}{2t}\right). \end{aligned}$$

(Characteristic fn of $N(0, \frac{1}{t})$.)

Thus,

$$K_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{2t}\right)$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp \exp\left(-\frac{t}{2}|p|^2\right) e^{ip \cdot x}$$

Taking $t \rightarrow 0$, obtain the desired formal identity.