Back to lattice YM. We apply the Word recursion to derive the master loop eq. First, given a string s, define

$$\mathbb{S}_\pm(s), \quad \mathbb{M}_\pm(s), \quad \mathbb{D}_\pm(s)$$

splitting merger deformation

as follows. Let e be the first edge of the first loop s_1 (can take general edge as well, but we fix the first edge for simplicity).

$\mathbb{S}_+(s)$ collects all cases:

$$s_1 = e s_{1,1} e s_{1,2} \mapsto e s_{1,1}, \; e s_{1,2}$$

$\mathbb{S}_-(s)$ collects all cases:

$$s_1 = e s_{1,1} e^{-1} s_{1,1} \mapsto s_{1,1}, s_{1,1}$$

$\mathbb{M}_+(s)$ collects all cases:

$$s_1 = e s_1', \; s_k = s_{k,1} e s_{k,2}$$

$$\mapsto e s_{k,2} s_{k,1} e s_1' = s_1 \oplus_e s_k$$

$\mathbb{M}_-(s)$ collects all cases:

$$s_1 = e s_1', \; s_k = s_{k,1} e^{-1} s_{k,2}$$

$$\mapsto s_1' s_{k,2} s_{k,1} = s_1 \ominus_e s_k$$

$\mathbb{P}_+(s)$ collects all cases:

$$s_1 = e s_1', \; p = e p' \mapsto e p' e s_1' = s_1 \oplus_e p$$

where p is an orient. plaq. containing the orient. edge e,

$\mathcal{P}_-(s)$ collects all cases:

$$s_1 = e s_1', \quad p = e^{-1} p' \mapsto s_1' p' = s_1 \ominus_e p,$$

where p is an orient. plaq. containing the orient. edge e^{-1}.

Let

$$\phi(s) = \langle W_s \rangle_{\Lambda, N, \beta} .$$

Prop [MLE] Let s be a string in Λ suff. far away from the bdry. Then for all $N \geq 1$, $\beta \geq 0$,

$$\phi(s) = \mp \sum_{s' \in \mathbb{S}_\pm(s)} \phi(s') \mp \frac{1}{N^2} \sum_{s' \in \mathbb{M}_\pm(s)} \phi(s')$$

$$\mp \beta \sum_{s' \in \mathbb{D}_\pm(s)} \phi(s').$$

Remark. This is linear in ϕ.

Pf. Recall that

$$\phi(s) = Z_{\Lambda,\beta}^{-1} \int W_s(U) \prod_{p \in \mathcal{P}_\Lambda} \exp(N\beta \operatorname{Tr}(U_p)) \prod_{e \in E_\Lambda^+} dU_e$$

$$= Z_{\Lambda,\beta}^{-1} \sum_{K : \mathcal{P}_\Lambda \to \mathbb{N}} \frac{(N\beta)^K}{K!} I(s, K)$$

where

$$I(s,K) = N^{-n} \int \mathrm{Tr}(U_{s_1}) \cdots \mathrm{Tr}(U_{s_n}) \prod_{p \in \mathcal{P}_\Lambda} \mathrm{Tr}(U_p)^{K(p)} \prod_{e \in E_\Lambda^+} dU_e .$$

Apply the word recursion. Obtain:

$$I(s,K) = \mp \sum_{s' \in \mathbb{S}_\pm(s)} I(s',K)$$

no $\frac{1}{N}$ prefactor b/c # loops increased by 1

$$\mp \frac{1}{N^2} \sum_{s' \in \mathbb{M}_\pm(s)} I(s',K)$$

loops decreased by 1

$$- \frac{1}{N} \sum_{p \ni e} \mathbb{1}(K(p) \geq 1)\, K(p) \times I(s \oplus_e p, K - \underbrace{\delta_p}_{\text{indicator at } p})$$

"p contains e"

$$+\frac{1}{N}\sum_{p\succ e^{-1}} \mathbb{1}(K(p)\geq 1)\,K(p) \times I(s\ominus e_p, K-\delta_p).$$

Inserting this, further obtain

$$\varphi(s) = Z_{\Lambda,\beta}^{-1}\sum_{K:P_\Lambda\to\mathbb{N}} \frac{(N\beta)^K}{K!} I(s,K)$$

$$= Z_{\Lambda,\beta}^{-1}\sum_{K:P_\Lambda\to\mathbb{N}} \frac{(N\beta)^K}{K!}\Big(\mp\sum_{s'\in S_\pm(s)} I(s',K)$$

$$\mp\frac{1}{N^2}\sum_{s'\in M_\pm(s)} I(s',K)$$

$$\mp\frac{1}{N}\sum_{p\succ e^{\pm}} \mathbb{1}(K(p)\geq 1)\,K(p)\,I(s\oplus e_p, K-\delta_p)\Big)$$

Have that

contr. $\mp \sum_{s' \in S_\pm(s)} \phi(s') \mp \frac{1}{N^2} \sum_{s' \in M_\pm(s)} \phi(s')$.

To finish, show that

contr. $\mp \beta \sum_{s' \in P_\pm(s)} \phi(s')$.

We focus on case $p \ni e$. Can write the sum as

$$-Z_{\Lambda,\beta}^{-1} \sum_{p \ni e} \sum_{\substack{K : P_\Lambda \to \mathbb{N} \\ K(p) \geq 1}} \frac{(N\beta)^K}{K!} \frac{1}{N}$$

$$\times K(p) I(s \oplus_e p, K - \delta_p)$$

[change var. $K \mapsto K - \delta_p$]

$$= -Z_{\Lambda,\beta}^{-1} \sum_{p \succeq e} \sum_{K : \mathcal{P}_\Lambda \to \mathbb{N}} \frac{(N\beta)^K}{K!} \frac{N\beta}{(K(p)+1)} \times \frac{1}{N} \times (K(p)+1) I(S \oplus_e p, K)$$

$$= -\beta Z_{\Lambda,\beta}^{-1} \sum_{p \succeq e} \sum_{K : \mathcal{P}_\Lambda \to \mathbb{N}} \frac{(N\beta)^K}{K!} I(S \oplus_e p, K)$$

$$= -\beta \sum_{p \succeq e} \phi(S \oplus_e p)$$

$$= -\beta \sum_{s' \in \mathbb{D}_\pm(s)} \phi(s'), \text{ as desired.} \quad \square$$

General question: using the MLE, what can we prove about Wilson loops?