Recall we introduced the Weingarten fn Wg_N as a matrix:

$$Wg_N(\sigma, \tau), \quad \sigma, \tau \in S_n,$$

(n-dep implicit).

In these notes, we discuss alternative viewpoints on Wg_N, which will lead to a recursion for Wg_N, which will lead to the Master loop eq. in lattice YM.

Group algebra

Define $\mathbb{C}[S_n]$ (the symmetric group algebra) as either:

(1) the set of functions $f: S_n \to \mathbb{C}$, equipped w/ convolution as product:

$$(fg)(\sigma) = \sum_{\tau \in S_n} f(\sigma\tau^{-1})\, g(\tau)$$

(2) the set of formal sums

$$f = \sum_{\sigma \in S_n} f(\sigma)\sigma,$$

equipped w/ product:

$$fg = \Big(\sum_{\sigma \in S_n} f(\sigma)\sigma\Big)\Big(\sum_{\sigma' \in S_n} g(\sigma')\sigma'\Big)$$

$$= \sum_{\sigma, \sigma' \in S_n} f(\sigma) g(\sigma') \sigma \sigma'$$

$$= \sum_{\sigma \in S_n} \left(\sum_{\tau \in S_n} f(\sigma \tau^{-1}) g(\tau) \right) \sigma .$$

These two viewpoints are equivalent.

Equip $\mathbb{C}[S_n]$ w/ the ℓ^1 norm:

$$\|f\| = \sum_{\sigma \in S_n} |f(\sigma)| .$$

Exercise:

$$\|fg\| \leq \|f\| \, \|g\| .$$

Consider now the element

$$G = \sum_{\sigma \in S_n} N^{\#\mathrm{cycles}(\sigma)} \sigma \in \mathbb{C}[S_n].$$

If G is invertible, i.e. $\exists\, W \in \mathbb{C}[S_n]$ st

$$GW = WG = \mathrm{id},$$

then we define the Weingarten fn

$$Wg_N = W \in \mathbb{C}[S_n].$$

To check that this would be equiv to the matrix def, note that:

$$\mathbb{1}(\sigma^{-1}\tau = \mathrm{id}) = (GW)(\sigma^{-1}\tau)$$

$$= \sum_{\sigma'} G(\sigma^{-1}\sigma') \, W((\sigma')^{-1}\tau)$$

$$= \sum_{\sigma'} N^{\#\text{cycles}(\sigma^{-1}\sigma')} \, W((\sigma')^{-1}\tau).$$

Recalling that we defined the Gram matrix

$$G(\sigma, \sigma') = N^{\#\text{cycles}(\sigma^{-1}\sigma')},$$

we see that defining,

$$Wg_N(\sigma', \tau) = W((\sigma')^{-1}\tau),$$

we have that

$$G \, Wg_N = I \quad .$$

Similarly, $Wg_N \, G = I$.

Thus, when G is invertible in $\mathbb{C}[S_n]$, the Gram matrix is an invertible matrix, and thus the Weingarten fn/matrix is the inverse of G / Gram matrix (if matrix invertible, then pseudo-inverse = inverse).

How to tell when G is invertible?

Fix $n \geq 1$. For $1 \leq k \leq n$, let

$$J_k = (1, k) + (2\ k) \cdots (k-1\ k) \in \mathbb{C}[S_n]$$

transpositions

In particular, $J_1 = 0$. These are the Jucys-Murphy elements.

Exercise. The $J_1, \ldots, J_n$ mutually commute.

Exercise. For any $N \in \mathbb{C}$, $n \geq 1$, have that

$$G = \sum_{\sigma \in S_n} N^{\#\text{cycles}(\sigma)} \sigma$$

$$= \prod_{k=1}^{n} \Big(\underset{\substack{\uparrow \\ \text{id}}}{N} + J_k \Big).$$

Note: J_k is a sum of $k-1$ terms.

Thus,

$$\|J_k\| \leq k-1.$$

Thus, if $|N| \geq k$, have that

$$(N+J_k)^{-1} = N^{-1}(\mathrm{id} + N^{-1}J_k)^{-1}$$

$$= N^{-1}\sum_{\ell=0}^{\infty}\left(-\frac{J_k}{N}\right)^{\ell}$$

converges in $\mathbb{C}[S_n]$,

i.e. $(N+J_k) \in \mathbb{C}[S_n]$ is invertible.

Thus, if $|N| \geq n$, have that G is invertible, and

$$Wg_N = G^{-1} = (N+J_1)^{-1} \cdots (N+J_n)^{-1}.$$

Exercise. Show that if $N<n$, then G is not invertible. (Reduce to showing that J_{N+1} is not invertible. Consider simple case of $N=1$, $n=2$.)

Let us suppose $|N| \geq n$ for now. We will return to the case of general N later.

Note:

$$\frac{1}{1+x} = 1 - x\frac{1}{1+x}.$$

Thus,

$$(N+J_n)^{-1} = N^{-1}\left(\mathrm{id} - N^{-1}J_n(\mathrm{id}+N^{-1}J_n)^{-1}\right)$$
$$= N^{-1} - N^{-1}J_n(N+J_n)^{-1}.$$

Inserting this, obtain Weingarten recursion:

$$\mathrm{Wg}_N = (N+J_n)^{-1}\cdots(N+J_1)^{-1}$$
$$= N^{-1}(N+J_{n-1})^{-1}\cdots(N+J_1)^{-1}$$
$$- N^{-1}J_n\,\mathrm{Wg}_N.$$

Note:

$$(N+J_{n-1})^{-1}\cdots(N+J_1)^{-1}$$

is supported on $\sigma \in S_n : \sigma(n) = n$.

For such σ, let $\sigma^{\downarrow} \in S_{n-1}$ obtained by deleting $\sigma(n) = n$. Then

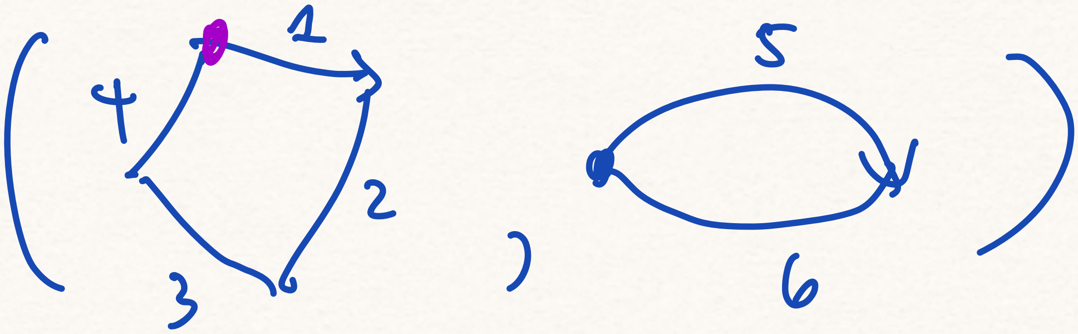
$$(N+J_{n-1})^{-1} \cdots (N+J_1)^{-1}(\sigma)$$
$$= Wg_{N,n-1}(\sigma^{\downarrow}).$$

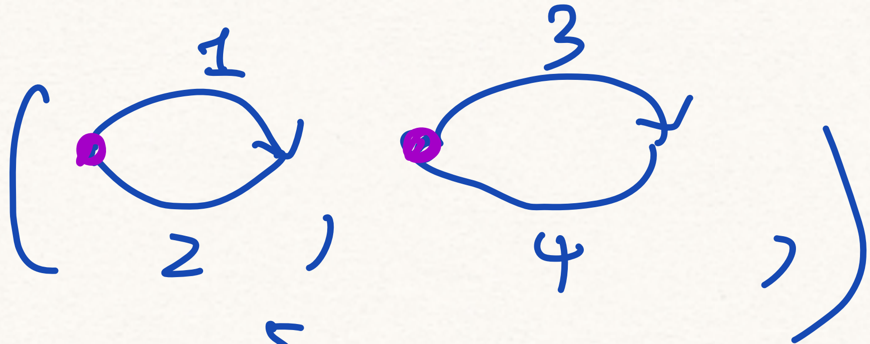
Equating both sides of the recursion ptwise, obtain that $\forall \sigma \in S_n$,

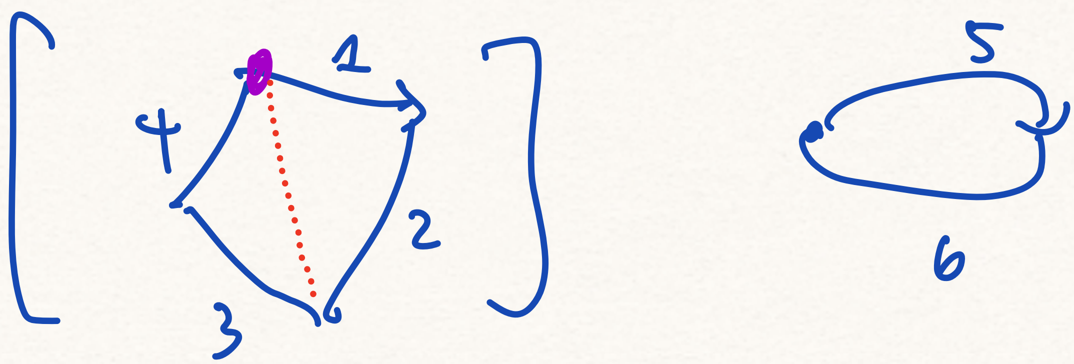
$$Wg_{N,n}(\sigma) = \mathbb{1}(\sigma(n)=n)\, N^{-1} Wg_{N,n-1}(\sigma^{\downarrow}) - N^{-1} \sum_{j=1}^{n-1} Wg_{N,n}((j\ n)\sigma).$$

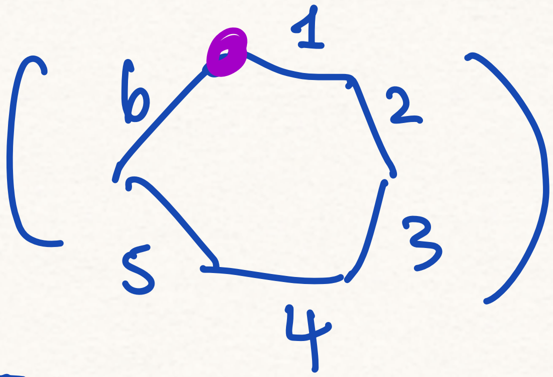
This turns out to imply the Master loop eq.

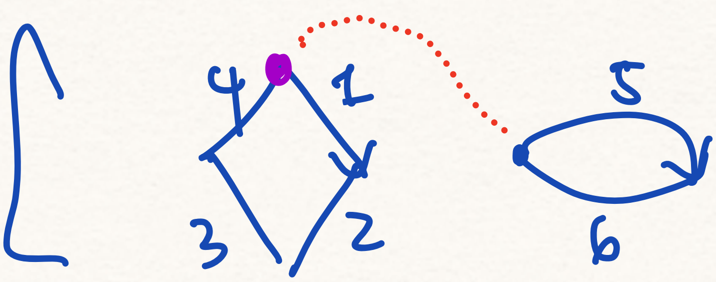
Visually, in terms of the blue faces, this recursion is interpreted as follows:

$\mathrm{Wg}_{N,3}$


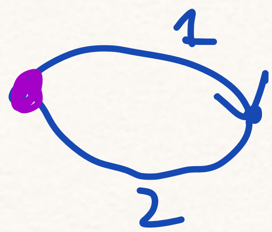
$= -\frac{1}{N}\,\mathrm{Wg}_{N,3}$


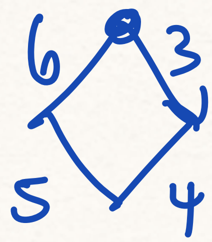


$-\frac{1}{N}\,\mathrm{Wg}_{N,3}$


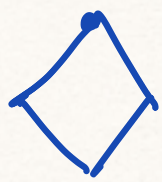


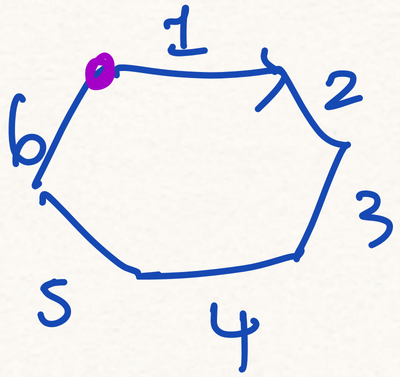
Another example:

$Wg_{N,3}\Big($ 

, 

$\Big)$

$= \frac{1}{N} Wg_{N,2}\Big($  $\Big)$

$- \frac{1}{N} Wg_{N,3}\Big($ 

$\Big)$.

Recall normalized Weingarten:

$$\overline{Wg_{N,n}(\sigma)} = N^{2n-\#\text{cycles}(\sigma)} Wg_{N,n}(\sigma).$$

Note:

$$2n - \#\text{cycles}(\sigma) = 2(n-1)+2-\#\text{cycles}(\sigma)$$

$$= 2(n-1)+2 - (\#\text{cycles}(\sigma^{\downarrow})+1)$$

$$= 2(n-1) - \#\text{cycles}(\sigma^{\downarrow}) + 1.$$

Thus, can rewrite Wg recursion in terms of normalized Wg fn:

$$\overline{Wg_{N,n}(\sigma)} = \mathbb{1}(\sigma(n)=n)\,\overline{Wg_{N,n-1}(\sigma^{\downarrow})} - \sum_{\substack{1\le j\le n \\ j,n \text{ in} \\ \text{same cycle of } \sigma}} \overline{Wg_{N,n}}((j\,n)\sigma) - \frac{1}{N^2}\sum_{\substack{1\le j\le n \\ j,n \text{ in diff.} \\ \text{cycles of } \sigma}} \overline{Wg_{N,n}}((j\,n)\sigma)$$

Note: this eq. has a large-N limit, and thus can expect that $\overline{W_{g_{N,n}}}$ to also have a large-N limit as well.

In the following, we will suppose that this reccursion also works when $N<n$. We will return to this at the end. [Short answer is that it still holds.]

Word recursion

Recall abstract setting:

- finite set of letters $\{\lambda_1, \dots, \lambda_L\}$
- finite collection of words $\Gamma = (\Gamma_1, \dots, \Gamma_k)$

$$\Gamma_i = \lambda_{c_i(1)}^{\varepsilon_i(1)} \cdots \lambda_{c_i(m_i)}^{\varepsilon_i(m_i)} .$$

Compute

$$\mathbb{E}[\mathrm{Tr}(U(\Gamma))] = \mathbb{E}[\mathrm{Tr}(U(\Gamma_1)) \cdots \mathrm{Tr}(U(\Gamma_n))]$$

$$= \sum_{\substack{(\sigma_i, \tau_i) \\ 1 \le i \le L}} N^{\#CC(\Gamma, (\sigma_i, \tau_i), i \in [L])} \prod_{i \in [L]} \mathrm{Wg}_N(\sigma_i^{-1} \tau_i),$$

here M is the assoc. map w/ external faces labelled $\Gamma_1, \dots, \Gamma_n$.

WLOG, let λ_1 be first letter of first word Γ_1. Insert Wg_N recursion for λ_1.

$$= \frac{1}{N} \sum_{\substack{(\sigma_i,\tau_i) \\ 1 \le i \le L}} N^{\#cc(\Gamma, (\sigma_i,\tau_i), i \in [L])} \prod_{2 \le i \le L} Wg_N(\sigma_i^{-1}\tau_i)$$

$$\mathbb{1}(\sigma_1(n_1) = \tau_1(n_1)) \, Wg_N((\sigma_1^{-1}\tau_1)^{\downarrow})$$

$$- \frac{1}{N} \sum_{j=1}^{n_1-1} \sum_{\substack{(\sigma_i,\tau_i) \\ 1 \le i \le L}} N^{\#cc(\Gamma, (\sigma_i,\tau_i), i \in [L])} \times \prod_{2 \le i \le L} Wg_N(\sigma_i^{-1}\tau_i)$$

$$\times \; Wg((j \; n_1)\sigma_1^{-1}\tau_1)$$

In 2^{nd} sum, change variables

$$\sigma_1 \mapsto (j\ n_1)\sigma_1^{-1}.$$

Then further obtain

$$= \frac{1}{N} \sum_{\substack{(\sigma_i,\tau_i)\\ 1\le i\le L}} N^{\#CC(I,(\sigma_i,\tau_i),\, i\in[L])} \prod_{2\le i\le L} Wg_N(\sigma_i^{-1}\tau_i)$$

$$\mathbb{1}(\sigma_1(n_1)=\tau_1(n_1))\, Wg_N((\sigma_1^{-1}\tau_1)^{\downarrow})$$

$$-\frac{1}{N} \sum_{j=1}^{n_1-1} \sum_{\substack{(\sigma_i,\tau_i)\\ 1\le i\le L}} N^{\#CC(I,((j\,n_1)\sigma_1,\tau_1),(\sigma_i,\tau_i),2\le i\le L)} \times \prod_{1\le i\le L} Wg_N(\sigma_i^{-1}\tau_i)$$

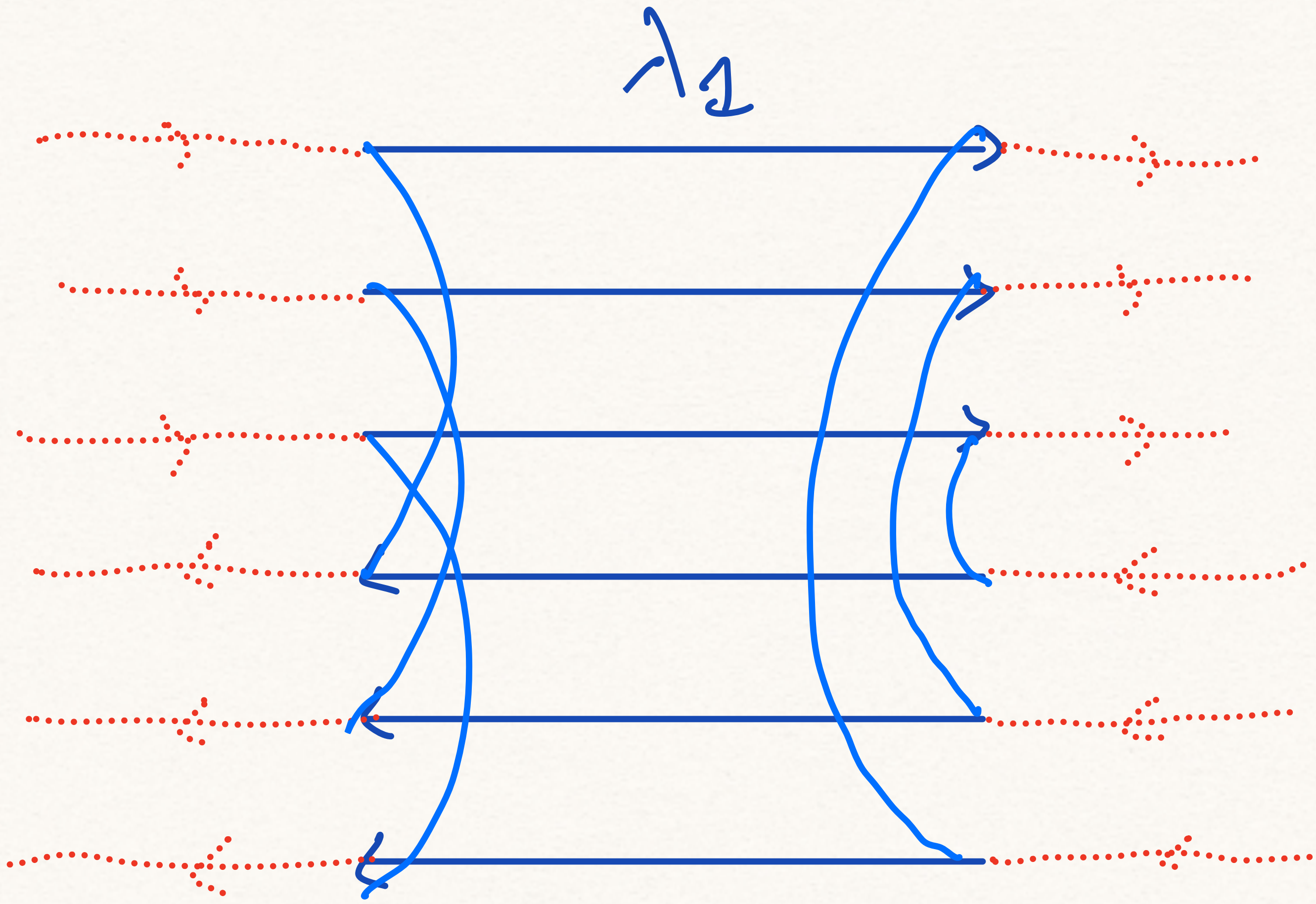
For each term, find new words I' st

$$\#CC(I,((j\,n_1)\sigma_1,\tau_1),(\sigma_i,\tau_i),\, 2\le i\le L)$$

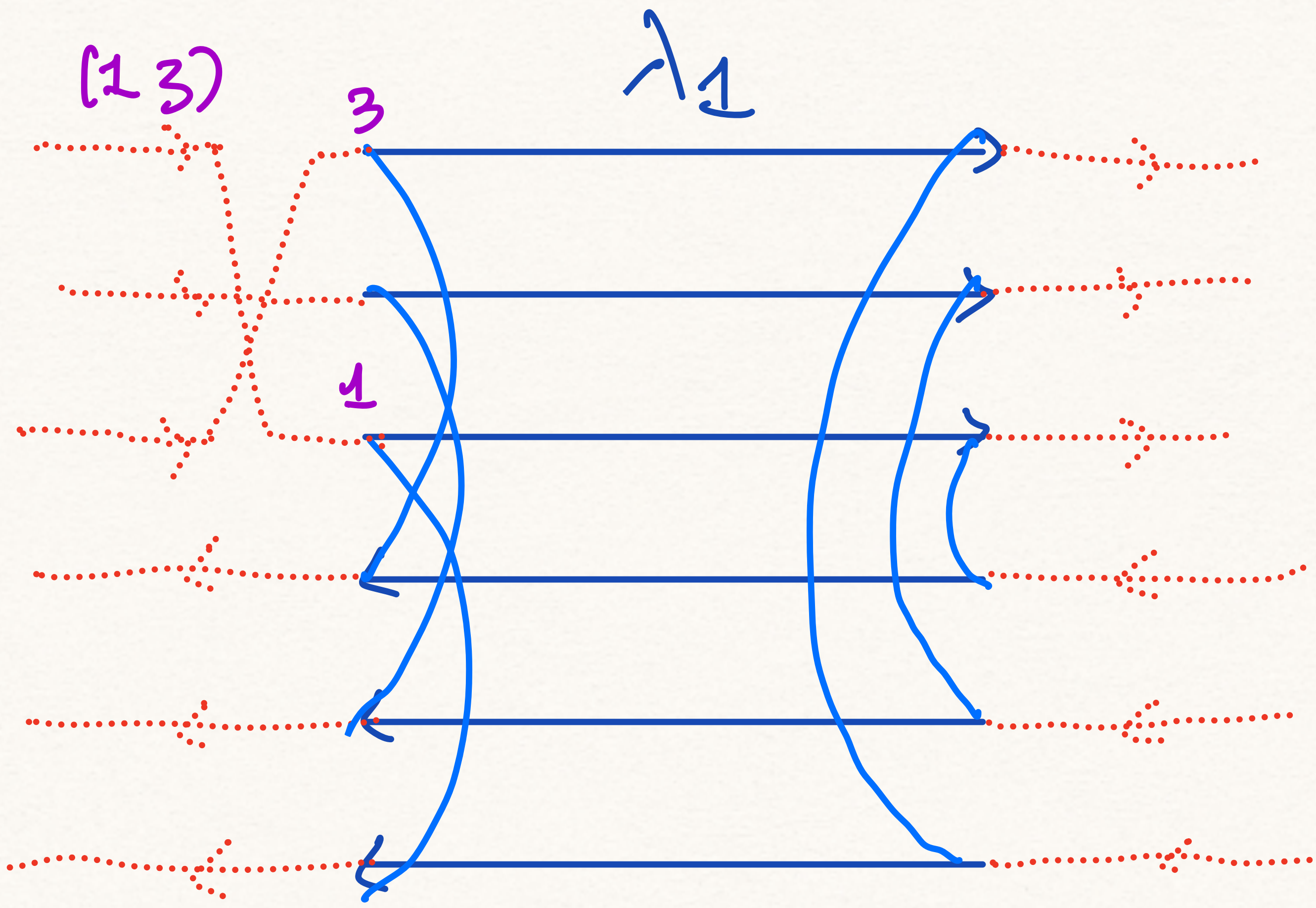
$$= \#CC(I',(\sigma_i,\tau_i),\, i\in[L])$$

Visualize as follows:

Case: $(\sigma_1^{-1}\tau_1)(n_1) \neq n_1$, i.e. $\sigma_1(n_1) \neq \tau(n_1)$



Think of $(j\ n_1)$ as swapping the top left dashed red w/ the jth from the top dashed red:

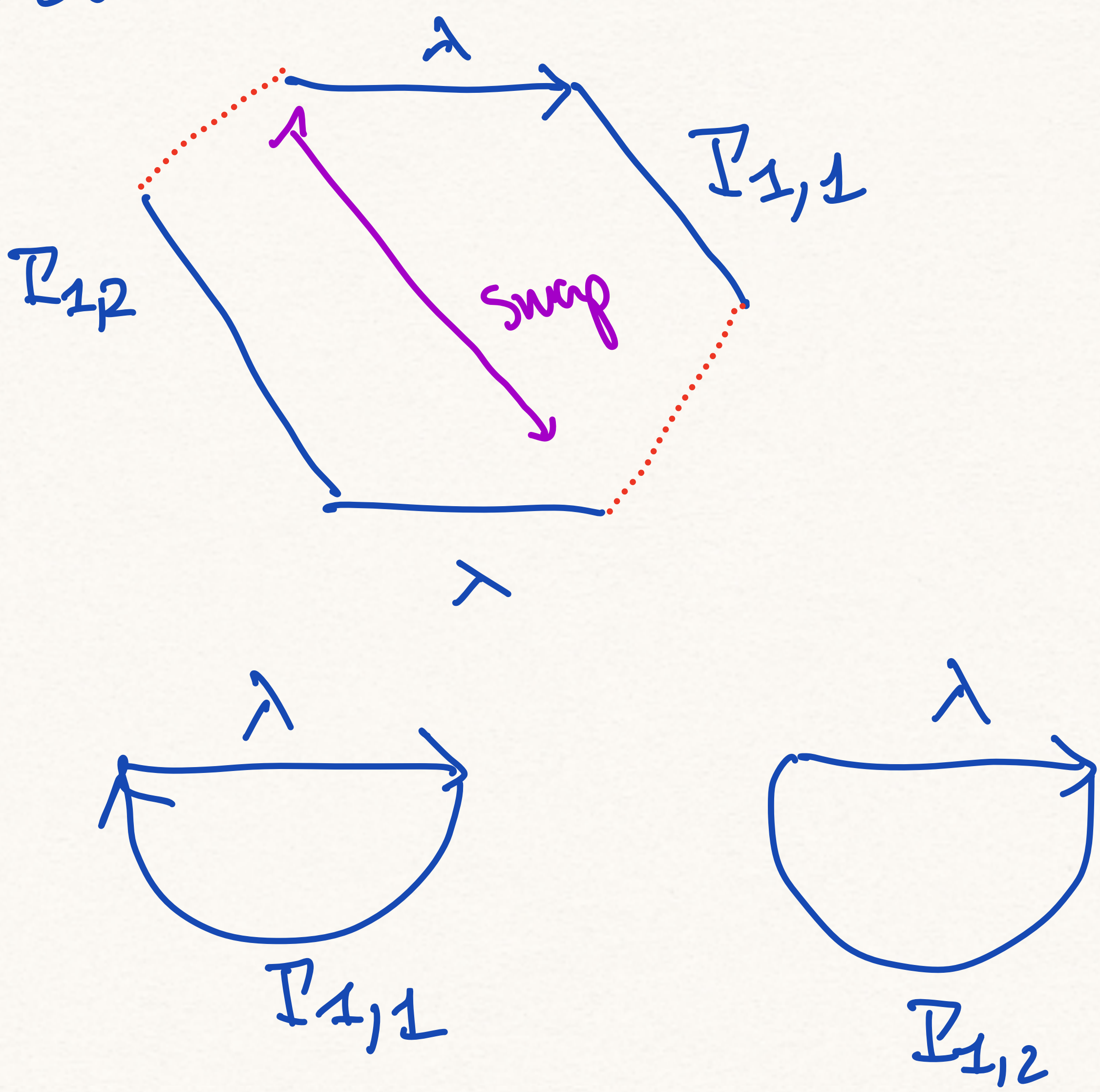


Recall: dashed red strands come from the words Γ. If change the strands, results in change in words. In what way do they change?

Case 1: two strands on same word Γ_1. Then $\Gamma_1 = \lambda \Gamma_{1,1} \lambda \Gamma_{1,2}$. By swapping, break Γ_1 into 2 words:

$$I_1 \mapsto I_{1,1}, \ I_{1,2} \ .$$

Visualize:

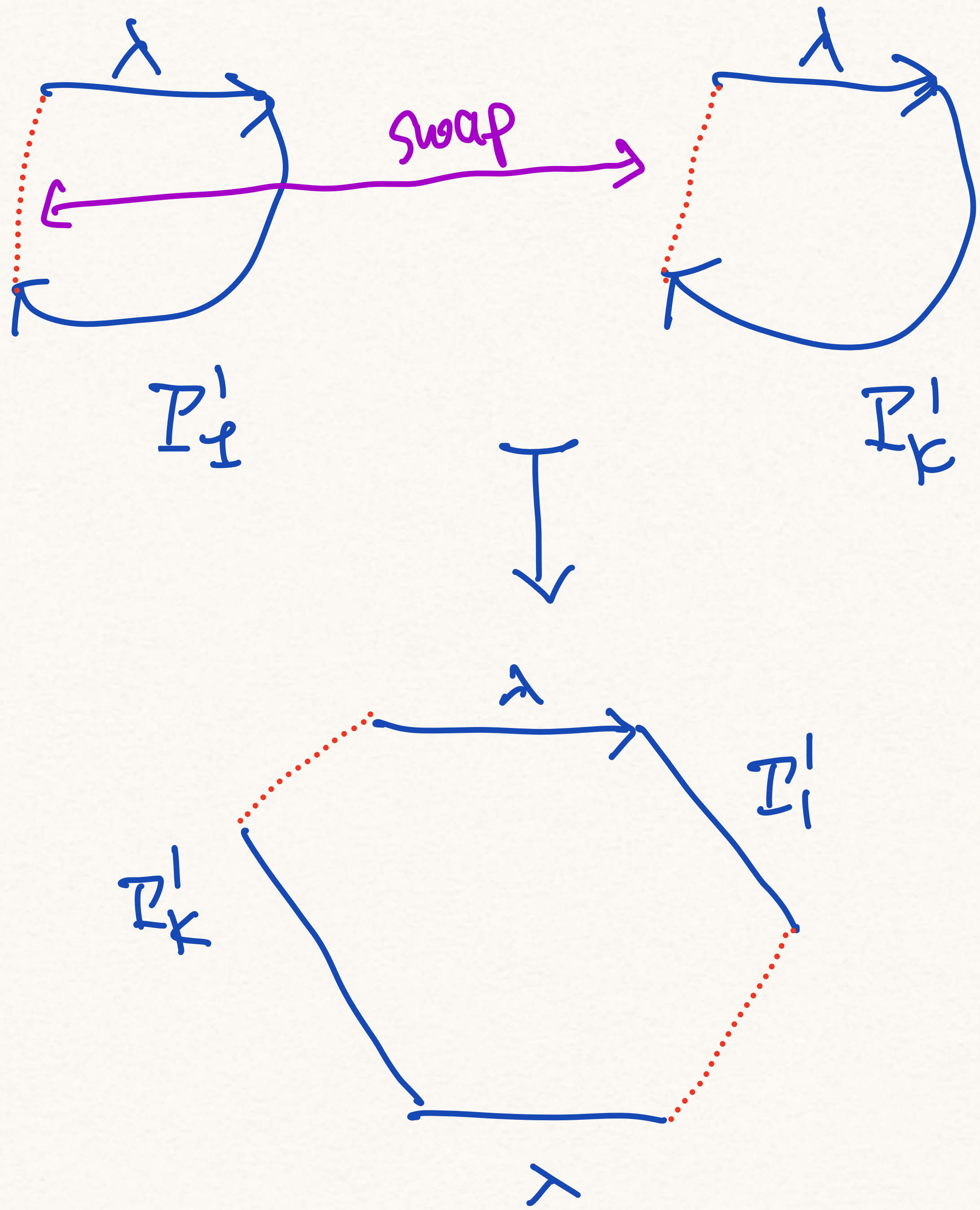


Thus,

$$I' = (I_{1,1}, I_{1,2}, I_2, \dots, I_n).$$

Case 2: two strands on different words, Γ_1, Γ_k. Then $\Gamma_1 = \lambda \Gamma_1'$, $\Gamma_k = \lambda \Gamma_k'$, then two words merge:

$$\Gamma_1, \Gamma_k \longmapsto \lambda \Gamma_k' \lambda \Gamma_1'$$



Thus,

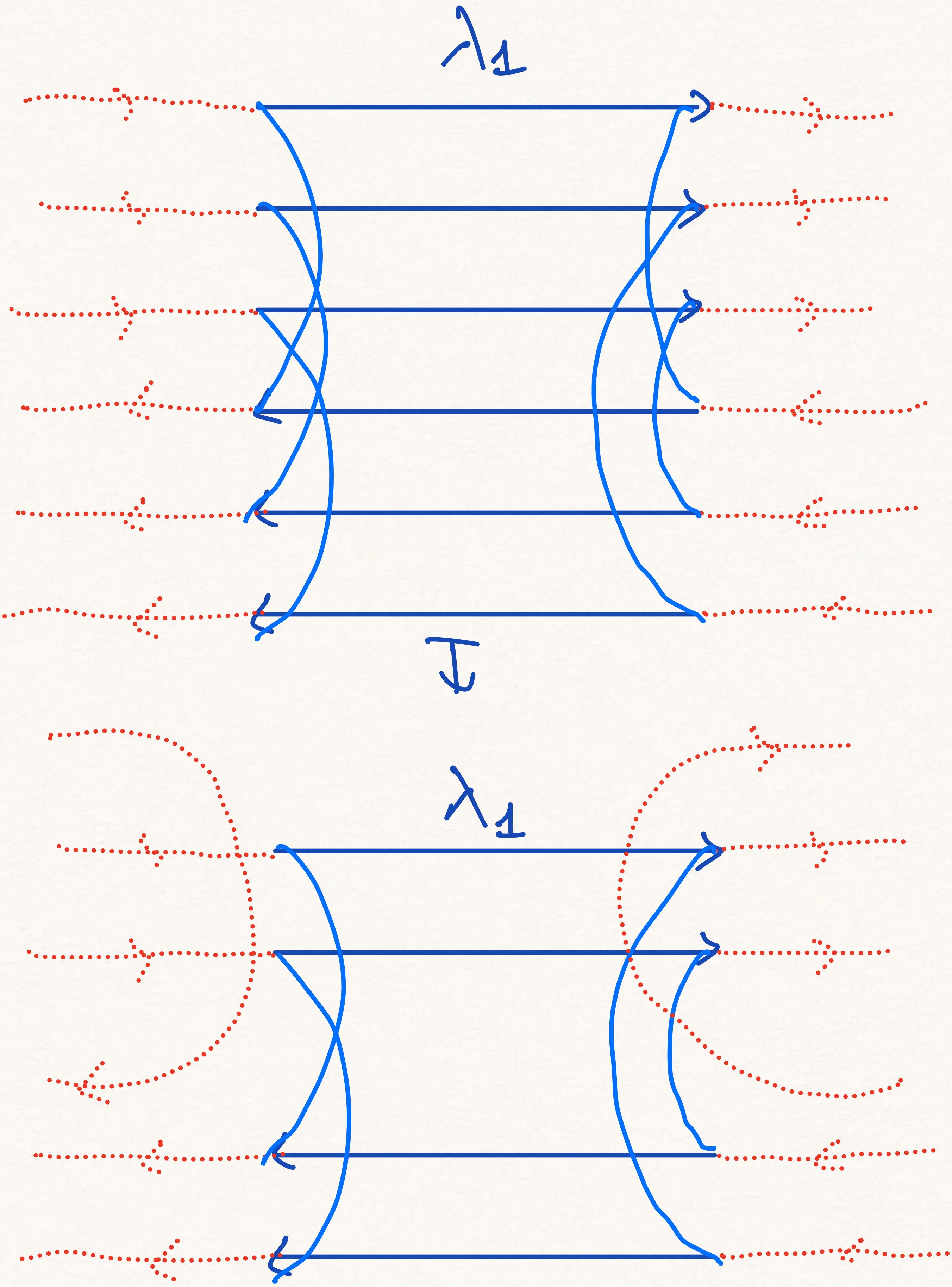
hat means omission

$$\Gamma' = (\lambda\Gamma'_k\lambda\Gamma'_1, \Gamma_2, \ldots, \widehat{\Gamma_k}, \ldots, \Gamma_n)$$

Case: $(\sigma_1^{-1}\tau_1)(n_1) = n_1$, i.e. $\sigma_1(n_1) = \tau_1(n_1)$.
Then recursion contains prev. cases as before, and an add'l case (from the term

$$\mathbb{1}(\sigma_1(n_1) = \sigma_2(n_2))\, Wg_{N, n_1-1}((\sigma_1^{-1}\tau_1)^{\downarrow})).$$

Visualize this case as follows:

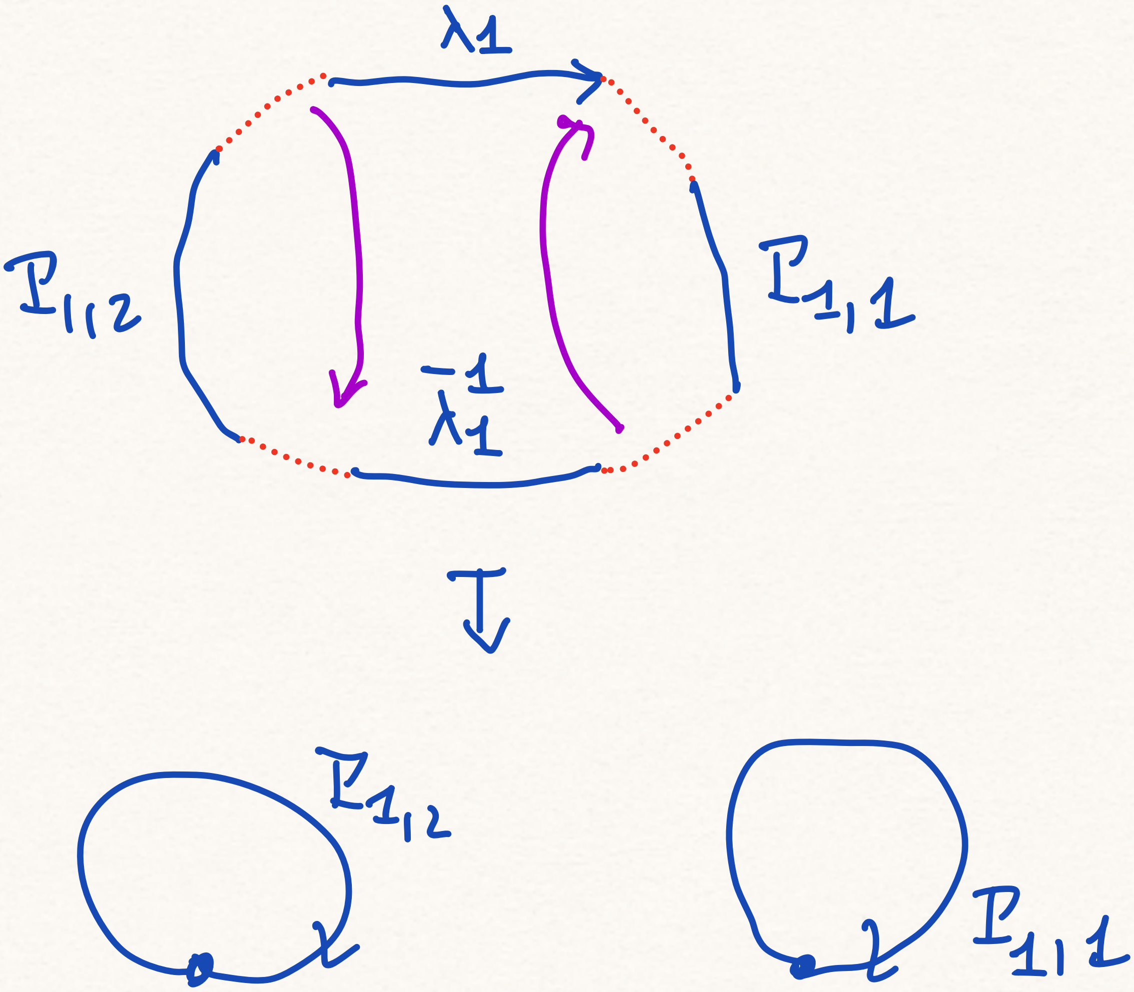


I.e., going from $\tilde{\sigma}_1^{-1}\tau_1$ to $(\tilde{\sigma}_1^{-1}\tau_1)^{\downarrow}$ corr. to deleting the topmost right-directed edge, as well the $\sigma(n_1) = \tau(n_2)$ from the top left-directed edge.

This again splits into two cases, dep. on whether these edges were on the same word or not.

Case 1: deleted strands on same word, Γ_1. Then $\Gamma_1 = \lambda_1 \Gamma_{1,1} \tilde{\lambda}_1^{-1} \Gamma_{1,2}$, then deleting results in splitting:

$$\Gamma_1 \mapsto \Gamma_{1,1}, \Gamma_{1,2}.$$

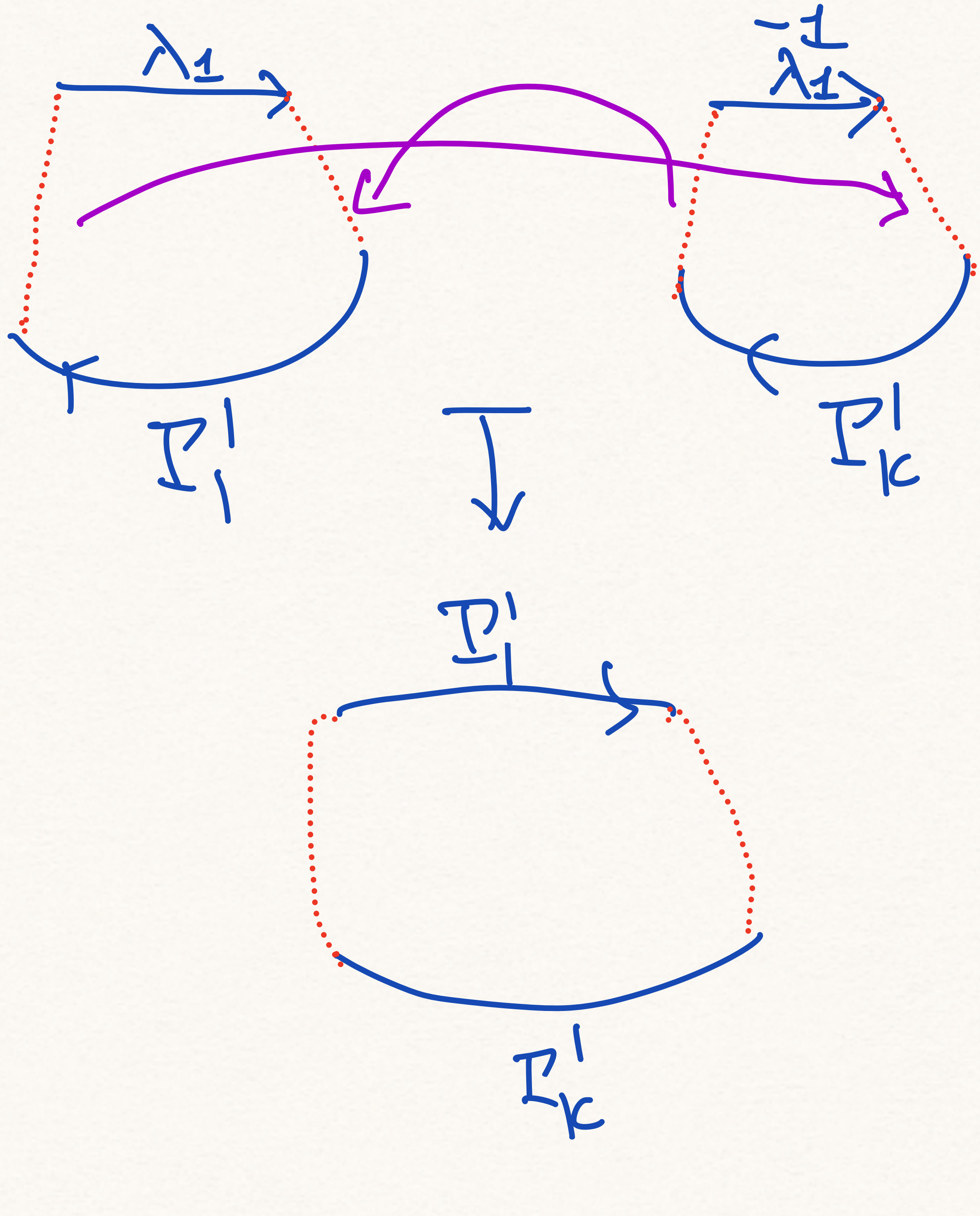


Thus, $I' = (I_{1,2}, I_{1,1}, I_2, \dots, I_n)$.
Let $\sigma_1', \tau_1' \in S_{n_1-1}$ be the perms obtained by starting from σ_1, τ_1 and omitting $\sigma(n_1), \tau(n_1)$. Then

$$\#CC(I, (\sigma_i, \tau_i), i \in [L])$$
$$= \#CC(I', (\sigma', \tau_1'), (\sigma_i, \tau_i), 2 \le i \le L).$$

Case 2: deleted strands on different words, $\Gamma_1 = \lambda \Gamma_1'$, $\Gamma_k = \lambda^{-1} \Gamma_k'$. Then deleting results in merger:

$$\lambda_1 \Gamma_1', \ \lambda_1^{-1} \Gamma_k' \mapsto \Gamma_1' \Gamma_k' .$$



In this case,

$$I' = (I_1' I_k', I_2, \dots, \hat{I}_k, \dots, I_n).$$

In summary, we obtain the following word recursion:

$$\boxed{\begin{aligned} &\mathbb{E}[\mathrm{Tr}(U(I))] \\ &= \mp \frac{1}{N} \sum_{I' \in \mathbb{S}_\pm(I)} \mathbb{E}[\mathrm{Tr}(U(I'))] \\ &\quad \mp \frac{1}{N} \sum_{I' \in \mathbb{M}_\pm(I)} \mathbb{E}[\mathrm{Tr}(U(I'))]. \end{aligned}}$$

Here, the sets $\mathbb{S}_\pm$ (pos. & neg. splittings), $\mathbb{M}_\pm$ (pos. & neg. mergers) are defined as:

$\mathbb{S}_+$ collects all cases

$$I_1 = \lambda_1 I_{1,1} \lambda_1 I_{1,2} \mapsto \lambda_1 I_{1,1}, \lambda_1 I_{1,2},$$

$\mathbb{S}_-$ collects all cases

$$I_1 = \lambda_1 I_{1,1} \lambda_1^{-1} I_{1,2} \mapsto I_{1,1}, I_{1,2},$$

$\mathbb{M}_+$ collects all cases

$$I_1 = \lambda_1 I_1', \; I_k = \lambda_1 I_k' \mapsto \lambda_1 I_k' \lambda_1 I_1',$$

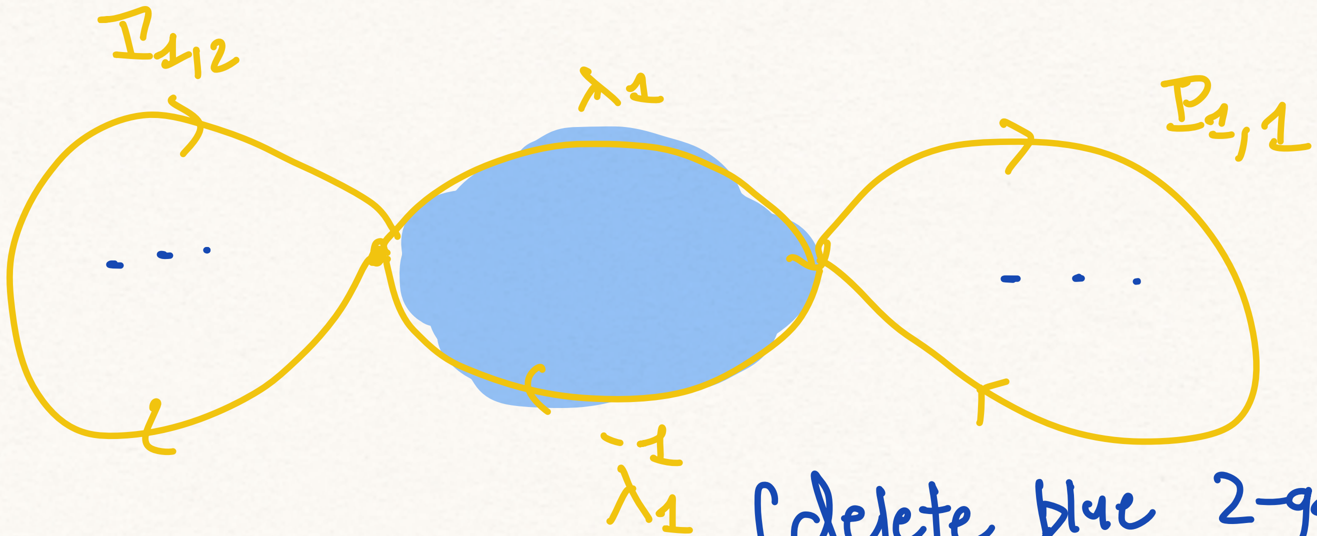
$\mathbb{M}_-$ collects all cases

$$I_1 = \lambda_1 I_1', \; I_k = \lambda_1^{-1} I_k' \mapsto I_1' I_k'.$$

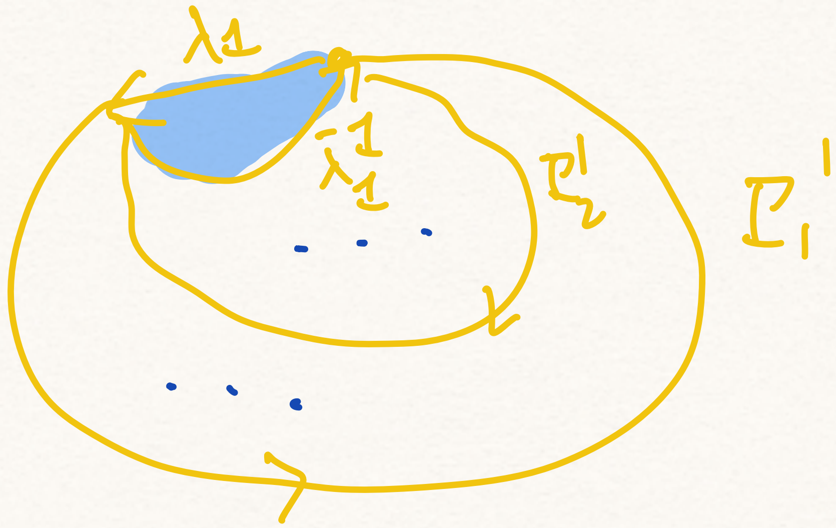
{all other words in I remain same}

Remark.

Can also visualize in terms of the map $\mathcal{M}$. When $\sigma(n) = \tau(n)$, then a blue 2-gon is glued to λ:



$\downarrow$ [delete blue 2-gon]



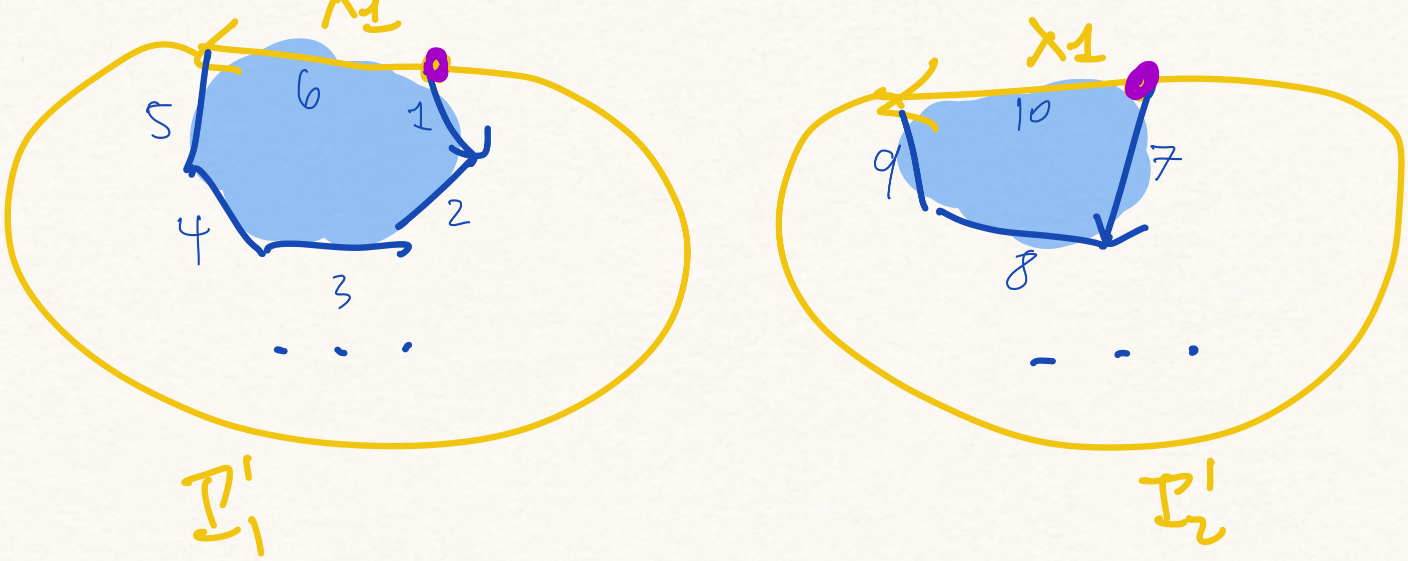
$\downarrow$ [delete blue 2-gon]

Exercise: draw pictures for the other cases.

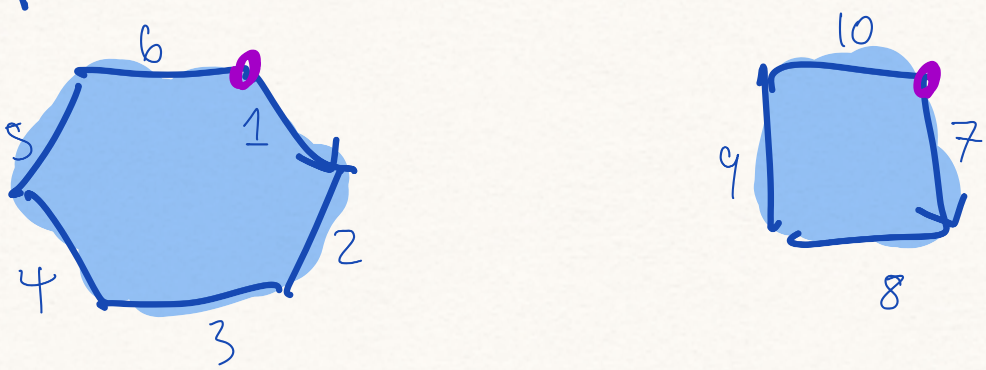
Note: this may be thought of as a surface exploration, i.e. peeling process. At each step, we select some bdry edge, reveal the blue face glued to that edge, then use the Wg_N recursion to replace the blue face w/ contr. of modified blue faces (obtained by splitting/merging blue faces, or deleting the blue face in the case it is a 2-gon).

Each possible operation results in a new map $\mathcal{M}'$ w/ a modified bdry.

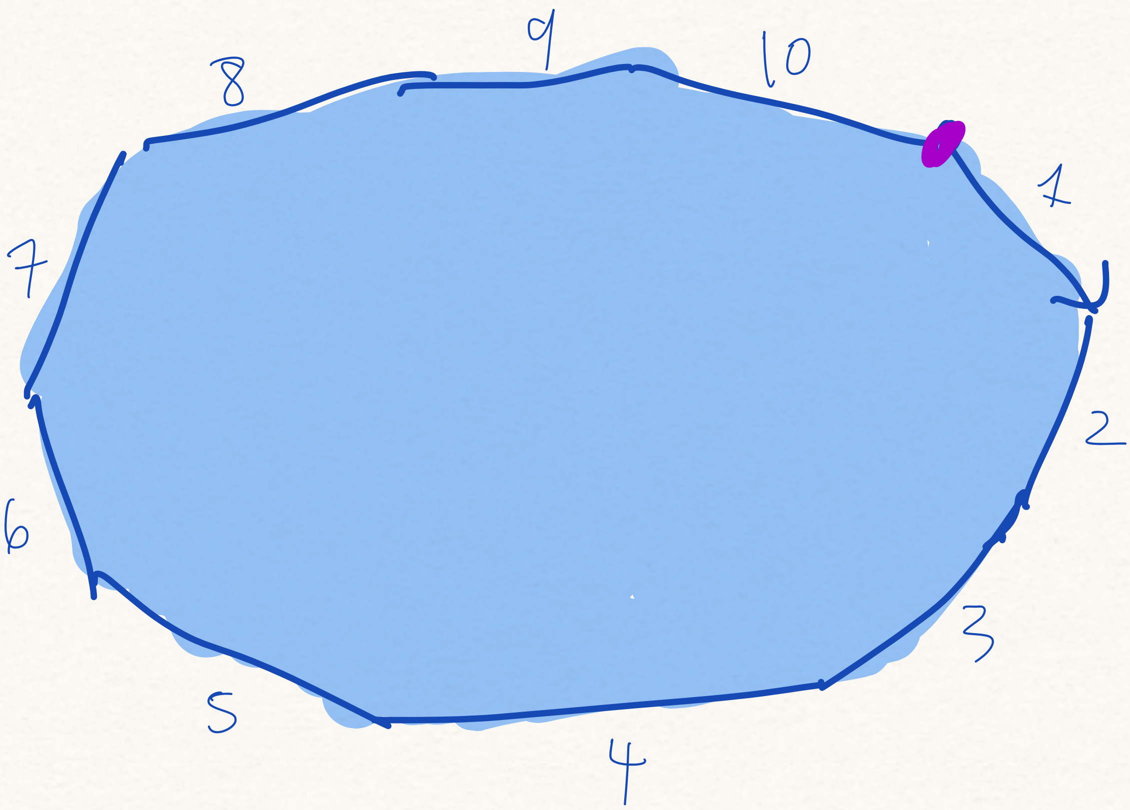
Another example (merger)



Replace

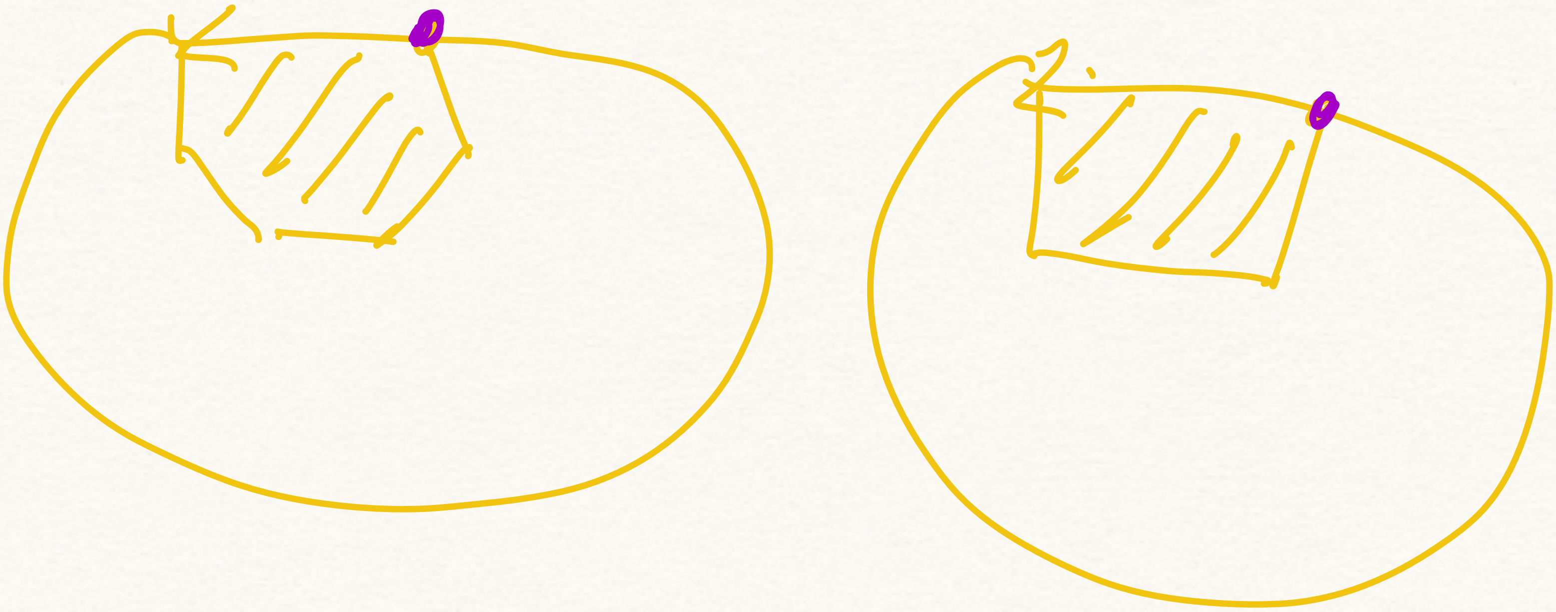


by

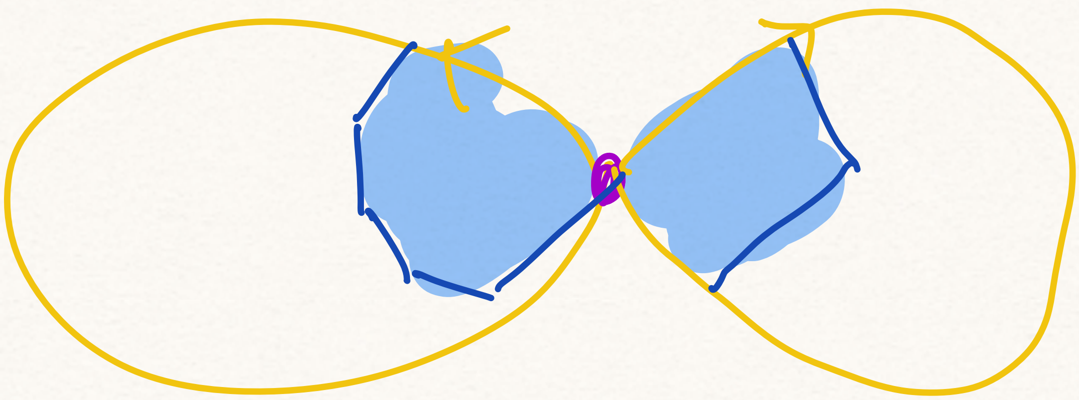


Think of resulting map $\mathcal{M}'$ as a map w/ bdry $\lambda \Sigma_1' \lambda \Sigma_2'$.

Visualize: first, cut out the blue faces:



Now glue in a new blue face connecting the two:



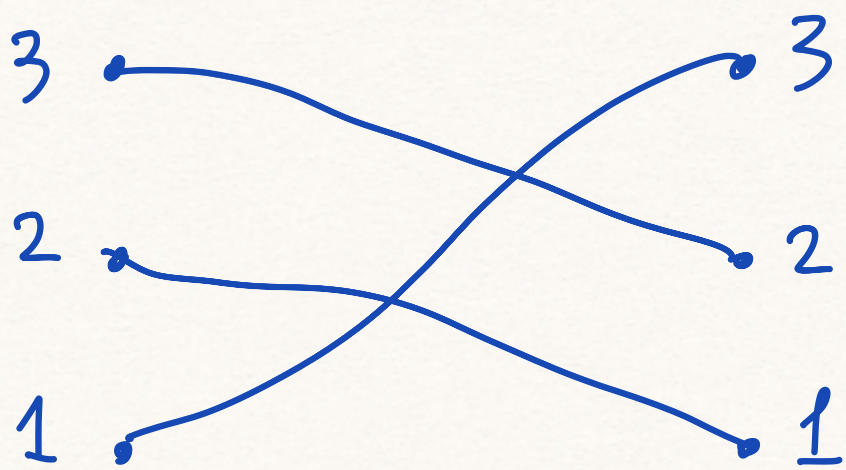
Recursion for general N

To discuss the Weingarten recursion for general N, we need some additional setup.

Define $\tilde{e} : S_n \to \mathrm{End}((\mathbb{C}^N)^{\otimes n})$ as follows. For $\sigma \in S_n$, $v_1, \ldots, v_n \in \mathbb{C}^N$, let

$$\tilde{e}(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

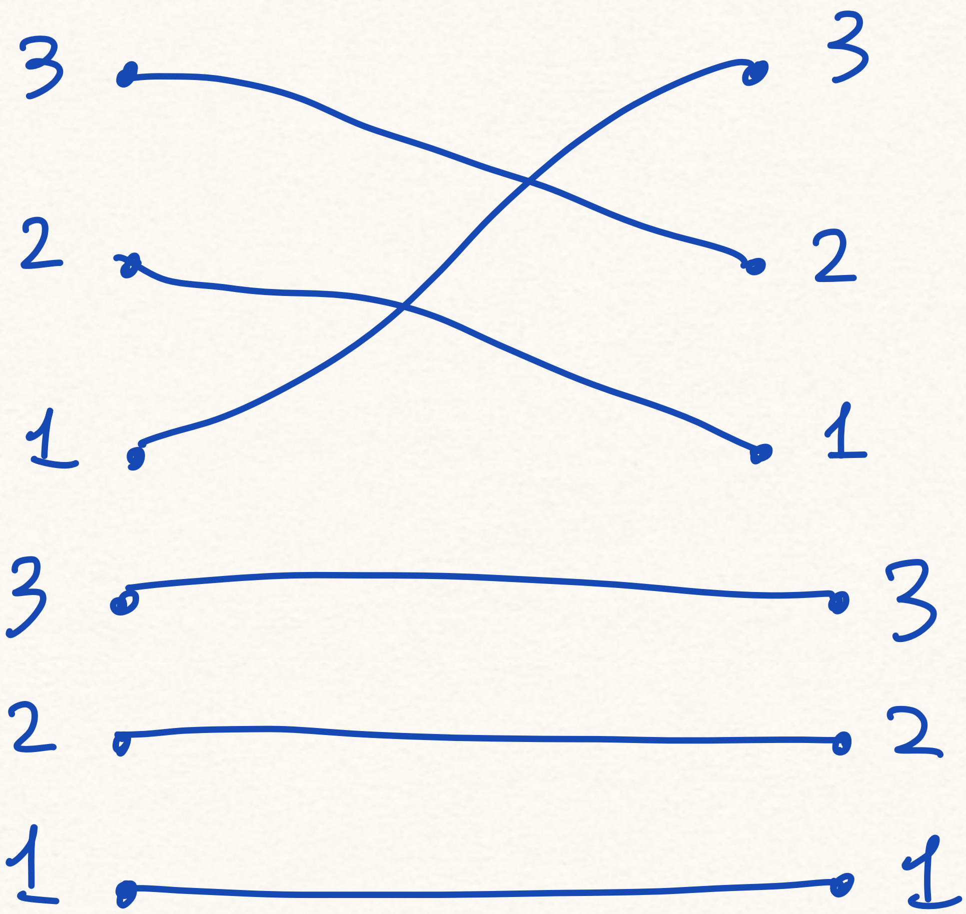
Visualize :



Define $e(\sigma) \in \mathrm{End}((\mathbb{C}^N)^{\otimes 2n})$ by

$$e(\sigma) = \tilde{e}(\sigma) \otimes I_N^{\otimes n},$$

here $I_N \in \mathrm{End}(\mathbb{C}^N)$ is the identity operator. Visualize:



.

Extend e to $\mathbb{C}[S_n]$ by linearity:

$$e\Big(\sum_{\sigma \in S_n} f(\sigma)\sigma\Big) = \sum_{\sigma \in S_n} f(\sigma) e(\sigma)$$

$$\in \mathrm{End}((\mathbb{C}^N)^{\otimes 2n}).$$

Recall that for general N, n,

$$N + J_n$$

may not be invertible in $\mathbb{C}[S_n]$.

Claim: for all $N, n \geq 1$

$$e(J_n) \geq -(N-1)I,$$

i.e. all eigenvals $\geq -(N-1)$.

In particular,

$$e(N + J_n)$$

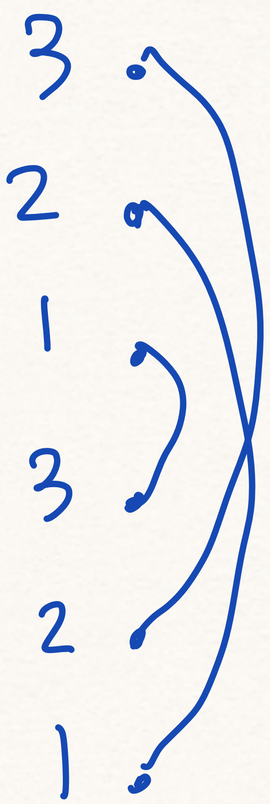
is always invertible.

We will not discuss the pf of this claim. It requires some rep theory (Schur-Weyl duality), see [CPS25, Section 2.3.1].

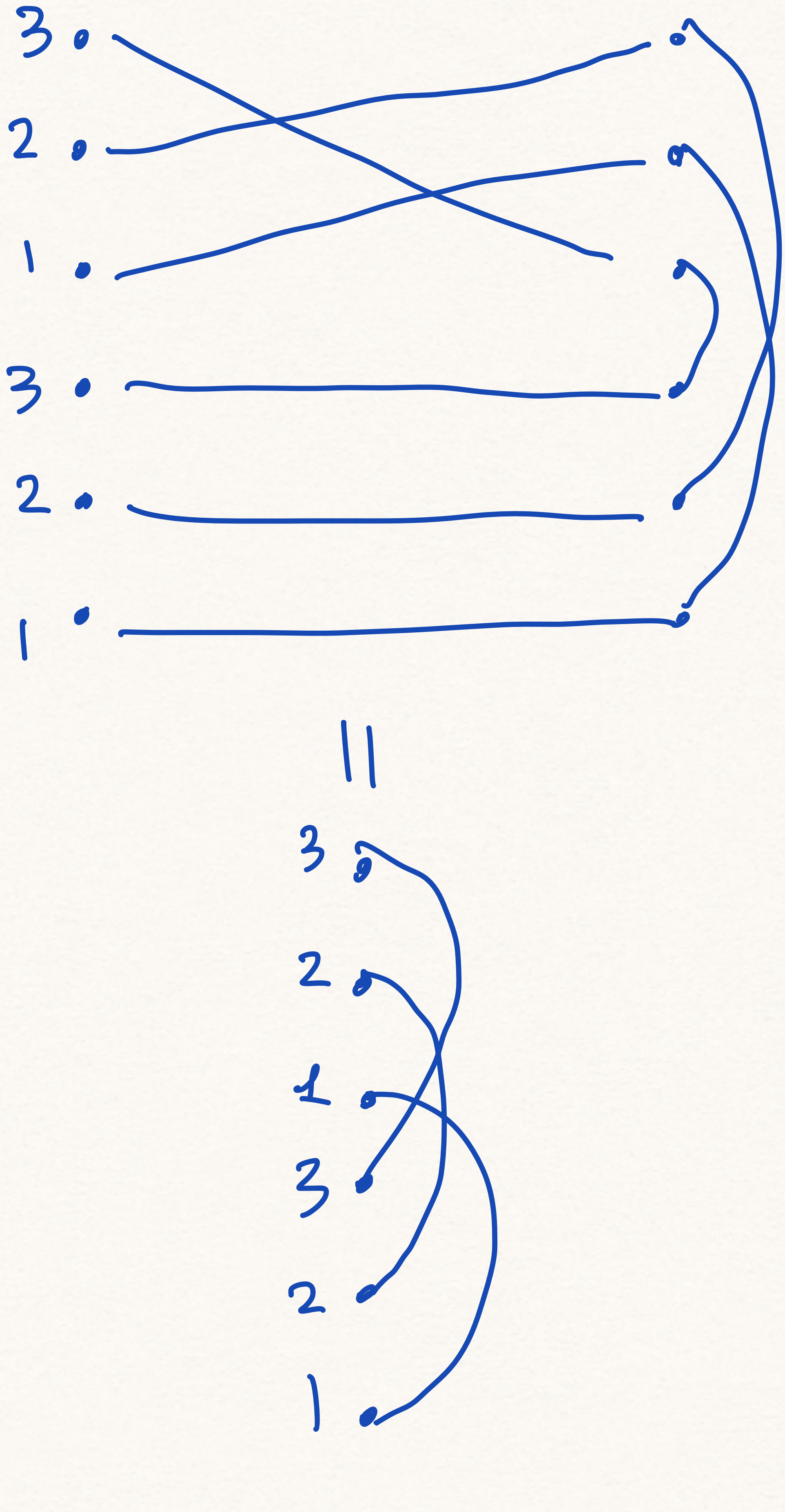
Given $\sigma \in S_n$, recall

$$v_\sigma = e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{i'_1} \otimes \cdots \otimes e_{i'_n} \, \delta^{i_1 i'_{\sigma(1)}} \cdots \delta^{i_n i'_{\sigma(n)}} \in (\mathbb{C}^N)^{\otimes n}$$

visualize:



Note: $\rho(\tau) v_\sigma = v_{\tau\sigma}$, i.e.



Lemma. For all $N, n \geq 1$, have that

$$\mathbb{E}[U^{\otimes n} \otimes \bar{U}^{\otimes n}]$$
$$= e(N+J_n)^{-1} \cdots e(N+J_1)^{-1} \sum_{\sigma \in S_n} v_\sigma v_\sigma^T .$$

Pf. Recall that

$$\mathbb{E}[U^{\otimes n} \otimes \bar{U}^{\otimes n}] = P,$$

where P is the orth. proj. onto $V = \text{span}(v_\sigma, \sigma \in S_n)$.

Thus, suff. to show that

$$\tilde{P} = e(N+J_n)^{-1} \cdots e(N+J_1)^{-1} \sum_{\sigma \in S_n} v_\sigma v_\sigma^T$$

is also the orth proj. onto V, i.e. show:

(1) $\tilde{P}^2 = \tilde{P}$

(2) $\tilde{P}^T = \tilde{P}$

(3) $\operatorname{Im}(\tilde{P}) = V$.

We leave (2) as an exercise and show (1), (3). For (3), let $\tau \in S_n$. Compute

$$\tilde{P} v_\tau = e(N+J_n)^{-1} \cdots e(N+J_1)^{-1} \sum_{\sigma \in S_n} v_\sigma v_\sigma^T v_\tau$$

$$= e(N+J_n)^{-1}\cdots e(N+J_1)^{-1}\sum_{\sigma\in S_n} v_\sigma N^{\#\mathrm{cycles}(\sigma^{-1}\tau)}$$

$$= e(N+J_n)^{-1}\cdots e(N+J_1)^{-1}\sum_{\sigma\in S_n} v_\sigma N^{\#\mathrm{cycles}(\tau\sigma^{-1})}$$

$$= e(N+J_n)^{-1}\cdots e(N+J_1)^{-1}\sum_{\sigma\in S_n} v_{\sigma^{-1}\tau} N^{\#\mathrm{cycles}(\sigma)}$$

$$= e(N+J_n)^{-1}\cdots e(N+J_1)^{-1}\sum_{\sigma\in S_n} e(\sigma^{-1}) N^{\#\mathrm{cycles}(\sigma)} v_\tau$$

$$= e(N+J_n)^{-1}\cdots e(N+J_1)^{-1}\sum_{\sigma\in S_n} e(\sigma) N^{\#\mathrm{cycles}(\sigma)} v_\tau .$$

Recall

$$(N+J_1)\cdots(N+J_n) = \sum_{\sigma\in S_n} N^{\#\text{cycles}(\sigma)}\,\sigma ,$$

thus

$$e(N+J_1)\cdots e(N+J_n) = \sum_{\sigma\in S_n} N^{\#\text{cycles}(\sigma)}\, e(\sigma) ,$$

thus further obtain

$$\tilde{P} v_\tau = v_\tau ,$$

and thus $\operatorname{Im}(\tilde{P}) = V$.

Similarly, for (1), we use (2) to compute

$$\tilde{P}^2 = \tilde{P} \sum_\tau v_\tau v_\tau^T\, e(N+J_n)^{-1} \cdots e(N+J_1)^{-1}$$

$$= \sum_\tau v_\tau v_\tau^T\, e(N+J_n)^{-1} \cdots e(N+J_1)^{-1}$$

$$= \tilde{P}, \text{ as desired.} \qquad \square$$

As before, using that

$$\frac{1}{1+x} = 1 - \frac{x}{1+x},$$

we have that $\forall\, N, n \geq 1$,

$$e(N+J_n)^{-1} = N^{-1} e(\mathrm{Id} + N^{-1} J_n)^{-1}$$

$$= N^{-1}\left(\mathrm{Id} - e(N^{-1}J_n)\, e(\mathrm{Id} + N^{-1}J_n)^{-1}\right)$$

$$= N^{-1}\mathrm{Id} - N^{-1} e(J_n)\, e(N + J_n)^{-1}.$$

From this, we may obtain the following result. This is the general Wg recursion.

Thm. For any $N, n \geq 1$, we have that

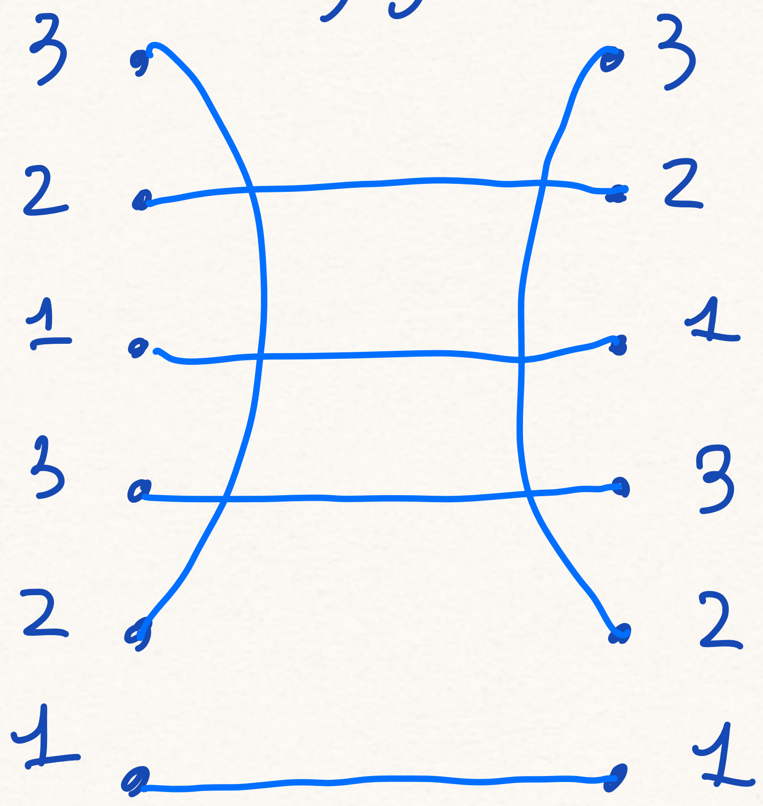
$$\mathbb{E}[U^{\otimes n} \otimes \bar{U}^{\otimes n}]$$

$$= -\frac{1}{N} \sum_{j=1}^{n-1} e((n\ j)) \mathbb{E}[U^{\otimes n} \otimes \bar{U}^{\otimes n}]$$

$$+ \frac{1}{N} \sum_{j=1}^{n} e(\langle n\ j \rangle) \mathbb{E}[U^{\otimes (n-1)} \otimes I_N \otimes \bar{U}^{\otimes (j-1)} \otimes I_N \otimes \bar{U}^{\otimes (n-j)}].$$

Here, $e(\langle n\ j \rangle) \in \mathrm{End}((\mathbb{C}^N)^{\otimes 2n})$ is defined visually as:

$n=3,\ j=2$



,

and in formulas by

$$e(\langle n\ j\rangle)(v_1\otimes\cdots\otimes v_{2n})$$

$$= \langle v_n, v_{n+j}\rangle \delta^{ab}\, v_1\otimes\cdots\otimes v_{n-1}\otimes e_a$$
$$\otimes v_{n+1}\otimes\cdots\otimes v_{n+j-1}\otimes e_b$$
$$\otimes v_{n+j+1}\otimes\cdots\otimes v_{2n}.$$

In words, we contract the n and

nt j comp., and replace by

$$\delta^{ab} e_a \otimes e_b .$$

Act as identity on all other comp.

Pf. Exercise using recursion [redacted] and lemma. $\square$