Ref: C.-Park-Sheffield "Random surfaces and lattice Yang-Mills"

Take $G = U(N)$ in following.

Recall $U(N)$ lattice YM theory:

$$d\mu_{\Lambda,N,\beta}(U)$$

$$= Z_{\Lambda,N,\beta}^{-1} \prod_{p \in \mathcal{P}_\Lambda} \exp(N\beta \operatorname{Tr}(U_p)) \prod_{e \in E_\Lambda^+} dU_e$$

oriented plaquettes   pos. oriented edges

This is a PM on $U(N)^{E_\Lambda^+}$.

For $e \in E_\Lambda^-$, define

Here, $U_e = U_{-e}^{-1}$.

$$Z_{\Lambda,N,\beta} = \int \prod_{p \in \mathcal{P}_\Lambda} \exp(N\beta \operatorname{Tr}(U_p)) \prod_{e \in E_\Lambda^+} dU_e$$

is the "partition fn".

Wilson loop observables: given loop

$$\gamma = e_1 \cdots e_n,$$

define

$$U_\gamma = U_{e_1} \cdots U_{e_n}$$

"holonomy of connection U around γ",

and define

$$W_\gamma(U) = \mathrm{tr}(U_\gamma) = \frac{1}{N}\mathrm{Tr}(U_\gamma).$$

Also, for collection of loops

$$s = (s_1, \ldots, s_n),$$

which we call a "string", we let

$$W_s(U) = W_{s_1}(U) \cdots W_{s_n}(U).$$

W_γ is a "Wilson loop observable", and W_s is a "Wilson string observable".

Main quantities of interest: Wilson loop & string expectations:

$$\langle W_\gamma \rangle_{\Lambda,N,\beta} = \mathbb{E}_{\Lambda,N,\beta}[W_\gamma(U)],$$

$$\langle W_s \rangle_{\Lambda,N,\beta} = \mathbb{E}_{\Lambda,N,\beta}[W_s(U)].$$

Thm. [C.-Park-Sheffield '25]
For any finite $\Lambda \subseteq \mathbb{Z}^d$, $N \geq 1$, $\beta \geq 0$, and string $s = (s_1, \ldots, s_n)$ contained in Λ, we have that

$$\langle W_s \rangle_{\Lambda,N,\beta}$$

Euler char.

$$= Z^{-1}_{\Lambda,N,\beta} \underbrace{\sum_{\mathcal{M}:\partial\mathcal{M}=s}}_{\text{sum over surfaces w/ bdry } s} \beta^{\text{area}(\mathcal{M})} N^{\chi(\mathcal{M})-n} w_N(\mathcal{M})$$

weight to be discussed later.

Additionally,

$$Z_{\Lambda,N,\beta} = \underbrace{\sum_{\mathcal{M}:\partial\mathcal{M}=\emptyset}}_{\text{sum over closed surfaces}} \beta^{\text{area}(\mathcal{M})} N^{\chi(\mathcal{M})} w_N(\mathcal{M}) .$$

In words, Wilson string expectations are expressed as ratios of weighted surface sums. These surfaces are embedded into $\mathbb{Z}^d$, and the weights depend on the area, Euler characteristic, and how the faces are glued together (the weight w_N — to be discussed later).

Comments.

- $\mathcal{M}$ may have many components. (Including those not touching S.)
- $\mathcal{M}$ is made out of gluing plaquettes to additional, abstract faces.
- $w(\mathcal{M})$ may be positive or negative.

The first main goal of the course will be to prove this result, and along the way, discuss the surface sums more precisely.

Beginning elements of the pf

We have that

$$\langle W_s \rangle_{\Lambda, N, \beta}$$

$$= \frac{1}{Z_{\Lambda, N, \beta}} \int W_s(U) \prod_{p \in P_\Lambda} \exp(N\beta \operatorname{Tr}(U_p)) \prod_{e \in E_\Lambda^+} dU_e.$$

For each $p \in P_\Lambda$, can expand out

$$\exp(N\beta \operatorname{Tr}(U_p)) = \sum_{K(p)=0}^{\infty} \frac{(N\beta)^{K(p)}}{K(p)!} \operatorname{Tr}(U_p)^{K(p)}.$$

Thus,

$$\prod_{p \in P_\Lambda} \exp(N\beta \operatorname{Tr}(U_p)) = \sum_{K : P_\Lambda \to \mathbb{N}} \prod_{p \in P_\Lambda} \frac{(N\beta)^{K(p)}}{K(p)!} \operatorname{Tr}(U_p)^{K(p)}.$$

Define

$$(N\beta)^K = \prod_{p \in \mathcal{P}_\Lambda} (N\beta)^{K(p)}$$

$$K! = \prod_{p \in \mathcal{P}_\Lambda} K(p)!,$$

so that

$$= \sum_{K : \mathcal{P}_\Lambda \to \mathbb{N}} \frac{(N\beta)^K}{K!} \prod_{p \in \mathcal{P}_\Lambda} \mathrm{Tr}(U_p)^{K(p)}.$$

Thus, have that

$$\int W_s(U) \prod_{p \in \mathcal{P}_\Lambda} \exp(N\beta \,\mathrm{Tr}(U_p)) \prod_{e \in E_\Lambda^+} dU_e$$

$$= \sum_{K : \mathcal{P}_\Lambda \to \mathbb{N}} \frac{(N\beta)^K}{K!} \int W_s(U) \prod_{p \in \mathcal{P}_\Lambda} \mathrm{Tr}(U_p)^{K(p)} \prod_{e \in E_\Lambda^+} dU_e.$$

Note: $|\mathrm{Tr}(U)| \leq N$, thus

$$\left| \int w_S(U) \prod_{p \in P_\Lambda} \mathrm{Tr}(U_p)^{K(p)} \prod_{e \in E_\Lambda^+} dU_e \right|$$

$$\leq N^K,$$

thus

$$\sum_{K : P_\Lambda \to \mathbb{N}} \frac{(N\beta)^K}{K!} N^K = \exp(N^2 \beta |P_\Lambda|) < \infty,$$

thus have absolute convergence.

Fix $K : P_\Lambda \to \mathbb{N}$. Our task now is to provide a surface sum formula for

$$I(s,K) = \int W_s(U) \prod_{p \in P_\Lambda} \mathrm{Tr}(U_p)^{K(p)} \prod_{e \in E_\Lambda^+} dU_e .$$

We will obtain a weighted sum over surfaces w/ bdry s and area $\sum_p K(p)$. Summing in K will then give the thm.

To obtain the formula, we will work in a more general setting, which we turn to next.

Unitary matrix integrals

Consider abstract setting:

- finite set of letters $\{\lambda_1, \dots, \lambda_L\}$
- finite collection of words $\Gamma = (\Gamma_1, \dots, \Gamma_k)$

$$\Gamma_i = \lambda_{c_i(1)}^{\varepsilon_i(1)} \cdots \lambda_{c_i(m_i)}^{\varepsilon_i(m_i)} .$$

Let $U_{\lambda_1}, \dots, U_{\lambda_L} \overset{iid}{\sim} U(N)$. Let

$$U(\Gamma_i) = U_{\lambda_{c_i(1)}}^{\varepsilon_i(1)} \cdots U_{\lambda_{c_i(m_i)}}^{\varepsilon_i(m_i)} .$$

$$U(\Gamma) = U(\Gamma_1) \cdots U(\Gamma_k)$$

Compute

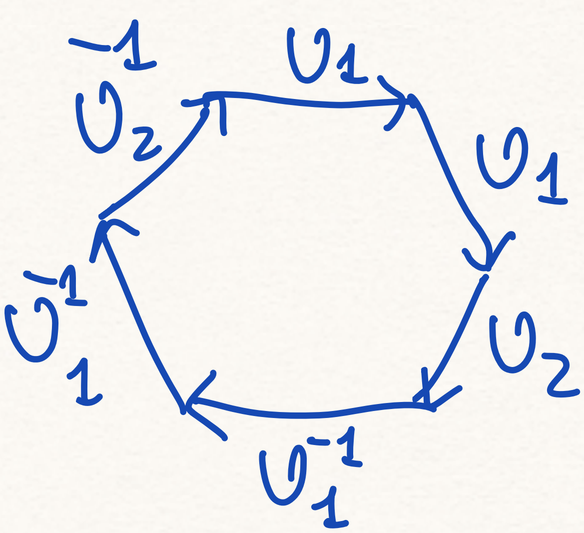
$$\mathbb{E}[\mathrm{Tr}(U(\Gamma))] = \mathbb{E}[\mathrm{Tr}(U(\Gamma_1)) \cdots \mathrm{Tr}(U(\Gamma_k))]$$

Step 1. Expand out all traces.
View trace as diagrammatic sum as follows.

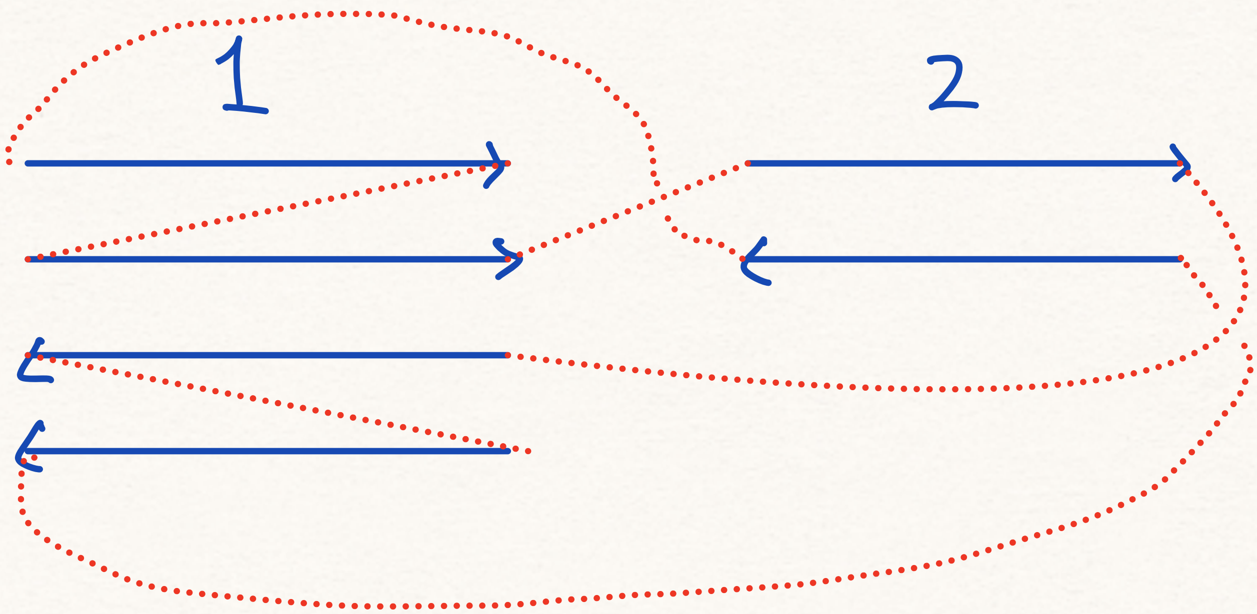
Consider explicit example

$$\Lambda = \{\lambda_1, \lambda_2\}$$
$$\Gamma = \lambda_1^2 \lambda_2 \lambda_1^{-2} \lambda_2^{-1}$$
$$\mathrm{Tr}(U(\Gamma)) = \mathrm{Tr}(U_1^2 U_2 U_1^{-2} U_2^{-1})$$

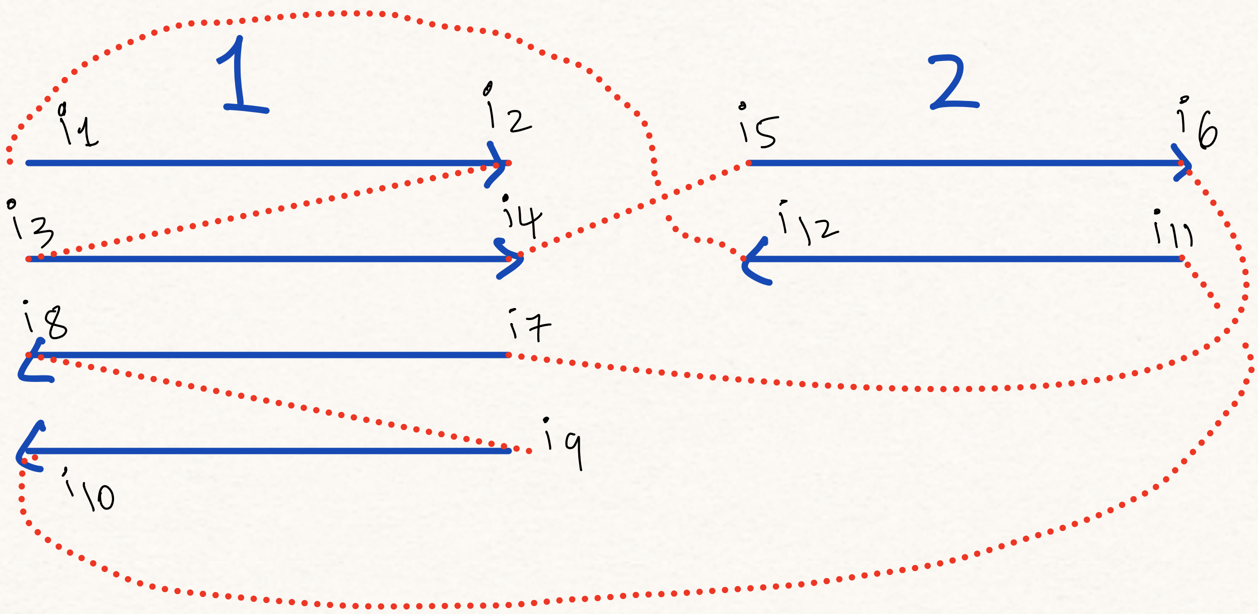
Start w/ single face



Draw corresponding diagram:



Label each vertex by index:



Then (where implicit summation over repeated indices)

$$\mathrm{Tr}(U[\Gamma]) = (U_1)_{i_1 i_2}\,\delta^{i_2 i_3}(U_1)_{i_3 i_4}\,\delta^{i_4 i_5}(U_2)_{i_5 i_6}\,\delta^{i_6 i_7}$$

$$(U_1^{-1})_{i_7 i_8}\,\delta^{i_8 i_9}(U_2^{-1})_{i_9 i_{10}}\,\delta^{i_{10} i_{11}}(U_2^{-1})_{i_{11} i_{12}}\,\delta^{i_{12} i_1}$$

$$= (U_1)_{i_1 i_2}\,\delta^{i_2 i_3}(U_1)_{i_3 i_4}\,\delta^{i_4 i_5}(U_2)_{i_5 i_6}\,\delta^{i_6 i_7}$$
$$\overline{(U_1)}_{i_8 i_7}\,\delta^{i_8 i_9}\,\overline{(U_2)}_{i_{10} i_9}\,\delta^{i_{10} i_{11}}\,\overline{(U_2)}_{i_{12} i_{11}}\,\delta^{i_{12} i_1}$$

Looking ahead, write this in a way that generaizes.

Let $B(\Gamma)=$ set of strands. Split

$$B(\Gamma) = B_+(\Gamma) \cup B_-(\Gamma)$$

right-directed strands

left-directed strands

In ongoing example:

$$B_+(\Gamma) = \{(1,2),\ (3,4),\ (5,6)\}$$

$$B_-(\Gamma) = \{(7,8),\ (9,10),\ (11,12)\}$$

Let $R(\Gamma)=$ dashed red edges. In

example:

$$R(\mathcal{I}) = \{(2,3), (4,5), (6,7), (8,9), (10,11), (12,1)\}.$$

Given collection of indices

$$i = (i_v, v \in V),$$

and an edge $e = (u,v)$, let

$$i_e = (i_u, i_v), \quad i_{-e} = (i_v, i_u),$$

$r(e)$ = letter that e corresponds to

[in example, $r(3,4) = \lambda_1$

$r(11,12) = \lambda_2$]

Then

$$\mathrm{Tr}(U(\mathcal{I})) = \prod_{e \in B_+(\mathcal{I})} (U_{r(e)})_{i_e} \prod_{e \in \underline{B}(\mathcal{I})} (\overline{U_{r(e)}})_{i_{-e}}$$

$$\prod_{e \in R(I)} \delta^{i_e}$$

Note: this generalizes to collections of words

$$I = (I_1, \ldots, I_k).$$

Have

$$\mathbb{E}[\mathrm{Tr}(U(I))] =$$

$$\mathbb{E}\left[\prod_{e \in B_+(I)} (U_{r(e)})_{i_e} \prod_{e \in \underline{B}(I)} (\overline{U_{r(e)}})_{\underline{i}_e}\right]$$

$$\times \prod_{e \in R(I)} \delta^{i_e}$$

(independence)

$$= \prod_{\ell \in [L]} \mathbb{E}\Big[\prod_{\substack{e \in B_+(I) \\ r(e) = \lambda_\ell}} \left((U_{r(e)})_{i_e} \right) \prod_{\substack{e \in B_-(I) \\ r(e) = \lambda_\ell}} \left((\overline{U_{r(e)}})_{i_{-e}} \right) \Big]$$

$$\times \prod_{e \in R(I)} \delta^{i_e}$$

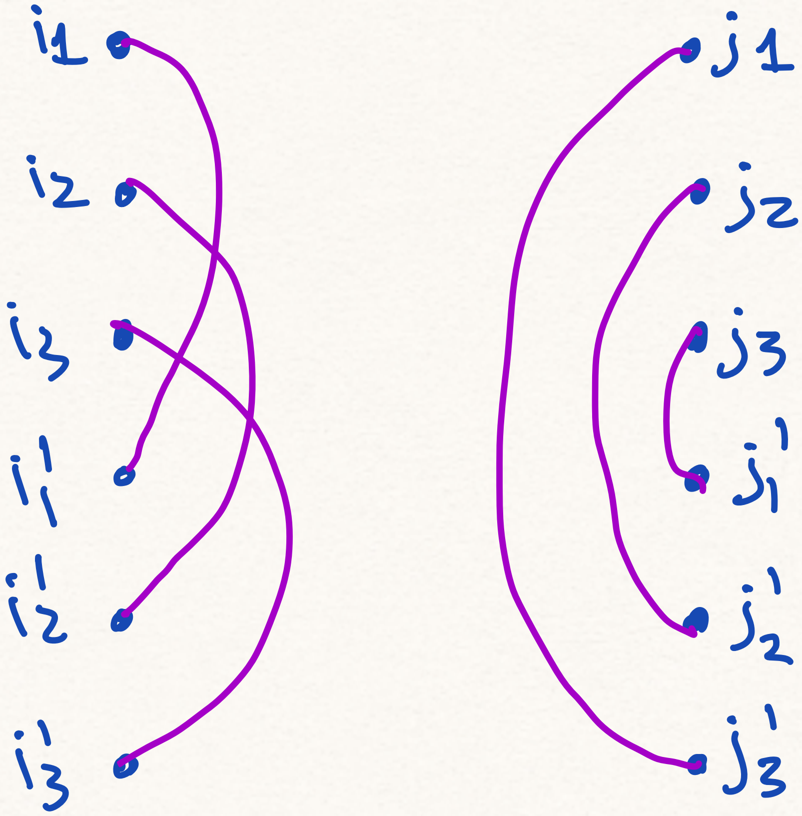
In summary: to compute expected trace, expand out and compute expectations of products of matrix entries.

Thm. [Matrix-entry Weingarten calculus]

$$\mathbb{E}\left[U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U_{i_1' j_1'}} \cdots \overline{U_{i_n' j_n'}} \right]$$

$$= \sum_{\sigma, \tau \in S_n} \delta_{i \sigma(i')} \delta_{j \sigma(j')} \mathrm{Wg}_N(\sigma^{-1} \tau)$$

Diagrammatic viewpoint:



$$\sigma = \mathrm{id} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Each σ, τ enforces conditions on left, right indices. Only those σ, τ which are "consistent" with the indices appear in sum.

Apply Weingarten calculus to each letter:

$$\mathbb{E}\Big[\prod_{\substack{e\in B_+(\Pi)\\ r(e)=\lambda_\ell}} (U_{r(e)})_{i_e} \prod_{\substack{e\in \underline{B}(\Pi)\\ r(e)=\lambda_\ell}} (\overline{U_{r(e)}})_{\underline{i}_e}\Big]$$

$$= \sum_{\sigma_\ell, \tau_\ell \in S_{n_\ell}} \delta_{\sigma_\ell(i(\ell)) i'(\ell)}\, \delta_{\tau_\ell(j(\ell)) j'(\ell)}\, \mathrm{Wg}_N(\sigma_\ell^{-1}\tau_\ell),$$

to obtain

$$\prod_{\ell\in[L]} \mathbb{E}\Big[\prod_{\substack{e\in B_+(\Pi)\\ r(e)=\lambda_\ell}} (U_{r(e)})_{i_e} \prod_{\substack{e\in \underline{B}(\Pi)\\ r(e)=\lambda_\ell}} (\overline{U_{r(e)}})_{\underline{i}_e}\Big]$$

$$\times \prod_{e\in R(\Pi)} \delta^{i_e}$$

$$= \sum_{\substack{\sigma_\ell, \tau_\ell \\ \ell \in [L]}} \prod_{\ell \in [L]} Wg_N(\sigma_\ell^{-1} \tau_\ell) \times$$

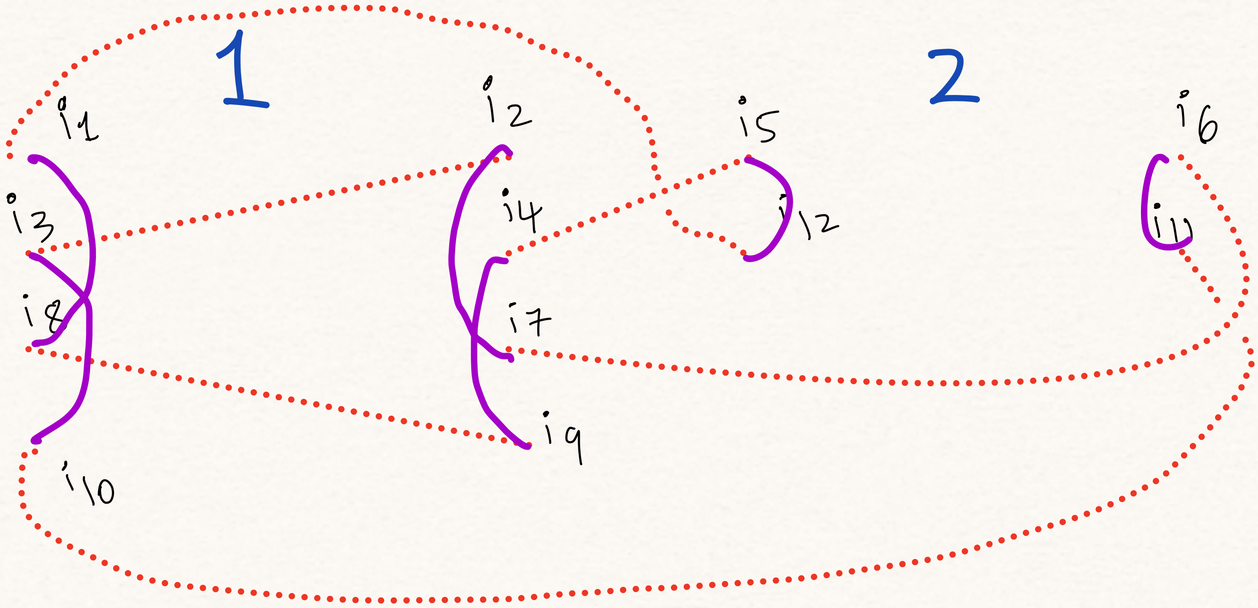
$$\prod_{\ell \in [L]} \delta_{\sigma_\ell(i(\ell)) i'(\ell)} \delta_{\tau_\ell(j(\ell)) j'(\ell)} \prod_{e \in R(\mathcal{I})} \delta^{i_e}$$

Having fixed σ_ℓ, τ_ℓ for each ℓ, this is a massive sum over indices, one for each vertex of the diagram, subject to certain constraints coming from:

(1) the dashed red lines, which are specified by the words $\mathcal{I}$

(2) the "interior connections" specified by σ_ℓ, τ_ℓ, $\ell \in [L]$.

Running example:



The sum over indices has value

$$N^{\#\text{connected components}}$$

[$= N^2$ in the example]

Thus, obtain

$$\mathbb{E}[\mathrm{Tr}(U(\Gamma))] = \sum_{\substack{\sigma_\ell, \tau_\ell \\ \ell \in [L]}} N^{\#CC(\Gamma, (\sigma_\ell, \tau_\ell), \ell \in [L])}$$

$$\times \prod_{\ell \in [L]} Wg(\sigma_\ell^{-1} \tau_\ell).$$

Finally, we show the matrix-entry Weingarten calculus, assuming some representation theory.

Let $(e_i, i \in [N])$ be standard basis of $\mathbb{C}^N$. Given $i = (i_1, \ldots, i_n) \in [N]^n$, let

$$e_i := e_{i_1} \otimes \cdots \otimes e_{i_n} \in (\mathbb{C}^N)^{\otimes n}.$$

Note $(e_i, i \in [N]^n)$ is a basis of $(\mathbb{C}^N)^{\otimes n}$.

Note that $(\mathbb{C}^N)^{\otimes n}$ has a natural inner product characterized by

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n \rangle$$

$$= \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle.$$

The collection $(e_i, i \in [N]^n)$ is ON wrt to this inner product.

Given matrices $M_1, \ldots, M_n \in \mathrm{End}(\mathbb{C}^N)$, form $M := M_1 \otimes \cdots \otimes M_n \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$, defined as map which sends

$$v_1 \otimes \cdots \otimes v_n \mapsto M_1 v_1 \otimes \cdots M_n v_n .$$

Matrix entries of M: for $i, j \in [N]^n$

$$\begin{aligned} M_{ij} &= \langle e_i, M e_j \rangle \\ &= \langle e_{i_1}, M_1 e_{j_1} \rangle \cdots \langle e_{i_n}, M_n e_{j_n} \rangle \\ &= (M_1)_{i_1 j_1} \cdots (M_n)_{i_n j_n} . \end{aligned}$$

Now given $U \in U(N)$, note that

$$U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U_{i_1' j_1'}} \, \overline{U_{i_n' j_n'}}$$

$$= \langle e_{i \cup i'} , U^{\otimes n} \otimes \bar{U}^{\otimes n} e_{j \cup j'} \rangle$$

I follow Zinn-Justin's paper "Jucys-Murphy elements and the Weingarten calculus"

We will first compute

$$P := \mathbb{E}[U^{\otimes n} \otimes \bar{U}^{\otimes n}].$$

Let $V \subseteq (\mathbb{C}^N)^{\otimes 2n}$ be the space of U-invariants, i.e.

$$V := \{ v \in (\mathbb{C}^N)^{\otimes 2n} : \forall U \in U(N), U^{\otimes n} \otimes \bar{U}^{\otimes n} v = v \}.$$

By properties of Haar measure,

$$\mathrm{Im}(P) = V, \quad P^2 = P, \quad P = P^*.$$

Thus, P is orthogonal projection onto V!

Abstractly, suppose I have a spanning set

$$\{ v_a, \ a \in I \} \text{ of } V.$$

I can make the guess that

$$P = \sum_{a,b} W_{a,b} v_a v_b^T = v_a W^{ab} v_b^T$$

[Einstein summation]

In order for P to be projection, need that

$$P v_c = v_c \quad \forall \ c \in I.$$

This is true iff $\forall \ c, d \in I$, have that

$$v_d^T P v_c = v_d^T v_c \ .$$

Let $G_{ab} = v_a^T v_b$ "Gram matrix".

Then, want that

$$v_d^T v_a W^{ab} v_b^T v_c = G_{dc}$$

$$\parallel$$

$$G_{da} W^{ab} G_{bc} .$$

I.e., as matrices, want

$$GWG = G .$$

If this holds, then $P v_c = v_c \; \forall c \in I$, thus

$$P^2 = P v_c W^{cd} v_d^T$$

$$= v_c W^{cd} v_d^T = P .$$

Moreover, if W is sym, then

$$P^T = (v_a W^{ab} v_b^T) = v_b W^{ab} v_a^T$$
$$= v_b W^{ba} v_a^T = P.$$

Thus, if W is an $\mathbb{R}$-valued matrix st

$$GWG = G,$$
$$W^T = W,$$

then

$$\mathbb{E}[U^{\otimes n} \otimes \bar{U}^{\otimes n}] = P = v_a W^{ab} v_b^T.$$

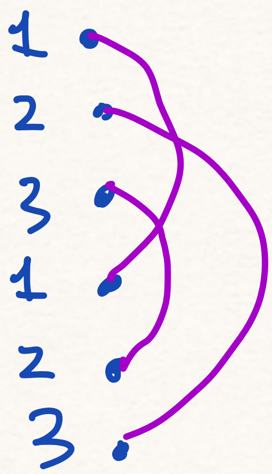
In general, does not uniquely specify W. To do so, impose additional condition that

$$WGW = W,$$

which implies that W is the pseudo-inverse of G. Pseudo-inverses are unique.

How to find a spanning set of U-invariants? This is given by the First Fundamental theorem of invariant theory, which says that the invariants are indexed by permutations $\sigma \in S_n$, visualized as matchings as follows:

$n=3$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

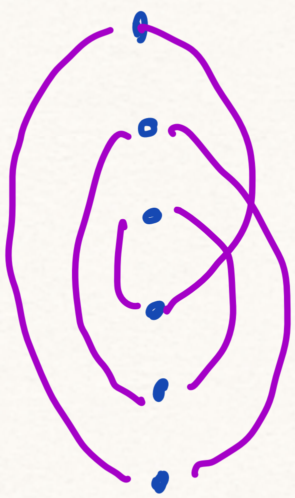
The corresponding

$$v_\sigma = e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{i'_1} \otimes \cdots \otimes e_{i'_n} \, \delta^{i_1 i'_{\sigma(1)}} \cdots \delta^{i_n i'_{\sigma(n)}} .$$

(note implicit summation over repeated indices).

Visually, the matching edges enforce identification of indices.

To compute $\langle v_\sigma, v_\tau \rangle$, visualize



$$\langle v_\sigma, v_\tau \rangle = N^{\#\text{components}},$$

where

$\#\text{components} = \#\text{cycles}(\sigma^{-1}\tau)$.

Thus,

$$G_{\sigma,\tau} = N^{\#\text{cycles}(\sigma^{-1}\tau)}$$

is precisely the Gram matrix we defined earlier! Thus

$$\mathbb{E}[U^{\otimes n} \otimes \bar{U}^{\otimes n}] = \sum_{\sigma,\tau} V_\sigma V_\tau^T \, \mathrm{Wg}_N(\sigma,\tau)$$

To finish, compute matrix entry:

$$(V_\sigma V_\tau^T)_{i \sqcup i', j \sqcup j'} = \delta_{i\sigma(i')} \delta_{j\sigma(j')} .$$

Remark. Turns out:

$$\mathrm{Wg}_N(\sigma,\tau) = \mathrm{Wg}_N(\sigma^{-1}\tau)$$

is just a fn of the "difference"

of the pair (σ, τ). Thus, can view

$$\mathrm{Wg}_N : S_n \to \mathbb{R}.$$

Moreover, under this viewpoint, Wg_N is a class fn, thus only depends on the cycle structure of the perm.

Back to YM

After that (rather long) detour into Unitary matrix integrals, we now finally return to YM. Recall our goal of computing:

$$I(s,K) = \int W_s(U) \prod_{p \in P_\Delta} \mathrm{Tr}(U_p)^{K(p)} \prod_{e \in E_\Delta^+} dU_e ,$$

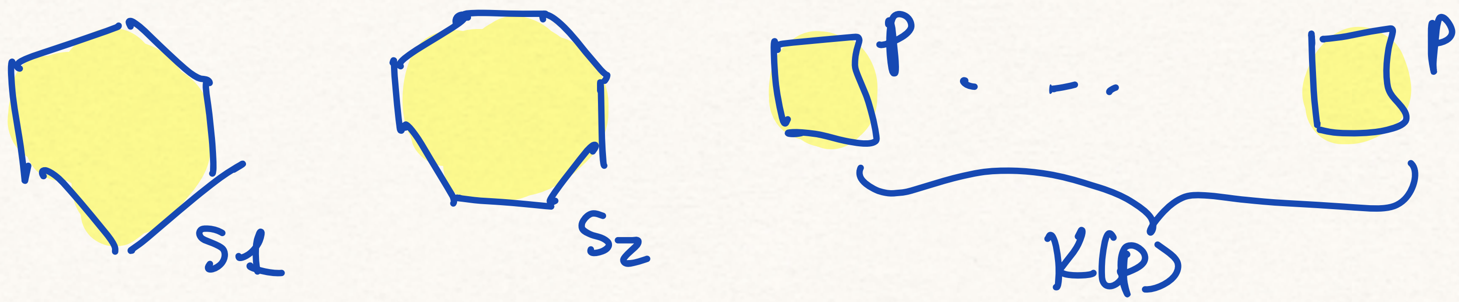
which is a unitary matrix integral.

By the result on unitary matrix integrals, we have that

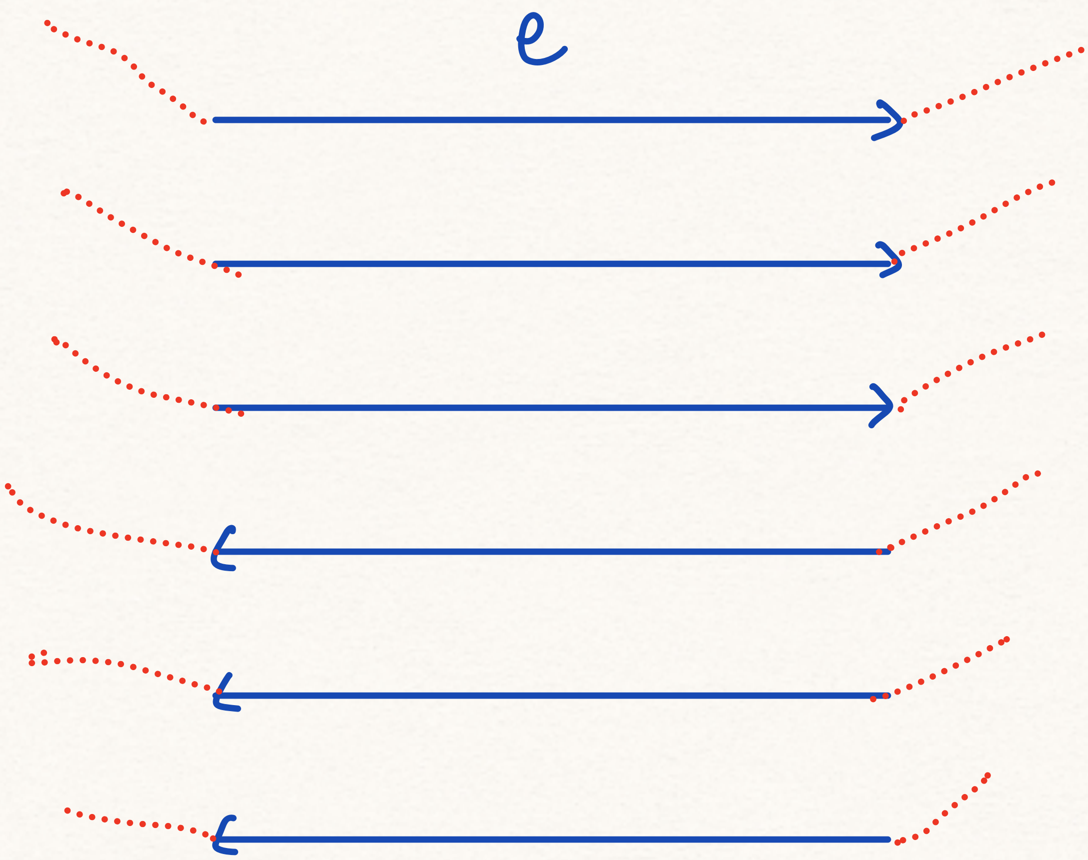
$$I(s,K) = \sum_{\substack{(\sigma_e,\tau_e) \\ e \in E_\Delta^+}} N^{\#CC(s,K,((\sigma_e,\tau_e), e\in E_\Delta^+)) - n} \prod_{e \in E_\Delta^+} Wg_N(\sigma_e^{-1}\tau_e) .$$

#strings

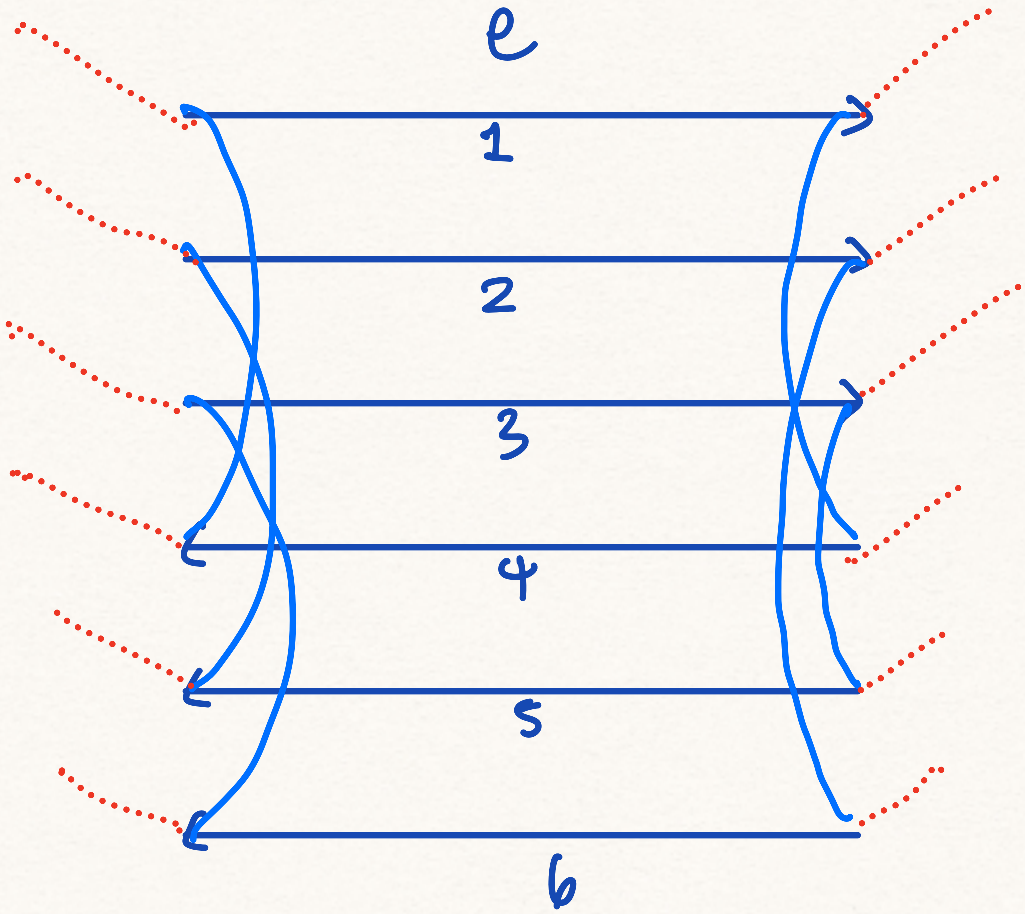
To visualize the computation, we have one face for each loop s_i in S, and $K(p)$ copies of 4-gons labeled by p for each $p \in P_\Delta$:



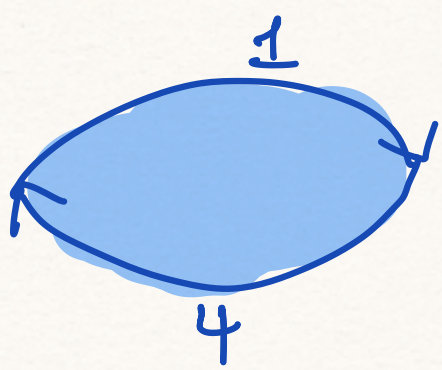
We now want to interpret each term of the diagrammatic sum as a "gluing" of these polygons. Towards this end, for each $e \in E_\Delta^+$, form the "local picture" consisting of all appearances of e or e^{-1} in the labeled polygons:

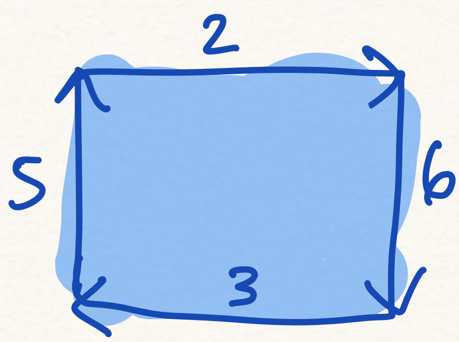


Recall from the Unitary matrix integration formula that we have to sum over all ways of putting in the "interior connections":



Each such choice gives another set of faces (by following "interior cycles"):

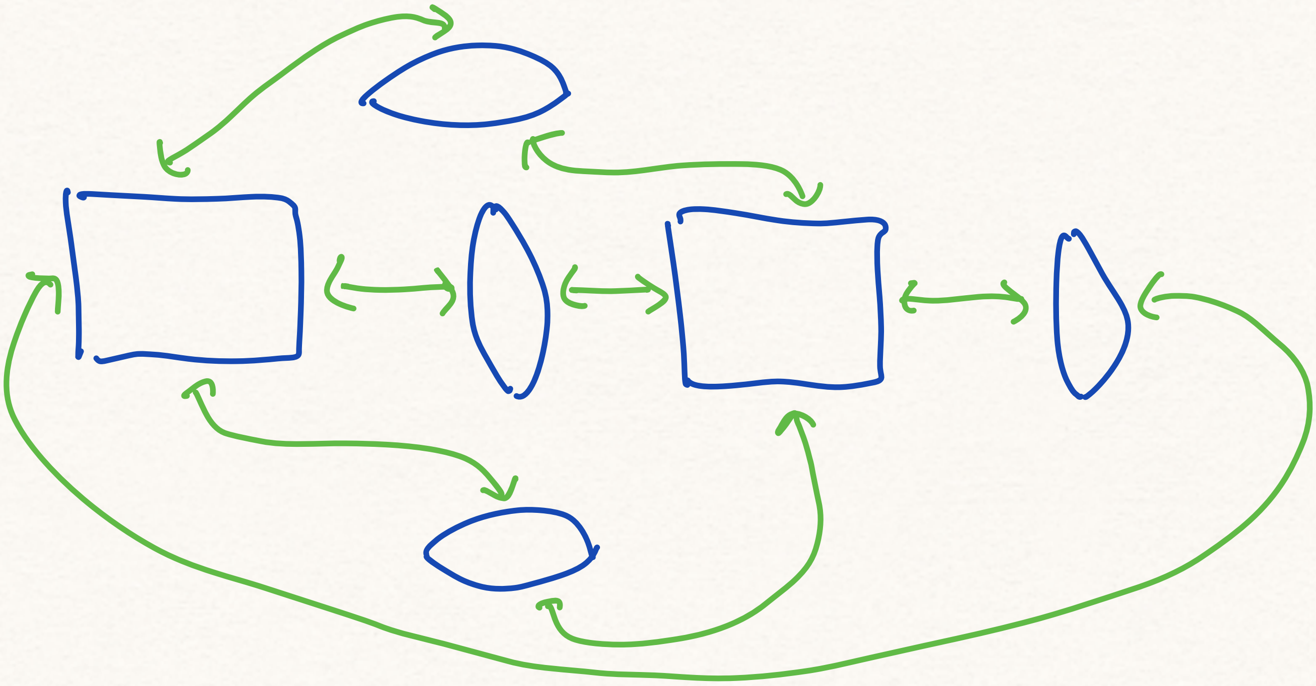




Moreover, the edges of these new faces are naturally paired w/ edges of the existing faces. That is b/c each edge in the "local picture" corr. to an edge of one of the existing faces.

In summary: for each choice of interior matchings (σ_e, τ_e) for each edge e, we have a collection of faces (some yellow, some blue) and a complete pairing of edges.

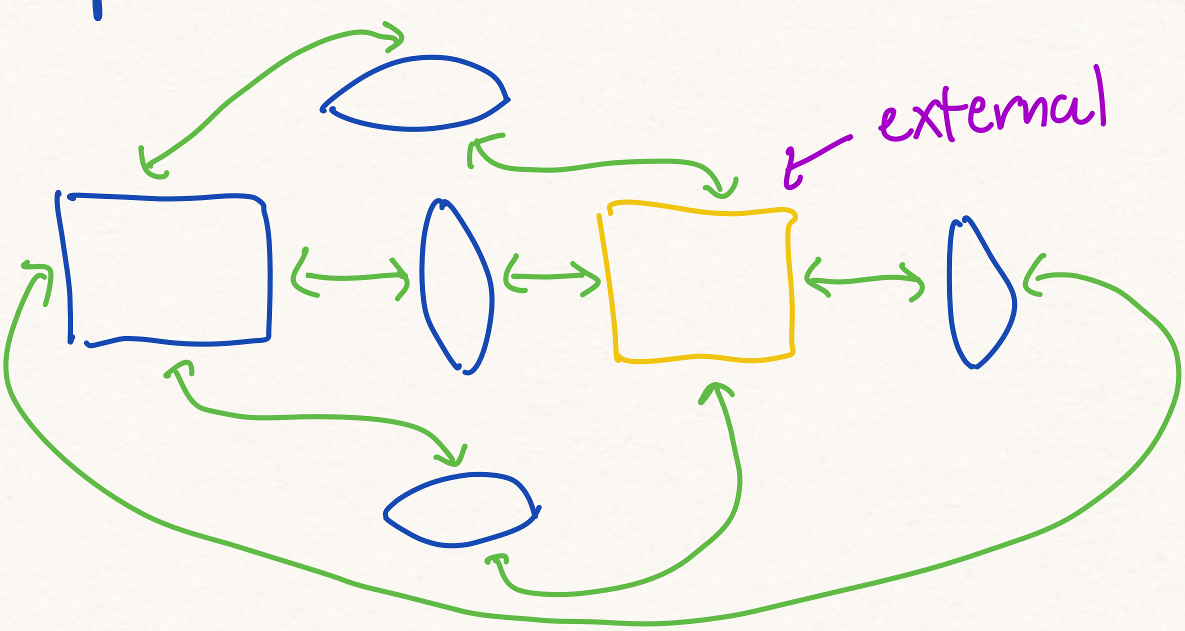
Def. A map $\mathcal{M}$ is a collection of polygonal faces with specified gluings.



Think of M as a discrete (2D) surface.

As defined, all maps are closed.

To specify maps w/ bdry, we consider a map w/ a collection of external faces



Thus, we may associate

$(s, K, ((\sigma_e, \tau_e), e \in E_\Delta^+))$

$\downarrow$

map $\mathcal{M}$ w/ external faces labelled by s, and which uses a total of $\sum_p K(p)$ plaquettes. Moreover, have a labelling ψ of the edges of $\mathcal{M}$ st:

(1) blue faces are labelled $e, \bar{e}^{1}, e, \bar{e}^{1}, \cdots, e, \bar{e}^{1}$, for some $e \in E_\Delta^+$.

(2) if two edges have been glued together, then for some $e \in E_\Delta^+$, one is labelled e, the other $\bar{e}^{1}$.

(3) non-external yellow faces are mapped by ψ to plaquettes. For each $p \in \mathcal{P}_\Delta$, the # of such faces mapped to p is $K(p)$.

(4) For $1 \le i \le n$, the ith external face is mapped by ψ to s_i.

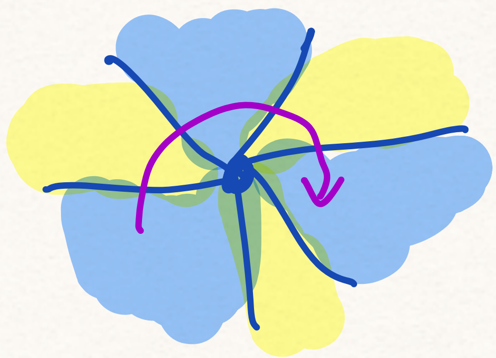
(5) Yellow faces are only glued to blue faces, and vice versa.

Claim: under this association,

$$\# CC(s, K, ((\sigma_e, \tau_e), e \in E_\Delta^+))$$

$$= V(\mathcal{M}) = \# \text{ vertices of } \mathcal{M}.$$

Basically, each connected comp. corr. to a "cycle" around a single vertex of $\mathcal{M}$:



So there is a 1-1 corr. between CCs and vertices.

For each edge $e \in E_\Delta^\dagger$, let

$n_e = n_e(s, K) =$ total # of appearances of e in s, K.

Then $2n_e$ is the total # of edges in the "local picture" at e, and

$$2\sum_{e \in E_\Delta^\dagger} n_e = E(\mathcal{M})$$

$$= \# \text{ edges of } \mathcal{M}.$$

What about $F(\mathcal{M})$ = # faces of $\mathcal{M}$?
Have that

$$F(\mathcal{M}) = \text{area}(\mathcal{M}) + \#\text{ blue faces}$$

$$= \sum_{p} K(p) + \sum_{e \in E_{\Delta}^{+}} \#\text{ blue faces labelled by } e$$

$$= \sum_{p} K(p) + \sum_{e \in E_{\Delta}^{+}} \#\text{cycles}(\sigma_e^{-1}\tau_e).$$

[The faces corr. to s are external, so we don't count them.]

Thus, we have that

$$\chi(\mathcal{M}) = V(\mathcal{M}) - E(\mathcal{M}) + F(\mathcal{M})$$

$$= \#CC(s, K, ((\sigma_e, \tau_e), e \in E_\Delta^+))$$

$$+ \sum_{e \in E_\Delta^+} \#\mathrm{cycles}(\sigma_e^{-1}\tau_e) - 2n_e(s, K)$$

$$+ \sum_p K(p).$$

Thus,

$$N^{\#CC} = N^{\chi(M)} \prod_{e \in E_\Delta^+} N^{2n_e(s,K) - \#\mathrm{cycles}(\sigma_e^{-1}\tau_e)}$$

$$\times \bar{N}^K .$$

Thus,

$I(s, K)$

$$= N^{-K} \sum_{\substack{(\sigma_e, \tau_e) \\ e \in E_\Delta^+}} N^{\chi(\mathcal{M})-n} \prod_{e \in E_\Delta^+} N^{2n_e - \#\mathrm{cycles}(\sigma_e^{-1}\tau_e)} \mathrm{Wg}_N(\sigma_e^{-1}\tau_e)$$

For $\sigma \in S_n$, define

$$\overline{\mathrm{Wg}_N(\sigma)} = N^{2n - \#\mathrm{cycles}(\sigma)} \mathrm{Wg}_N(\sigma).$$

"normalized Weingarten fn". This is a natural normalization b/c it has a nontrivial large-N limit.

Recall also that Wg_N, thus also $\overline{\mathrm{Wg}_N}$, is a class fn, i.e. it only dep. on the cycle structure of σ.

In terms of the blue faces,

cycle structure of $\sigma_e^{-1}\tau_e$

= partition of $2n_e$ given by the degrees of the blue faces at e.

Call the partition on the RHS

$$\mu_e(\mathcal{M}, \Psi).$$

We thus have:

$$I(S, K) = N^{-K} \sum_{\substack{(\sigma_e, \tau_e) \\ e \in E_1^+}} N^{\chi(\mathcal{M}) - n} \prod_{e \in E_1^+} \overline{Wg_N(\mu_e(\mathcal{M}, \Psi))},$$

and thus summing in K, we obtain

$$Z_{\Lambda,N,\beta} \langle W_s \rangle_{\Lambda,N,\beta}$$

$$= \sum_{K : P_\Lambda \to \mathbb{N}} \sum_{\substack{(\sigma_e, \tau_e) \\ e \in E_\Lambda^+}} \frac{\beta^{\mathrm{area}(M,\psi)}}{(\psi^{-1})!} N^{\chi(M) - n} \prod_{e \in E_\Lambda^+} \overline{Wg_N(M_e(M,\psi))}$$

Here, $(\psi^{-1})! = \prod_{p \in P_\Lambda} |\psi^{-1}(p)|! = K!$

All weights on the RHS are expressed purely in terms of the map (M,ψ).

This gives a precise meaning to the surface sum

$$\sum_{\mathcal{H}:\partial\mathcal{H}=s} \beta^{\mathrm{area}(\mathcal{H})} N^{\chi(\mathcal{H})-n} w_N(\mathcal{H})$$

that appeared at the beginning.

Exercise. Try to express the sum itself in terms of surfaces. I.e., instead of summing over K, (σ_e, τ_e), $e \in E_\Lambda^+$, just sum over some class of surfaces. This is basically about going from ordered objects to unordered objects, so there is some combinatorics involved.