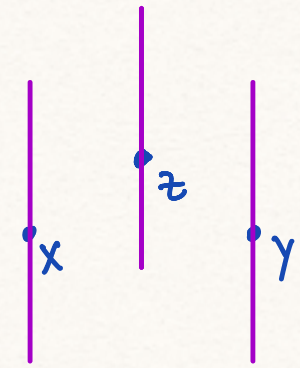
Ref: Chatterjee Yang-Mills for probabilists

YM: study of geometric objects known as connections (on principle bundles or vector bundles).

Vector bundle:



Setup: G a compact Lie grp (say, $U(N)$, i.e. $N \times N$ unitary matrices).

Let $\mathfrak{g}$ be the corresponding Lie algebra. If $G = U(N)$, then

$$\mathfrak{g} = \mathfrak{u}(N) = \{ X \in \underbrace{M_N(\mathbb{C})}_{\text{N×N complex matrices}} : \overset{\text{conjugate transpose}}{X^{*}} = -X \}.$$

Let $d \geq 2$ spacetime dim.

Concretely, a connection one-form (AKA a gauge field) on $\mathbb{R}^d$ is:

$$A(x) = \underbrace{A_j(x)\, dx^j}$$

Einstein summation convention.
Implicit sum over $j \in [d]$.

Alternatively, think of $A = (A_1, \ldots, A_d)$ as

$$A : \mathbb{R}^d \to \mathfrak{g}^d .$$

Think of as $\mathfrak{g}$-valued one-form, i.e. for all $x \in \mathbb{R}^d$, have a linear map

$$A(x) : \underset{\substack{\cong \\ \mathbb{R}^d}}{T_x \mathbb{R}^d} \to \mathfrak{g}$$

$$v \mapsto A(x)\cdot v = A_j(x)v^j \,,$$
$$\underset{\mathbb{R}^d}{\overset{\cap}{x}}$$

Let V be a vector space on which G acts (if $G = U(N)$, could take $V = \mathbb{C}^N$). Can form (trivial) vector bundle

$$E = \mathbb{R}^d \times V \,.$$

Choice of A induces geometry on E (analogous to choice of metric in Riemannian geometry). Given A, can:

(1) identify fibers above neighboring pts ("parallel transport")

(2) differentiate V-valued fns $\phi : \mathbb{R}^d \to V$

(more generally, "sections" of E)

Given $\phi : \mathbb{R}^d \to V$, covariant derivative given by:

$$D_A \phi = (D_{A,j} \phi,\ j \in [d]),$$

$$D_{A,j} \phi = D_A^j \phi = \partial^j \phi + A^j \phi.$$

Nontrivial geometry: in general, covariant derivatives do not commute:

$$(D_A^i D_A^j - D_A^j D_A^i)\phi$$

$$= (\partial^i A^j - \partial^j A^i + [A^i, A^j])\phi$$

[Exercise.] We define the curvature $F_A = ((F_A)_{ij},\ i,j \in [d])$:

$$(F_A)_{ij} = F_A^{ij} = \partial^i A^j - \partial^j A^i + [A^i, A^j],$$

so that

$$(D_A^i D_A^j - D_A^j D_A^i)\phi = F_A^{ij}\phi.$$

The curvature $F_A(x)$ captures how "non-trivial" the geometry associated to A is at the pt x.

Note: $F_A^{ij} \in \mathfrak{g} \subseteq M_N(\mathbb{C})$. For general matrices $X \in M_N(\mathbb{C})$, define Hilbert-Schmidt norm

$$|X| = \mathrm{Tr}(X^* X)^{\frac{1}{2}}.$$

Rmk. If $X \in \mathfrak{g}$, then $X^* = -X$, so

$$|X|^2 = \mathrm{Tr}(X^* X) = -\mathrm{Tr}(X^2).$$

Define

$$|F_A|^2 = \frac{1}{2}\sum_{i,j\in[d]} |F_A^{ij}|^2$$

$$= -\frac{1}{2}\mathrm{Tr}\left(F_A^{ij}(F_A)_{ij}\right).$$

The YM action:

$$S_{YM}(A) = \frac{1}{2}\int_{\mathbb{R}^d} |F_A(x)|^2\, dx.$$

Euclidean YM theory: For $\beta \geq 0$,

$$d\mu_{YM}(A) = Z_{YM}^{-1} \exp(-\beta S_{YM}(A))\, dA$$

PM on space of connections.

Central question: rigorously construct μ_{YM} for $d=4$. Would be key step towards YM millenium problem.

Some immediate issues:

(1) dA is formally Leb measure on an ∞-dim. space. Does not exist.

(2) S_{YM} invariant under ∞-dim grp of symmetries. For $g: \mathbb{R}^d \to G$ (say C^∞), let

$$g \cdot A = g A g^{-1} - dg\, g^{-1}$$

another connection one-form. Then (exercise)

$$F_{g\cdot A} = g F_A g^{-1},$$

so $|F_{g\cdot A}|^2 = |F_A|^3$, thus

$$S_{YM}(g\cdot A) = S_{YM}(A).$$

Fin-dim analogy: cannot define

$$d\mu(x,y) = Z^{-1} \exp(-(x-y)^2)\, dx\, dy$$

as PM on $\mathbb{R}^2$, since $(x-y)^2$ is inv. under a 1-dim. symmetry:

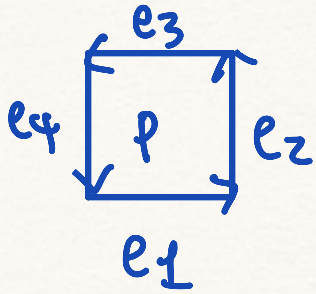
$$(x,y) \mapsto (x+c, y+c), \quad c \in \mathbb{R}.$$

Lattice YM

Obtain well-defined fin-dim approx. by going to a lattice. Let $\varepsilon > 0$ (mesh size), and let $\Lambda \subseteq (\varepsilon \mathbb{Z})^d$ be a (large) finite box.

Let:

- E_Λ set of oriented edges of Λ
- E_Λ^+ set of pos. oriented edges
- $\mathcal{P}_\Lambda$ set of orientend plaquettes



- $\mathcal{P}_\Lambda^+$ set of pos. orient. plaquettes.

Lattice discretization of gauge field:

a fn $U : E(\Lambda) \to G$ st

$$U(e^{-1}) = U(e)^{-1} \quad \forall \; e \in E(\Lambda).$$

So basically, enough to assign one matrix $U_e \in G$ to each pos. orient. edge $e \in E_\Lambda^+$.

Think of

$$U(x, x+\varepsilon e_i) = e^{\varepsilon A_i(x)},$$

[Here $e^X = \sum_{j=0}^{\infty} \frac{X^j}{j!}$. If $X \in \mathfrak{g}$, then $e^X \in G$ (exercise).]

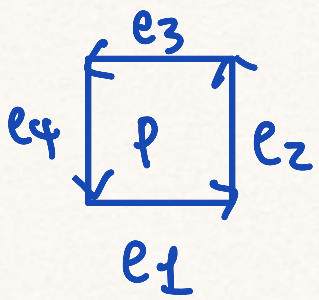
which is an approx. of parallel transport of A along edge $(x, x+\varepsilon e_i)$.

Lattice version of curvature: for plaquette $p \in \mathcal{P}_{\Lambda}$, take

$$\frac{1}{\varepsilon^2}\left(\underset{\sim}{\mathrm{Id}} - U_p\right),$$

N×N id. matrix

where $U_p = U_{e_1} U_{e_2} U_{e_3} U_{e_4}$, if p is like:



Note U_p is an approx. of parallel transp. of A along bdry of p, which is a square of area ε^2. The reason why corr. to the curvature is b/c curvature is "infinitesimal holonomy" [Ambrose-Singer thm].

Lattice action:

$$\tilde{S}_{YM}(U) = \varepsilon^d \sum_{p \in P_\Lambda^+} \Big(\varepsilon^{-2} \overbrace{\|Id - U_p\|}^{\text{HS norm}}\Big)^2$$

$$= \varepsilon^{d-4} \sum_{p \in P_\Lambda^+} \|Id - U_p\|^2.$$

Lattice YM theory:

$$d\tilde{\mu}_{YM}(U) = \tilde{Z}_{YM}^{-1} \exp\Big(-\beta \varepsilon^{d-4} \sum_{p \in P_\Lambda^+} \|Id - U_p\|^2\Big) \prod_{e \in E_\Lambda^+} \underbrace{dU_e}_{\text{Haar measure}}$$

So $\tilde{\mu}_{YM}$ is a PM on $G^{E_\Lambda^+}$. Perfectly well-defined.

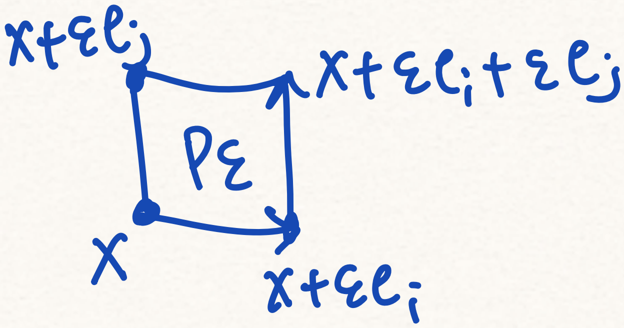
Prop. Let $A : \mathbb{R}^d \to \mathfrak{g}^d$ be smooth. Let $x \in \mathbb{R}^d$, $i, j \in [d]$. Then

$$(F_A)_{ij}(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-2}(\mathrm{Id} - U_{P_\varepsilon}),$$

where

$$U_{P_\varepsilon} = e^{\varepsilon A_i(x)} e^{\varepsilon A_j(x+\varepsilon e_i)} e^{-\varepsilon A_i(x+\varepsilon e_j)} e^{-\varepsilon A_j(x+\varepsilon e_j)}.$$

Visualize:



Pf. See Chatterjee (YM for probabilists).

Problem. [Continuum limit] Take $\varepsilon \downarrow 0$ and show that $\tilde{\mu}_{\Lambda_\varepsilon}$ converge to some PM on connections (after approp. rescaling).

There are also questions about the long-distance behavior. In the following, take $\varepsilon = 1$, so unit lattice.

We first rewrite $\tilde{\mu}$. Note: $\forall g \in G$, have that $g^* g = \mathrm{Id}$. Thus

$$\|\mathrm{Id} - U_p\|^2 = \mathrm{Tr}\left((\mathrm{Id} - U_p)^* (\mathrm{Id} - U_p)\right)$$

$$= \mathrm{Tr}(\mathrm{Id}) + \mathrm{Tr}(\mathrm{Id}) - \left(\mathrm{Tr}(U_p^*) + \mathrm{Tr}(U_p)\right)$$

Thus,

$$\exp\left(-\beta \sum_{p \in \mathcal{P}_\Lambda^+} \|\mathrm{Id} - U_p\|^2\right)$$

$$= C_{\Lambda, G, \beta} \exp\left(\beta \sum_{p \in \mathcal{P}_\Lambda} \mathrm{Tr}(U_p)\right)$$

[note $\mathrm{Tr}(U_p^*) = \mathrm{Tr}(U_p^{-1})$ for $p \in \mathcal{P}_\Lambda^+$].

Thus, can let

$$d\mu_{\Lambda, N, \beta}(U) = Z_{\Lambda, N, \beta}^{-1} \exp\left(\beta \sum_{p \in \mathcal{P}_\Lambda} \mathrm{Tr}(U_p)\right) \prod_{e \in E_\Lambda^+} dU_e$$

.

Have that $\tilde{\mu} = \mu$, by prev. discussion.

Rmk. The action

$$S_{YM}(U) = \beta \sum_{p \in \mathcal{P}_\Lambda} \mathrm{Tr}(U_p)$$

is the Wilson action. Could also take other actions.

Observables

For loops $\gamma = e_1 \cdots e_n$ in $\mathbb{Z}^d$, let

$$U_\gamma = U_{e_1} \cdots U_{e_n}$$

("holonomy of U along γ").

Define Wilson loop observable:

$$W_\gamma(U) = \mathrm{tr}(U_\gamma) = \frac{1}{N} \mathrm{Tr}(U_\gamma).$$

Let

$$\langle W_\gamma \rangle_{\Lambda, G, \beta} = \mathbb{E}_{\Lambda, G, \beta} [W_\gamma(U)].$$

Problem [Wilson's area law/Quark confinement]. Let $d=4$, $G = SU(3)$ (or any other non-Abelian grp w/ non-trivial center). For any $\beta \geq 0$, there exist $C, c > 0$ st for any rectangular loop γ and any $\Lambda \supseteq \gamma$, have that

$$|\langle W_\gamma \rangle_{\Lambda, G, \beta}| \leq C \exp(-c\,\text{area}(\gamma)).$$

Rmk. Perimeter decay is known. Area decay known for $\beta \ll 1$.

Problem [Mass gap] Let $d=4$, $G=SU(3)$. For any $\beta \geq 0$, there exist $C, c > 0$ st st for any observables f, g which are fns of edges E_f, E_g, have that for any $\Delta \supseteq E_f, E_g$,

$$|\mathrm{Cov}_{\Delta, G, \beta}(f,g)| \leq C e^{-c\, \mathrm{dist}(E_f, E_g)}.$$

I.e., $\mu_{\Delta, G, \beta}$ exhibits exp. decay of correlations.

The parameter $\beta \geq 0$ governs amount of order/disorder at short distances. The last two problems are saying that, no matter how much order is imposed at short distances, the system looks very disordered at large enough distances.