Extremal Graph Theory

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If you have any questions/comments or find any mistakes, please contact me at shint@mit.edu.
This handout is linked at `web.mit.edu/shint/www/handouts/ExtremalGraphTheory.pdf`.

Contents

1 Introduction

Extremal graph theory is a rich branch of combinatorics which deals with how general properties of a graph (eg. number of vertices and edges) controls the local structure of the graph.

Other parts of graph theory including **regularity** and **pseudorandomness** are built upon extremal graph theory and can be extended into the world of **additive combinatorics**. Compare, for example, the following two statements:

Theorem: Mantel

Every graph on n vertices with edge density greater than $\frac{1}{2} \cdot \frac{n}{n-1}$ contains a triangle.

Theorem: Roth

Every subset of $\mathbb{N}$ with density greater than 0 contains a 3-term arithmetic progression.

These theorems are samples of extremal graph theory and additive combinatorics, respectively, and have similar-looking statements.

In this handout, we will examine some key results in extremal graph theory. The main question we will look at is:

Given a positive integer n and graph H, what is the maximum possible number of edges in a graph on n vertices with no copies of H (such a graph is called ***H-free***)?

Here is a slightly different question in the same spirit:

Question. Given a positive integer n, what is the maximum number of edges in a graph on n vertices with no cycles?

Instead of looking at H-free graphs for a specific H, we are looking for cycle-free graphs in general.

We know by definition that a cycle-free graph is a forest. This has some number of connected components, each of which has one less edge than number of vertices. Convince yourself that the number of edges is maximized when our forest is in fact a tree, so the answer is $n - 1$.

This is perhaps the simplest example of an "extremal graph theory question." In the next few sections, we look at similar and more advanced questions.

2 Avoiding cliques

An important question with applications in many other parts of math is how to avoid cliques.

2.1 Mantel's theorem

The first result in this manner is Mantel's Theorem.

Theorem 2.1: Mantel (1907)

Any triangle-free graph on n vertices has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.

Proof. Let G be a triangle-free graph with n vertices.

Observation. For any edge uv, u and v have no common neighbors. Thus $(\deg u - 1) + (\deg v - 1) \leq n - 2$.

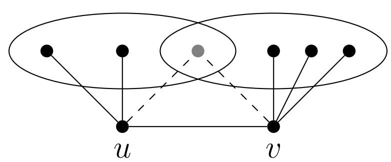


Figure 1: A common neighbor would imply the existence of a triangle.

So

$$\sum_{uv \in E} \deg u + \deg v \leq e(G)n.$$

But the sum counts $\deg v$ for each neighbor u of v, for a total of $\deg v$ times. Thus

$$\sum_{uv \in E} \deg u + \deg v = \sum_{v \in V} (\deg v)^2.$$

Combining these and using Cauchy-Schwarz,

$$e(G)n^2 \geq n \sum_{uv \in E} \deg u + \deg v = \left(\sum_{v \in V} 1\right)\left(\sum_{v \in V} (\deg v)^2\right) \geq \left(\sum_{v \in V} \deg v\right)^2 = (2e(G))^2$$

by the Handshake Lemma. Thus $e(G) \leq \frac{n^2}{4}$. ■

This proof is straightforward and elegant, but it doesn't provide an immediate equality case (though one can work through the conditions to deduce it). The main reason for this is because the proof's main part is algebraic, revealing little about the graph theoretic structure of an equality case. Here is a "better" proof:

Proof. Let $A \subseteq V$ be a maximum independent set[1].

Observation. The neighborhood of any vertex is independent. Thus $\deg v \leq |A|$.

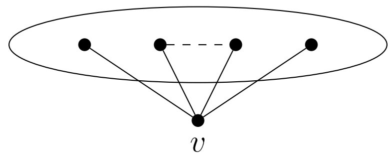


Figure 2: Two neighbors being adjacent would imply the existence of a triangle.

Consider the complement $B = V \setminus A$ of A. Every edge must have a vertex in B because A is independent. Thus

$$e(G) \leq \sum_{v \in B} \deg v \leq |A||B| \leq \left(\frac{|A| + |B|}{2} \right)^2 = \frac{n^2}{4}$$

by AM-GM. ■

If we try analyzing the equality case now, we get:

- $|A| = |B|$ from AM-GM
- $e(G) = \sum_{v \in B} \deg v$ implies no edge is from B to B, so all edges are between A and B
- $\deg v = |A|$ for all $v \in B$

Together, these conditions immediately imply that equality only occurs at the complete bipartite graph $T_{\frac{n}{2}, \frac{n}{2}}$.

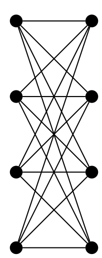


Figure 3: The triangle-free graph on 8 vertices with maximal number of edges.

Remark. If n is odd, the same argument with a bit more work implies that the edge count is maximized at $T_{\frac{n-1}{2}, \frac{n+1}{2}}$ with $\frac{n^2-1}{4} = \lfloor \frac{n^2}{4} \rfloor$ edges.

Remark. That is not to discount the first proof, however. In addition to being quite straightforward, we can modify the proof a bit to deduce the following result:

[1]An independent set is a set of vertices with no edges between them.

Fact 2.2

A graph with n vertices and m edges has at least $\frac{4m}{3n}(m - \frac{n^2}{4})$ triangles.

2.2 Turán's theorem

Motivated by Mantel's theorem, one might guess that the maximum number of edges in a graph with no K_{r+1} would be an "equitable complete r-partite graph." We first provide some definitions to formalize what we mean.

Definition. An ***k-partite graph*** is one whose vertex set V can be partitioned into sets $V_1 \sqcup \cdots \sqcup V_k$ such that every edge has vertices in different parts. A ***complete k-partite graph*** $K_{s_1,\dots,s_k}$ is the graph on $s_1 + \cdots + s_k$ vertices formed by partitioning the vertices into sets V_i with $|V_i| = s_i$ and creating edges between every pair of vertices in different parts.

Remark. For the same reasons as before, a graph G is k-partite if and only if $\chi(G) \le k$.

If the sets have roughly the same size, the partition is called equitable. These graphs have a name:

Definition. The ***Turán graph*** $T_{n,r}$ is the complete r-partite graph $K_{s_1,\dots,s_r}$ with $s_1 + \cdots + s_r = n$ and each s_i being either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

Example 2.1

We have $T_{8,3} = K_{2,3,3}$ and $T_{15,5} = K_{3,3,3,3,3}$.

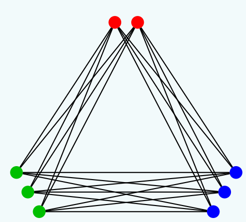


Figure 4: The Turán graph $T_{8,3}$.

As we have hypothesized, the Turán graphs give the most edges in a K_{r+1}-free graph:

Theorem 2.3: Turán (1941)

Any K_{r+1}-free graph on n vertices has at most $e(T_{n,r}) \le (1 - \frac{1}{r})\frac{n^2}{2}$ edges.

Proof. We proceed by strong induction on n. If $n \le r$, then $T_{n,r} = K_{1,\dots,1} = K_n$. Now assume that $n > r$ and that the theorem holds for all graphs with less than n vertices. Let

G be a K_{r+1}-free graph with n vertices, and let us assume G has the maximal number of edges. This implies that G has a copy of K_r — if not, then we can add an edge without creating K_{r+1}, contradicting maximality. Let A be a set of vertices that form a K_r, and look at its complement $B = V \setminus A$.

Observation. Every vertex $v \in B$ has at most $r-1$ neighbors in A.

Thus

$$e(G) \leq \binom{|A|}{2} + (r-1)|B| + e(B) \leq \binom{r}{2} + (r-1)(n-r) + e(T_{n-r,r}) \leq e(T_{n,r}). \qquad \blacksquare$$

Notice how this proof examined a specific substructure of the graph, removed it, and continues after this reduction. This is an example of a ***local*** approach. Proofs that use local methods often employ induction to repeatedly remove substructures.

Contrast this with our ***global*** proofs of Mantel's theorem which (mostly) looked at the graph as a whole and used inequalities in that form. But noticeably, the local approach provides more graph theoretical information. Our first proof of Mantel was almost entirely global and algebraic and told us little information. The second proof was less global (looks at one substructure and distinguishes it, but uses this information directly) and provided good information on the equality case and the underlying graph structure. And finally, our proof of Turán directly gives us the equality case by induction.

2.3 Remarks on Turán

As we might expect, K_{r+1}-free graphs that are close to the Turán bound are nearly r-partite.

Theorem 2.4: Erdős-Simonovits (1991)

Every K_{r+1}-free graph on n vertices with at least $e(T_{n,r}) - k$ edges can be made r-partite by removing at most k edges.

The triangle case is not too hard and follows from ideas above, but the theorem is trickier to prove for general cliques.

If we have passed the Turán bound, we might get many copies of the clique. As an example, adding one edge to $T_{n,2}$ produces $\lfloor \frac{n}{2} \rfloor$ triangles. In fact, this is true as long as we have the correct edge count, regardless of the structure of the graph.

Theorem 2.5: Rademacher (1941)

Every graph on n vertices and $\lfloor \frac{n^2}{4} \rfloor + 1$ edges contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

This can be proven by induction on n by deleting either a single vertex with small degree or two vertices with combined small degree.

3 Avoiding general subgraphs

A natural question to ask is what happens if we try to forbid other subgraphs besides cliques.

Definition. Given a positive integer n and graph H, define the ***extremal number of H*** (on graphs with n vertices), denoted $\operatorname{ex}(n, H)$, to be the maximum possible number of edges in a H-free graph on n vertices.

We will generally only care about the asymptotics of $\operatorname{ex}(n, H)$ as n grows large. So Turán states that

$$\operatorname{ex}(n, K_{r+1}) = e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

Now, suppose that the chromatic number $\chi(H) = r + 1$ for some graph H. Then $T_{n,r}$ is H-free — since $\chi(T_{n,r}) = r$, we can r-color $T_{n,r}$, and if H is a subgraph of $T_{n,r}$ then this induces an r-coloring of H too. Thus

$$\operatorname{ex}(n, H) \geq e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

One might think that we can do better, because we wastefully tack on extra edges to avoid. But in fact, this is the best asymptotic.

Theorem 3.1: Erdős-Stone-Simonovits (1946)

$$\operatorname{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}$$

There are many proofs of this theorem (for example by a graph counting lemma derived by Szemerédi's graph regularity lemma), but all are either quite long or quite advanced so we will black-box the result here.

Remark. Erdős-Stone-Simonovits can be written as

$$\lim_{n\to\infty} \frac{\operatorname{ex}(n, H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H) - 1}.$$

This can be interpreted as the statement that asymptotically, the maximum possible ***edge density*** in graphs that avoid H is $1 - \frac{1}{\chi(H)-1}$.

This result is quite surprising. For example, consider the Petersen graph with chromatic number 3. The threshold of edge density to mandate a certain subgraph is the same for triangles as it is for such a complicated graph as the Petersen graph.

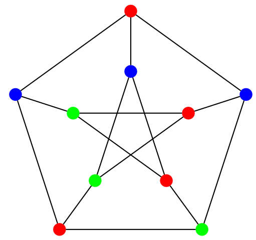


Figure 5: The complicated Petersen graph, with chromatic number 3.

3.1 Remarks on extremal numbers

As before, if we have passed the extremal bound, we will get many copies of the graph. This idea is known as ***supersaturation***.

Theorem 3.2: Erdős-Simonovits (1983)

For all graphs H and real numbers $\epsilon > 0$, there exists a constant $c > 0$ such that for all sufficiently large $n \in \mathbb{N}$, every graph with at least $\operatorname{ex}(n, H) + \epsilon\binom{n}{2}$ edges contains at least $c\binom{n}{|V(H)|}$ copies of H.

4 Avoiding bipartite graphs

The Erdős-Stone-Simonovits resolves the question of (asymptotically) determining $\operatorname{ex}(n, H)$ for many graphs H, but says nothing about bipartite graphs H. Indeed, if $\chi(H) = 2$ then the result is that $\operatorname{ex}(n, H) = o(n^2)$, which is very imprecise. What is the correct asymptotic order of $\operatorname{ex}(n, H)$?

Before we start, we state an easy result which will allow us to compare the extremal numbers of related graphs.

Lemma 4.1

If H_1 is a subgraph of H_2, then $\operatorname{ex}(n, H_1) \leq \operatorname{ex}(n, H_2)$.

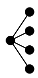
Proof. Suppose this is false. Let G be an H_1-free graph on n vertices with maximal number of edges. Then $e(G) = \operatorname{ex}(n, H_1) > \operatorname{ex}(n, H_2)$, so G has a copy of H_2. But H_1 is a subgraph of H_2, so G has a copy of H_1, contradiction. □

We can start our analysis by tackling complete bipartite graphs. That is the subject of the **Zarankiewicz problem**, which asks for precise asymptotic orders of $\operatorname{ex}(n, K_{s,t})$ for $s \leq t$. This problem is still open, but many levels of progress have been made.

Theorem 4.2: Kővári-Sós-Turán (1954)

$$\operatorname{ex}(n, K_{s,t}) = O(n^{2-\frac{1}{s}})$$

Proof. Let G be a $K_{s,t}$-free graph with n vertices and m edges.

First, assume that all vertices in G have degree at least $s-1$. We will count the number of $K_{1,s}$, e.g.  if $s = 4$, in G.

First, for any set A of s vertices, count the number of $K_{1,s}$ which use A as the part of size s. There is one for each common neighbor of A. But A has less than t common neighbors, otherwise A and t of their common neighbors form a $K_{s,t}$. So there are at most $t-1$ of these $K_{1,s}$ implying $\#K_{1,s} \le \binom{n}{s}(t-1) \le \frac{n^s(t-1)}{s!}$.

Now, for each vertex v, count the number of $K_{1,s}$ which use v as the part of size 1. There is one for each set of s neighbors of v for a total of $\binom{\deg v}{s}$. Thus

$$\#K_{1,s} = \sum_{v \in V} \binom{\deg v}{s} \ge n\binom{\frac{1}{n}\sum \deg v}{s} = n\binom{2m/n}{s} \ge \frac{n(\frac{2m}{n} - s + 1)^s}{s!}$$

by convexity[2] and the Handshake Lemma.

Combining these two bounds gives

$$m \le \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n = O(n^{2-\frac{1}{s}}).$$

Now, assume that G has vertices of degree less than $s-1$. Consider the graph G' formed by adding arbitrary edges to each vertex v with $\deg v < s-1$ until $\deg v = s-1$. The new graph G' is $K_{s,t}$-free (how could a $K_{s,t}$ be formed?) so it satisfies $e(G') = O(n^{2-\frac{1}{s}})$ from the previous work. Since $e(G) \le e(G')$, we are done. ∎

Remark. Because every bipartite graph is a subgraph of a complete bipartite graph, Kővári-Sós-Turán gives an upper bound on $\operatorname{ex}(n, H)$ for every bipartite graph H.

This has implications in areas besides graph theory. For example, here is a geometric application.

Fact 4.3

Given n points in the plane, at most $2n^{\frac{3}{2}}$ pairs are distance 1 apart.

Proof. The graph on the points with edges indicating unit distance is $K_{2,3}$-free because circles only intersect twice. So apply the bound obtained in Kővári-Sós-Turán. ∎

[2]The function $\binom{x}{s} = \frac{x(x-1)\cdots(x-s+1)}{s!}$ is convex for $x \ge s-1$.

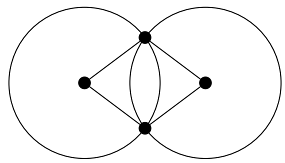


Figure 6: The unit distance graph is $K_{2,3}$-free.

Kővári-Sós-Turán actually gives the upper bound of $\frac{1}{\sqrt{2}}n^{\frac{3}{2}} + \frac{1}{2}n = (\frac{1}{\sqrt{2}} + o(1))n^{\frac{3}{2}}$. The best known upper bound, due to Spencer-Szemerédi-Trotter (1984), is $O(n^{\frac{4}{3}})$. For reference, the current best lower bound (i.e. an upper bound cannot be improved past this point), due to Erdős (1946), is $n^{1+\Omega((\log\log n)^{-1})}$.

4.1 Lower-bounding $\mathrm{ex}(n, K_{s,t})$

It is believed that the bound in Kővári-Sós-Turán is tight.

Conjecture 4.1: KST Conjecture

$$\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-\frac{1}{s}})$$

This is proven for some values of s, t but still open for most cases where t is not too much bigger than s.

Theorem 4.4: Erdős-Rényi-Sós (1966)

$$\mathrm{ex}(n, K_{2,2}) \geq \left(\frac{1}{2} - o(1)\right) n^{\frac{3}{2}}$$

Proof. Let $p = (1 - o(1))\sqrt{n}$ be a prime in the interval $[x - x^{0.525}, x]$ for $x = \sqrt{n}$ (possible due to a result of Baker-Harman-Pintz in 2001). Construct the graph G on n vertices with $p^2 - 1$ vertices labeled with coordinates (a, b) for $a, b = 0, 1, \ldots, p-1$ except for $(0, 0)$, as well as $n - p^2 + 1$ other vertices. Put an edge between (a, b) and (x, y) if $ax + by \equiv 1 \pmod p$ (the $n - p^2 + 1$ other vertices have degree 0). If (x, y) is a common neighbor of (a, b) and (c, d), then

$$\begin{aligned} ax + by &\equiv 1 \pmod p \\ cx + dy &\equiv 1 \pmod p \end{aligned}$$

which has at most one solution for (x, y)[3]. So G is $K_{2,2}$ free. But each vertex with a

[3] If there are multiple, then the linear map $(x, y) \to (ax + by, cx + dy)$ in $\mathbb{F}_p^2$ is not injective so its image is a one-dimensional subspace of $\mathbb{F}_p^2$. This subspace must be when both coordinates are equal, so $(a, b) = (c, d)$.

coordinate has degree at least $p-1$[4], so

$$e(G) = \frac{1}{2}\sum_{v\in V} \deg v \geq \frac{1}{2}(p^2-1)(p-1) = \left(\frac{1}{2} - o(1)\right)p^3 = \left(\frac{1}{2} - o(1)\right)n^{\frac{3}{2}}. \qquad \blacksquare$$

Theorem 4.5: Brown (1966)

$$\text{ex}(n, K_{3,3}) \geq \left(\frac{1}{2} - o(1)\right)n^{\frac{5}{3}}$$

The proof is similar to that of Erdős-Rényi-Sós but with more details and is omitted here.

These theorems imply that the KST conjecture is true for $s = 2, 3$. But what about $s \geq 4$?

It turns out that the KST conjecture for $K_{4,4}$ is still open. But we have results for all $t \geq f(s)$ for some function f.

Theorem 4.6: Alon-Kollár-Rónyai-Szabó (1996)

$$\text{ex}(n, K_{s,t}) = \Theta(n^{2-\frac{1}{s}}) \quad \text{if } t \geq (s-1)! + 1$$

The proof of this theorem uses a graph formed by the norm map of the finite field $\mathbb{F}_{p^s}$; this graph provides a construction implying the desired lower bound[5].

4.2 Avoiding sparse bipartite graphs

Kővári-Sós-Turán provides an upper bound on $\text{ex}(n, H)$ for every bipartite graph H, but the bound is quite wasteful for bipartite H that are not close to complete. For example, trees are far from complete.

First, let us get a lower bound on the extremal numbers of trees. If we consider the graph with $\frac{n}{k-1}$ connected components, each of which is a copy of K_{k-1}, then there is no copy of T where T is a tree on k vertices. This graph has $\frac{k-2}{2}n$ edges. Thus $\text{ex}(n, T) \geq \frac{k-2}{2}n$.

Theorem 4.7

$$\text{ex}(n, T) \leq (k-2)n - 1$$

[4] It should have degree p by counting solutions to $ax + by \equiv 1 \pmod p$, but sometimes $(x, y) = (a, b)$ is a solution (corresponding to a self-loop) so we might need to subtract one.

[5] This method actually only works for $t \geq s! + 1$ but can be modified to prove the theorem by using a projective norm map on $\mathbb{F}_{p^{s-1}} \times \mathbb{F}_p^\times$ instead.

We first prove a lemma.

Lemma 4.8

Every graph G contains a subgraph with minimum degree more than half of the average degree of G.

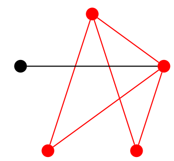


Figure 7: The colored subgraph has the desired property.

Proof. Let $t = \frac{e(G)}{|V(G)|}$ be half the average degree of G (by the Handshake Lemma). Define a sequence of graphs as follows: Let $G_0 = G$. If G_i is empty (no vertices) or has minimum degree more than t, stop. Otherwise, remove a vertex of minimal degree from G_i (and all its edges) and call the new graph G_{i+1}. The average degree of G_{i+1} is at least that of G_i.

If we stop at an empty graph, then we stop at $G_{|V(G)|}$ because G_i has $|V(G)| - i$ vertices. At each step, we remove at most t edges, with the last step removing no edges (because a graph on 1 vertex has no edges). So we remove at most $(|V(G)| - 1)t$ edges, but there are $|V(G)|t$ edges, contradiction. Thus we stop at a non-empty graph.

So the graph that we stopped at has minimum degree more than t as desired. $\square$

Now we prove the inequality on trees.

Proof. Let G be a graph with at least $(k-2)n$ edges. By the lemma, G has a subgraph H with minimal degree more than $k-2$, so every vertex in H has degree at least $k-1$. It suffices to show that H has a copy of T.

Root the tree T and let $v_1, \ldots, v_k$ be a topological sort of T. Choose any vertex u_1 of H. For $i = 2, \ldots, k$, do the following recursion: Let v_j be the parent of v_i, with $j < i$. If $j = 1$, then $\deg u_j \geq k-1 > i-2$ so there is some child c of u_1 that is not in $\{u_2, \ldots, u_{i-1}\}$. Otherwise $j > 1$ so v_j has a parent $v_{j'}$ and $\deg u_j - 1 \geq k-2 > i-3$ so there is some child c of u_j that is not in $\{u_1, \ldots, u_{i-1}\} \setminus \{u_j, u_{j'}\}$. Either way, set u_i to be the vertex c.

Thus we have embedded T into H, which is embedded into G, so G has a copy of T. $\blacksquare$

So $\operatorname{ex}(n, T) = \Theta(n)$. In fact, the lower bound is correct, but the proof is long and is omitted here.

Theorem 4.9: Ajtai-Komlós-Simonovits-Szemerédi (2015)

$$\mathrm{ex}(n, T) = \frac{k-2}{2}n + o(1)$$

Another example of a sparse bipartite graph is the even cycle C_{2k}.

Theorem 4.10: Bondy-Simonovits (1974)

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}})$$

This proof is quite advanced and is omitted here.

Once again, it is still an open problem if Bondy-Simonovits is the correct asymptotic. We know that $\mathrm{ex}(n, C_{2k}) = \Theta(n^{1+\frac{1}{k}})$ for $k = 2, 3, 5$ but not for any other values of k.

5 Concluding remarks

Throughout this handout, we examined several results about forbidding certain patterns in graphs. More generally, we consider a property and ask how big we can make our objects before these objects are forced to have the property.

As we saw in the intro, this is similar in nature to some questions asked in additive combinatorics. When attempting to prove Roth's theorem by converting the setting to a graph, a natural question to ask might be: What is the maximum number of edges in a triangle-free graph on n vertices? Mantel's theorem answers this. However, it does not imply the bound we want in Roth's theorem.

Instead, the correct question to ask is: What is the maximum number of edges in a graph on n vertices where every edge is in a unique triangle? Using Szemerédi's graph regularity lemma, one can show that the answer is $o(n^2)$. This in turn implies Roth's theorem.

The natural generalization of Roth's theorem is true, for roughly the same reasons.

Theorem 5.1: Szemerédi (1975)

Every subset of $\mathbb{N}$ with positive upper density contains arbitrarily long arithmetic progressions.

What is surprising, though, is a recent celebrated result of Green-Tao.

Theorem 5.2: Green-Tao (2008)

The prime numbers contain arbitrarily long arithmetic progressions.

The proofs of these theorems combine intuition from extremal graph theory, methods from regularity, and careful understanding of the graph theoretical structure behind these properties to prove some of the simplest statements in additive combinatorics.

If you want to learn more about these vast fields mentioned, their connections, and more, I encourage you to check out Yufei Zhao's subject on Graph Theory and Additive Combinatorics, available on MIT OpenCourseWare (as taught in Fall 2019).