

ELIMINATION OF THE PARAMETERS IN AN ELLIPTIC PARAMETRIZATION OF THE SCHWARZ DIAMOND SURFACE

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Date: September 15, 2026.

1991 *Mathematics Subject Classification.* Primary 53A10; Secondary 33E05, 53C42.

Key words and phrases. Triply periodic, Schwarz D surface, Schwarz diamond surface, Schwarz P surface, Schwarz primitive surface, separable minimal surface, Jacobi elliptic functions, conjugate minimal surfaces.

ABSTRACT. Let Ω be the curvilinear square bounded by the four circular arcs $|\zeta \mp 1| = \sqrt{2}$, $|\zeta \mp i| = \sqrt{2}$, and let $W = \sqrt{1 + 14\zeta^4 + \zeta^8}$. Put

$$\begin{aligned}\theta(\zeta) &= \arcsin \frac{2(1+i)\zeta}{\sqrt{1+4i\zeta^2-\zeta^4}}, \\ f &= \frac{1}{4}(-iF[\theta, \tfrac{1}{4}] + F[\theta, \tfrac{3}{4}]), \quad g = \frac{1}{4}(iF[\theta, \tfrac{1}{4}] + F[\theta, \tfrac{3}{4}]), \\ \mathfrak{h} &= (2 - \sqrt{3})F[\arcsin(i(2 + \sqrt{3})\zeta^2), (2 - \sqrt{3})^4].\end{aligned}$$

These are, verbatim, the three functions of our companion paper [P] on the primitive surface P, cited here as [P]: the two papers start from the same point, and differ only in which real parts are taken. With $X = (\frac{1}{2} + \kappa \operatorname{Im} f, \frac{1}{2} + \kappa \operatorname{Im} g, -\frac{1}{2} + \kappa \operatorname{Re} \mathfrak{h})$ and $\kappa = 2/K[1/4]$, the map X parametrizes a fundamental patch of Schwarz's diamond surface D, the conjugate of P.

Part I (§§1–6) builds the algebraic apparatus, and is common property with [P]. The differentials are algebraic on the genus-3 curve $W^2 = 1 + 14\zeta^4 + \zeta^8$, namely $f' = (1 - \zeta^2)/W$, $g' = i(1 + \zeta^2)/W$, $\mathfrak{h}' = 2i\zeta/W$ (Proposition 2.2). The three rotated coordinates $x + y - 1$, $x - y$, $z + \frac{1}{2}$ are the imaginary or real parts of *single* elliptic integrals $\mathcal{P} = \frac{1}{2}F[\theta, \frac{3}{4}]$, $\mathcal{N} = \frac{1}{2}F[\theta, \frac{1}{4}]$, $\mathfrak{h}$, and all three admit *rational* Jacobi dictionaries at the *same* modulus -3 :

$$\operatorname{sn}(\mathcal{P}, -3) = \frac{(1+i)\zeta}{1-i\zeta^2}, \quad \operatorname{sn}(2\mathfrak{h}, -3) = \frac{2\zeta^2}{1+\zeta^4}, \quad \operatorname{sn}(2f, -3) = \frac{2\zeta(1+\zeta^2)}{W},$$

$\mathfrak{h} := -i\mathfrak{h}$, with cn , dn likewise rational in ζ and in $r_{\pm} = \sqrt{1 \pm 4i\zeta^2 - \zeta^4}$, $r_+ r_- = W$. The normalizing constant is identified exactly: $\varpi := \kappa^{-1} = \frac{1}{2}K[1/4]$, and $K[-\frac{1}{3}] = \sqrt{3}\varpi$, $K[(2 - \sqrt{3})^4] = \frac{2+\sqrt{3}}{2}\varpi$.

Part II (§§7–12) carries out the elimination that brings about

$$(\star) \quad \operatorname{cd}(\sqrt{3}\varpi(x+y-1), -\tfrac{1}{3}) \operatorname{dc}(\sqrt{3}\varpi(x-y), -\tfrac{1}{3}) \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(\sqrt{3}\varpi z, -\tfrac{1}{3}) = -1,$$

equivalently $\mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(1-z) + \mathcal{T}(x) + \mathcal{T}(y) + \mathcal{T}(1-z) = 0$ with $\mathcal{T} = \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(\sqrt{3}\varpi \cdot, -\frac{1}{3})$, equivalently $\Phi(x)\Phi(y) + 1 = \Phi(z)(\Phi(x) + \Phi(y))$ with $\Phi = |\mathcal{T}|$. The decisive one-variable step is an identity valid at an arbitrary modulus, $S_a D_a C_b + S_b D_b C_a = \operatorname{sn}(a+b) \operatorname{dn}(a-b)(1 - m \operatorname{sn}^2 a \operatorname{sn}^2 b)$, which collapses $(\star)$ to a relation in the three separated coordinates alone. Complexification ($\omega := \bar{\zeta}$) makes the first two arguments half-differences and the third a half-difference carrying a half-period shift; the swap $\zeta \leftrightarrow \omega$ acts by $w_q \mapsto K - w_q$. Because the invariant $\bar{\Theta}$ satisfies $\bar{\Theta}(K - w) = \bar{\Theta}(w)$, the combination $\omega \partial_{\zeta} + \zeta \partial_{\omega}$ eliminates the height entirely, leaving a single *antisymmetric rational* identity (D0) in the field $\mathbb{K} = \mathbb{C}(\zeta, \omega)(r_{\pm}(\zeta), r_{\pm}(\omega))$ of degree ≤ 16 — an exact analog of [P]'s (P0). We verify (D0) on six algebraic curves and in low homogeneous degrees; a proof of (D0) in general, and with it a certificate-based proof of $(\star)$, is left open.

Part III (§§13–18) supplies the complete proof, and the clean form of the equation. Setting $D(t) := \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(2\varpi t, \frac{1}{4})$ and $\Psi := \log D$, we show that $(\star)$ is *additively separable*, $\Psi(x) + \Psi(y) + \Psi(z) = 0$, hence equivalent to the trilinear *product* identity

$$D(x)D(y)D(z) = 1$$

on the region where all three factors are positive. The separating function satisfies $\Psi'^2 = 4\varpi^2(1 + 2 \cosh 2\Psi)$, against [P]'s $1 + 2 \cos 2\Psi$: the two surfaces are the two signs of one separation constant, and D is the isotropic member of the branch $\kappa > 0$ of the classification of Kim and Ogata [13]. The resulting surface is minimal and everywhere regular, whence the local converse; and Björling's theorem along the straight segment $\{(\frac{1}{2}, t, -\frac{1}{2})\}$ identifies it with the Weierstrass patch, which proves $(\star)$ outright (Theorem 1.3). Finally we identify Nitsche's transcendental function [14, §83] exactly as $\frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(\cdot, \frac{1}{4})$, so that $D(x)D(y)D(z) = 1$ is his equation (38): the product form is classical, due to Schwarz [17] and Cayley [4], and priority is recorded in §18.

For an overview of this paper and [P], please see [arXiv:2609.14206](#).

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PART I. THE PARAMETRIZATION AND ITS ALGEBRAIC DICTIONARY

1. INTRODUCTION; THE MAIN THEOREM

1.1. **Data and elliptic conventions.** Throughout, for $m \in \mathbb{C}$ and admissible φ ,

$$F[\varphi, m] = \int_0^{\sin \varphi} \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-m\tau^2}}, \quad K[m] = F[\pi/2, m], \quad (1.1)$$

and $\operatorname{sn}(u, m)$, $\operatorname{cn}(u, m)$, $\operatorname{dn}(u, m)$ are the Jacobi functions in the *parameter* convention $m = k^2$ (so $k' = \sqrt{1-m}$), as in [2]; thus

$$\operatorname{sn}(F[\varphi, m], m) = \sin \varphi, \quad \operatorname{sn}^2 + \operatorname{cn}^2 = 1, \quad \operatorname{dn}^2 + m \operatorname{sn}^2 = 1. \quad (1.2)$$

We use the quotient abbreviations $\operatorname{sd} = \operatorname{sn} / \operatorname{dn}$, $\operatorname{cd} = \operatorname{cn} / \operatorname{dn}$, $\operatorname{nd} = 1 / \operatorname{dn}$, $\operatorname{cs} = \operatorname{cn} / \operatorname{sn}$, $\operatorname{sc} = \operatorname{sn} / \operatorname{cn}$, $\operatorname{nc} = 1 / \operatorname{cn}$, $\operatorname{dc} = \operatorname{dn} / \operatorname{cn}$, $\operatorname{ns} = 1 / \operatorname{sn}$, $\operatorname{ds} = \operatorname{dn} / \operatorname{sn}$. These are the conventions of [P, §1.1] and of Mathematica's `EllipticF`, `EllipticK`, `JacobiSN`.

Throughout,

$$\beta := 2 + \sqrt{3}, \quad \beta + \beta^{-1} = 4, \quad \beta - \beta^{-1} = 2\sqrt{3}, \quad \beta^2 + \beta^{-2} = 14, \quad \beta^2 - 1 = 2\sqrt{3}\beta, \quad (1.3)$$

$$W(\zeta) := \sqrt{1 + 14\zeta^4 + \zeta^8}, \quad W(0) = +1, \quad (1.4)$$

and we fix once and for all the auxiliary quantities

$$\pi_{\pm} := 1 \pm i\zeta^2, \quad r_{\pm} := \sqrt{1 + 4i\zeta^2 - \zeta^4}, \quad s_{\pm} := \sqrt{1 + \beta^{\pm 2}\zeta^4}, \quad E := 1 + \zeta^4, \quad (1.5)$$

all square roots being the branches equal to $+1$ at $\zeta = 0$ (Convention 1.2). The domain is

$$\Omega := \{\zeta \in \mathbb{C} : |\zeta - 1| \leq \sqrt{2}, |\zeta + 1| \leq \sqrt{2}, |\zeta - i| \leq \sqrt{2}, |\zeta + i| \leq \sqrt{2}\}, \quad (1.6)$$

the same domain as in [P, §1.1]. Finally

$$\kappa := \frac{2}{K[1/4]} = 1.1864152923\dots, \quad \varpi := \kappa^{-1} = \frac{1}{2}K[1/4] = 0.8428751774\dots, \quad (1.7)$$

and

$$K_1 := K[-\tfrac{1}{3}] = \sqrt{3}\varpi = 1.4599026\dots \quad (1.8)$$

(Proposition 5.5 and Table 5.1).

Remark 1.1 (the normalizing constant, and the bridge to [P]). In [P, §1.1] the constant $\varpi := \kappa^{-1}$ equals $K[-3] = \frac{1}{2}K[3/4] = \frac{1}{2}K'[1/4]$; here it equals $\frac{1}{2}K[1/4]$. Thus in *both* papers ϖ is half the quarter period of the horizontal modulus, and

$$\frac{\varpi_P}{\varpi_D} = \frac{K'[1/4]}{K[1/4]} = 1.2792615711710064662\dots \quad (1.9)$$

is Schwarz's constant of §12.1 — not an artifact of either normalization, but a consequence of the definition $\varpi = \kappa^{-1}$ in each. Every other symbol of this paper is that of [P]; Table 1.1 lists the few that carry different *values*.

TABLE 1.1. The notation bridge. Rows above the rule are shared verbatim, symbol and meaning; rows below are the symbols whose *value* differs between the two papers.

symbol	meaning	where
θ, s	the amplitude (1.10) and its sine	[P, §1.2]
$f, g, \mathfrak{h}$	the three integrals (1.11)–(1.13)	[P, §1.2]
$\mathfrak{h} = -i\mathfrak{h}$	the real height	[P, (1.14)]
$W, \beta, \pi_{\pm}, r_{\pm}, s_{\pm}, E$	(1.3), (1.4), (1.5)	[P, §1.1]
Ω, ζ_k, ι	domain, corners, arc involution	[P, §§1.1, 3, 6]
$\mathcal{P}, \mathcal{N}, A, B$	$\mathcal{P} = A = f + g, \mathcal{N} = i(f - g), B = -i\mathcal{N} = f - g$	[P, §2.3]
p, q, Z	the separated coordinates	(1.17); cf. [P, (2.11)]
$\mathbb{K}, W_{\zeta}, W_{\omega}$	the field, and W at ζ, ω	[P, §4]
Ψ	the separating function of Part III	(14.1); cf. [P, §14]
F, G, Q	defect, separated defect, clean function	§16; cf. [P, §16]
$\varpi = \kappa^{-1}$	$\frac{1}{2}K[1/4]$ here; $\frac{1}{2}K[3/4] = K[-3]$ in [P]	(1.7)
κ	$2/K[1/4]$ here; $3/(2K[1/9])$ in [P]	(1.7)
horizontal modulus	$-\frac{1}{3}$ ($\sim \frac{1}{4}$) here; -3 ($\sim \frac{3}{4}$) in [P]	§5
$\mathcal{T}$	$\frac{\text{sn dn}}{\text{cn}}(K_1 \cdot, -\frac{1}{3})$; the $\mathcal{T}$ of [P, Table 12.1]	Definition 4.10
D	$\frac{\text{sn dn}}{\text{cn}}(2\varpi \cdot, \frac{1}{4})$; the lattice function of Part III	(4.16)

1.2. **The three functions.** Let $\zeta = u + iv$ and

$$\theta(\zeta) := \arcsin(s(\zeta)), \quad s(\zeta) := \frac{2(1+i)\zeta}{\sqrt{1+4i\zeta^2-\zeta^4}} = \frac{2(1+i)\zeta}{r_+}. \quad (1.10)$$

Define

$$f(\zeta) = \frac{1}{4} \left(-i F\left[\theta(\zeta), \frac{1}{4}\right] + F\left[\theta(\zeta), \frac{3}{4}\right] \right), \quad (1.11)$$

$$g(\zeta) = \frac{1}{4} \left(i F\left[\theta(\zeta), \frac{1}{4}\right] + F\left[\theta(\zeta), \frac{3}{4}\right] \right), \quad (1.12)$$

$$\mathfrak{h}(\zeta) = \beta^{-1} F\left[\arcsin(i\beta\zeta^2), \beta^{-4}\right], \quad \beta^{-1} = 2 - \sqrt{3}. \quad (1.13)$$

Equations (1.10)–(1.13) are the starting point of the paper. They are, verbatim, the three functions and the amplitude of [P, §1.2]: *the diamond and the primitive surface are built from the same three integrals*, and differ only in which real parts are taken (Remark 2.4).

Convention 1.2 (branches). All square roots in (1.5) — in particular $W, r_{\pm}, s_{\pm}$ and the radical inside (1.10) — are the analytic continuations from $\zeta = 0$ of the branches equal to +1 there, along paths in Ω° . By (1.1) the composite $F[\theta(\zeta), m]$ depends on θ only through $s = \sin \theta$, so it is to be read as

$$F[\theta(\zeta), m] = \int_0^{s(\zeta)} \frac{d\tau}{\sqrt{(1-\tau^2)(1-m\tau^2)}}, \quad (1.14)$$

the path of integration being the continuous deformation, as ζ moves in Ω° from 0, of the segment $[0, s]$; and similarly for $F[\arcsin(i\beta\zeta^2), \beta^{-4}]$. Equivalently — and this is the form we use — $f, g, \mathfrak{h}$ are the primitives (2.5) of the algebraic differentials (2.4) vanishing at $\zeta = 0$.

Near $\zeta = 0$ all of these agree with Mathematica's principal branches. Farther out the principal **ArcSin** and **EllipticF** jump: r_+ vanishes at two of the four corners of Ω and r_- at the other two (Lemma 4.1), so $s \rightarrow \infty$ there; and $i\beta\zeta^2 \rightarrow \pm 1$ on parts of $\partial\Omega$. At such points (1.14), i.e.,

(2.5), is the intended continuation. This is the branch convention of [P, §1.2] verbatim, and the point at which the present paper is more delicate than [P]: the boundary curves of §10 reach $|\omega| \approx 1$, well beyond the branch points, and require the leg-by-leg transport of Appendix A, Cell A4.

1.3. The map. Put

$$X(u, v) = (x, y, z) := \left(\frac{1}{2} + \kappa \operatorname{Im} f, \frac{1}{2} + \kappa \operatorname{Im} g, -\frac{1}{2} + \kappa \operatorname{Re} \mathfrak{h} \right). \quad (1.15)$$

This is the system of three equations in five unknowns of the problem. With $\mathfrak{h} := -i\mathfrak{h}$, so that

$$\operatorname{Re} \mathfrak{h} = -\operatorname{Im} \mathfrak{h}, \quad (1.16)$$

(1.15) reads $X = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) + \kappa(\operatorname{Im} f, \operatorname{Im} g, -\operatorname{Im} \mathfrak{h})$: all three coordinates are imaginary parts of the *same* Weierstrass triple, which is what makes the surface minimal (Proposition 2.3). The base point is $X(0) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

1.4. The theorem. Put

$$p := x + y - 1, \quad q := x - y, \quad Z := z + \frac{1}{2}, \quad (1.17)$$

so that $p = q = Z = 0$ at the base point, exactly as in [P, (2.11)].

Theorem 1.3 (implicit representation of D). *For every $\zeta \in \Omega$ the point $(x, y, z) = X(\zeta)$ satisfies*

$$\boxed{\operatorname{cd}(K_1 p, -\frac{1}{3}) \cdot \operatorname{dc}(K_1 q, -\frac{1}{3}) \cdot \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(K_1 z, -\frac{1}{3}) = -1} \quad (1.18)$$

where $K_1 = K[-\frac{1}{3}] = \sqrt{3}\varpi$ and $\varpi = \kappa^{-1}$ (Proposition 5.5). We call this representation $(\star)$.

The proof is completed in §18 (Theorem 18.4), by Björling's theorem along the straight segment $\{(\frac{1}{2}, t, -\frac{1}{2})\}$ of Proposition 6.7, using the minimality and regularity of the separable surface established in §17. Part II reduces (1.18) instead to a single symmetric rational identity (D0) (Theorem 9.10), which remains open; see Remark 18.5 for why the two routes are ordered oppositely to those of [P], and §12 for the status of every step.

Remark 1.4 (equivalent forms). Three rewritings of (1.18) are used throughout.

(a) *The symmetric form.* With

$$\mathcal{T}(\xi) := \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(K_1 \xi, -\frac{1}{3}) = -\frac{d}{d\xi} \frac{1}{K_1} \log \operatorname{cn}(K_1 \xi, -\frac{1}{3}), \quad (1.19)$$

(1.18) is equivalent to

$$\mathcal{T}(x) \mathcal{T}(y) \mathcal{T}(1-z) + \mathcal{T}(x) + \mathcal{T}(y) + \mathcal{T}(1-z) = 0, \quad (1.20)$$

a fully symmetric relation in the three quantities $\mathcal{T}(x)$, $\mathcal{T}(y)$, $\mathcal{T}(1-z)$ (Proposition 6.1).

(b) *The Φ -form.* With $\Phi := |\mathcal{T}|$, equivalently

$$\Phi(\xi) = \sqrt{\frac{1 - \operatorname{cn}(2K_1 \xi, -\frac{1}{3})}{1 + \operatorname{cn}(2K_1 \xi, -\frac{1}{3})}} \quad (\geq 0), \quad (1.21)$$

and on the box $[0, 1]^2 \times [-1, 0]$ that contains $X(\Omega)$ (Proposition 6.15), (1.20) is equivalent to

$$\Phi(x)\Phi(y) + 1 = \Phi(z)(\Phi(x) + \Phi(y)). \quad (1.22)$$

This is the sign-resolved shadow of (1.20): it is the form in which the face-centered cubic invariance is provable over $\mathbb{R}$ (Corollary 6.3), and the absolute value is genuinely needed there — in contrast to [P], whose relation is sign-honest [P, Prop. 6.3(c)].

(c) *The product form.* With $D(t) := \frac{\text{sn dn}}{\text{cn}}(2\varpi t, \frac{1}{4})$ and $\Psi := \log D$, (1.18) is equivalent to the additively separable relation $\Psi(x) + \Psi(y) + \Psi(z) = 0$, i.e., to

$$D(x) D(y) D(z) = 1, \quad (1.23)$$

wherever the three factors are positive (Theorem 16.3). Equation (16.12) is *classical*: it is equation (38) of [14, §83], where the surface is attributed to Schwarz [17, vol. I, pp. 1–125] and the classification of the surrounding class to Cayley [4]. What Part III adds is the identification of Nitsche's implicitly defined function as $\frac{\text{sn dn}}{\text{cn}}(\cdot, \frac{1}{4})$, the exact constants, the equivalence with (1.18), and the bridge to the Weierstrass data. See §18.

Remark 1.5 (why $(\star)$ looks the way it does). Comparison with [P, (1.15)], $\text{sn}(\varpi p, -3) \text{sn}(\varpi q, -3) = \frac{1}{3} \text{sn}(3\varpi Z, \frac{1}{9})$, is instructive. Both relations are products of three functions, one of each separated coordinate. Three differences, all traceable to the single $\text{Re} \leftrightarrow \text{Im}$ swap of Remark 2.4, are:

- (i) *The first two factors are reciprocal in type.* Here they are cd and dc ; in [P] both are sn . Equivalently (1.18) says that a certain *quotient* $\text{cn}(w_p)/\text{cn}(w_n)$ is a function of the height, whereas $(\star)$ of [P] says that a *product* is. This is the reason the height is eliminated here by $\omega\partial_\zeta + \zeta\partial_\omega$ rather than by $\omega\partial_\zeta - \zeta\partial_\omega$ (§9).
- (ii) *The height enters uncentered.* The third argument is $K_1 z$, not $K_1 Z$; since $z = Z - \frac{1}{2}$, this exhibits the half-period shift $-\frac{1}{2}K_1$ which [P, Rem. 7.3] identifies as the structural difference between the two papers. In particular both sides of (1.18) are nonzero at the base point, where the three factors are 1, 1, -1 ; the first-order test at the base point is vacuous, and the content appears at second order (Proposition 6.5).
- (iii) *One modulus, not two.* All three factors sit at $-\frac{1}{3}$, whereas [P] needs -3 horizontally and $\frac{1}{9}$ vertically. The reason is again the shift: no half-argument step is required on the height side here. Conversely, [P]'s relation has no exceptional set, whereas (16.12) must be restricted by a positivity condition (Theorem 16.3).

1.5. What is proved here.

- (a) The algebraic form of the differentials (Proposition 2.2), derived from (1.10)–(1.13); and the full dictionary (§4): $\mathcal{P}$, $\mathcal{N}$, $\mathbf{h}$ and $2f$, $2g$ all at modulus -3 , with rational $\text{sn}, \text{cn}, \text{dn}$; plus the height dictionary of Theorem 4.7. The exact identification $\varpi = \kappa^{-1} = \frac{1}{2}K[1/4]$, $K_1 = \sqrt{3}\varpi$, $K[(2 - \sqrt{3})^4] = \frac{2+\sqrt{3}}{2}\varpi$ is §5.
- (b) The transformation law under the arc reflection, the four straight boundary segments, the corner images $(1, 1, -1), (0, 1, 0), (1, 0, 0), (0, 0, -1)$ — the vertices of a regular tetrahedron of edge $\sqrt{2}$ — the convex-hull bound $X(\Omega) \subset [0, 1]^2 \times [-1, 0]$, and the face-centered cubic invariance of (1.22) (§6).
- (c) The key one-variable identity $S_a D_a C_b + S_b D_b C_a = \text{sn}(a+b) \text{dn}(a-b) \Delta_{ab}$ at an arbitrary modulus, and the collapse of (1.20) to a relation in p, q, z alone (§8).
- (d) Reduction of Theorem 1.3 to one *symmetric* rational identity (D0) in a field of degree ≤ 16 (§9, Theorem 9.10).
- (e) Independent verifications of (D0): on the six algebraic curves (10.1) and in low homogeneous degrees (§10); and the jets of orders 2, 4, 6, 8, which force the horizontal modulus through $3m^2 + 10m + 3 = 0$ (§11).
- (f) The separable form (§§13–16); the minimality and everywhere regularity of the resulting surface, and the local converse (§17); the dichotomy theorem, which identifies D and P as the only two isotropic separable minimal surfaces (Theorem 17.7); and the proof of Theorem 1.3 by Björling (§18).

Left open: (D0), and the global converse. §12 and §18 state the status precisely and record the relation to [P].

2. THE WEIERSTRASS DATA

This section removes all transcendence from (1.11)–(1.13): the three functions have *algebraic* derivatives on the curve $W^2 = 1 + 14\zeta^4 + \zeta^8$. Every statement is that of [P, §2]; only §2.3 differs, and there only by the $\text{Re} \leftrightarrow \text{Im}$ swap.

2.1. The amplitude.

Lemma 2.1 (Legendre data of θ and of $\arcsin(i\beta\zeta^2)$). *With s as in (1.10) and $\sigma := i\beta\zeta^2$,*

$$s^2 = \frac{8i\zeta^2}{r_+^2}, \quad 1 - s^2 = \frac{r_-^2}{r_+^2}, \quad 1 - \frac{1}{4}s^2 = \frac{\pi_+^2}{r_+^2}, \quad 1 - \frac{3}{4}s^2 = \frac{\pi_-^2}{r_+^2}, \quad s' = \frac{2(1+i)E}{r_+^3}, \quad (2.1)$$

$$1 - \sigma^2 = s_+^2, \quad 1 - \beta^{-4}\sigma^2 = s_-^2, \quad s_+s_- = W, \quad \sigma' = 2i\beta\zeta. \quad (2.2)$$

All branches are +1 at $\zeta = 0$ (Convention 1.2).

Proof. Since $(1+i)^2 = 2i$ and $r_+^2 = 1 + 4i\zeta^2 - \zeta^4$, the first formula of (2.1) is immediate, and then

$$1 - s^2 = \frac{r_+^2 - 8i\zeta^2}{r_+^2} = \frac{1 - 4i\zeta^2 - \zeta^4}{r_+^2} = \frac{r_-^2}{r_+^2}, \quad 1 - \frac{1}{4}s^2 = \frac{r_+^2 - 2i\zeta^2}{r_+^2} = \frac{1 + 2i\zeta^2 - \zeta^4}{r_+^2} = \frac{\pi_+^2}{r_+^2},$$

using $\pi_+^2 = (1 + i\zeta^2)^2 = 1 + 2i\zeta^2 - \zeta^4$; and likewise $1 - \frac{3}{4}s^2 = (1 - 2i\zeta^2 - \zeta^4)/r_+^2 = \pi_-^2/r_+^2$. For s' : with $(r_+^2)' = 8i\zeta - 4\zeta^3$,

$$s' = 2(1+i)\left(r_+^{-1} - \frac{1}{2}\zeta r_+^{-3}(r_+^2)'\right) = \frac{2(1+i)}{r_+^3}\left(r_+^2 - \frac{1}{2}\zeta(8i\zeta - 4\zeta^3)\right) = \frac{2(1+i)(1 + \zeta^4)}{r_+^3},$$

because $1 + 4i\zeta^2 - \zeta^4 - 4i\zeta^2 + 2\zeta^4 = 1 + \zeta^4 = E$. For (2.2): $1 - \sigma^2 = 1 + \beta^2\zeta^4 = s_+^2$ and $1 - \beta^{-4}\sigma^2 = 1 + \beta^{-2}\zeta^4 = s_-^2$, whose product is $1 + (\beta^2 + \beta^{-2})\zeta^4 + \zeta^8 = W^2$ by (1.3). $\square$

2.2. The differentials.

Proposition 2.2 (algebraic form of the differentials). *On Ω° , with the branches of Convention 1.2,*

$$\frac{d}{d\zeta}F\left[\theta, \frac{1}{4}\right] = \frac{2(1+i)\pi_-}{W}, \quad \frac{d}{d\zeta}F\left[\theta, \frac{3}{4}\right] = \frac{2(1+i)\pi_+}{W}, \quad (2.3)$$

and therefore

$$\boxed{f'(\zeta) = \frac{1 - \zeta^2}{W}, \quad g'(\zeta) = \frac{i(1 + \zeta^2)}{W}, \quad h'(\zeta) = \frac{2i\zeta}{W}, \quad h'(\zeta) = \frac{2\zeta}{W}.} \quad (2.4)$$

Consequently

$$f(\zeta) = \int_0^\zeta \frac{1 - \eta^2}{W(\eta)} d\eta, \quad g(\zeta) = \int_0^\zeta \frac{i(1 + \eta^2)}{W(\eta)} d\eta, \quad h(\zeta) = \int_0^\zeta \frac{2\eta d\eta}{W(\eta)}, \quad (2.5)$$

the integrals being taken along any path in Ω° from 0.

Proof. By (1.14) and the chain rule, $\frac{d}{d\zeta}F[\theta, m] = s'[(1 - s^2)(1 - ms^2)]^{-1/2}$. For $m = \frac{1}{4}$, Lemma 2.1 gives $(1 - s^2)(1 - \frac{1}{4}s^2) = r_-^2\pi_+^2/r_+^4$, whose square root (+1 at $\zeta = 0$) is $r_-\pi_+/r_+^2$; hence

$$\frac{d}{d\zeta}F\left[\theta, \frac{1}{4}\right] = \frac{2(1+i)E/r_+^3}{r_-\pi_+/r_+^2} = \frac{2(1+i)E}{r_+r_-\pi_+} = \frac{2(1+i)\pi_+\pi_-}{W\pi_+} = \frac{2(1+i)\pi_-}{W},$$

using $r_+r_- = W$ and $E = \pi_+\pi_-$ (Lemma 4.1). The computation for $m = \frac{3}{4}$ is identical with π_+ replaced by π_- , giving $2(1+i)\pi_+/W$. This is (2.3).

Now differentiate (1.11). Since $-i(1+i) = 1-i$,

$$f' = \frac{1}{4} \cdot \frac{2(1+i)}{W} [-i\pi_- + \pi_+] = \frac{1}{2W} [(1-i)\pi_- + (1+i)\pi_+] = \frac{1}{2W} \cdot 2(1-\zeta^2) = \frac{1-\zeta^2}{W},$$

because $(1+i)(1+i\zeta^2) + (1-i)(1-i\zeta^2) = [(1+i) + (1-i)] + \zeta^2[i(1+i) - i(1-i)] = 2 - 2\zeta^2$. Likewise, since $i(1+i) = -(1-i)$,

$$g' = \frac{1}{2W} [-(1-i)\pi_- + (1+i)\pi_+] = \frac{1}{2W} \cdot 2i(1+\zeta^2) = \frac{i(1+\zeta^2)}{W},$$

because $(1+i)(1+i\zeta^2) - (1-i)(1-i\zeta^2) = 2i + \zeta^2[i(1+i) + i(1-i)] = 2i + 2i\zeta^2$.

For $\mathfrak{h}$: by (2.2) the product $(1-\sigma^2)(1-\beta^{-4}\sigma^2) = s_+^2s_-^2 = W^2$, so (1.13) and the chain rule give $\mathfrak{h}' = \beta^{-1}\sigma'/W = \beta^{-1} \cdot 2i\beta\zeta/W = 2i\zeta/W$; and $\mathfrak{h}' = -i\mathfrak{h}' = 2\zeta/W$ by (1.16). Finally (2.5) follows since all three functions vanish at $\zeta = 0$. (Verified numerically to 16 digits at four points: Appendix A, Cell A1.) $\square$

Proposition 2.3 (conformality; the associate family). *Let $\phi := (f', g', \mathfrak{h}') = (\frac{1-\zeta^2}{W}, \frac{i(1+\zeta^2)}{W}, \frac{2\zeta}{W})$. Then*

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0, \quad |\phi|^2 = \frac{2(1+|\zeta|^2)^2}{|W|^2} > 0, \quad (2.6)$$

so for each Bonnet angle ψ the map $X_\psi := \operatorname{Re} \int e^{i\psi} \phi$ is a conformal minimal immersion of Ω° . The choice $\psi = \pi/2$, i.e., $X_{\pi/2} = \operatorname{Re} \int i\phi = -\operatorname{Im} \int \phi$, is (1.15) up to the sign of the third slot and the base point; $\psi = 0$ gives P [P, Prop. 2.3]. The Gauss map is $G = \zeta$ and the unit normal

$$N = \frac{1}{1+|\zeta|^2} (2u, 2v, 1-|\zeta|^2), \quad (2.7)$$

independent of ψ ; the height differential is $dh = 2\zeta d\zeta/W$. Consequently D and P are conjugate, hence isometric.

Proof. $(1-\zeta^2)^2 - (1+\zeta^2)^2 + 4\zeta^2 = 0$, which is (2.6)(i); and $|1-\zeta^2|^2 + |1+\zeta^2|^2 + 4|\zeta|^2 = 2 + 2|\zeta|^4 + 4|\zeta|^2 = 2(1+|\zeta|^2)^2$. Since $\phi \propto (1-G^2, i(1+G^2), 2G)$ with $G = \zeta$, the normal is (2.7); the rest is the Weierstrass–Enneper theorem [14, §3.3]. For the identification of (1.15) with $\psi = \pi/2$: $\operatorname{Re}(i\phi_j) = -\operatorname{Im} \phi_j$, so $X_{\pi/2} = -(\operatorname{Im} f, \operatorname{Im} g, \operatorname{Im} \mathfrak{h})$, and by (1.16) the third slot is $-\operatorname{Im} \mathfrak{h} = \operatorname{Re} \mathfrak{h}$, implying $\phi_3 = -2\zeta/W$; changing the sign of the first two slots is the reflection $(x, y, z) \mapsto (-x, -y, z)$, and adding $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ places the base point. $\square$

Remark 2.4 (D versus P in one line). Writing $\mathcal{P} := f + g$, $\mathcal{N} := i(f - g)$ (Proposition 2.5), the two surfaces are

$$D : (\operatorname{Im} \mathcal{P}, \operatorname{Re} \mathcal{N}, \operatorname{Im} \mathfrak{h}), \quad P : (\operatorname{Re} \mathcal{P}, \operatorname{Im} \mathcal{N}, \operatorname{Re} \mathfrak{h}),$$

up to the base point and the reflections just noted. The functions $f, g, \mathfrak{h}$ of (1.11)–(1.13) are the same in both papers; *every* Re becomes Im and conversely. This single swap is responsible for all the differences cataloged in Table 12.2: for the interchange $\frac{1}{4} \leftrightarrow \frac{3}{4}$ of the horizontal modulus, for the half-period shift on the height (Remark 1.5(ii)), for the straight boundary segments here against planar geodesics in [P], for the tetrahedron here against a square there, and for the cosh against the cos of §14.

2.3. Separation of the coordinates.

Proposition 2.5 (separation). *Put $\mathcal{P} := f + g$, $\mathcal{N} := i(f - g)$ and*

$$A := \mathcal{P} = f + g, \quad B := -i\mathcal{N} = f - g, \quad (2.8)$$

so that $A + B = 2f$ and $A - B = 2g$. Then:

(a) $\mathcal{P}$ and $\mathcal{N}$ are halves of the two incomplete integrals of (1.11)–(1.12):

$$\boxed{\mathcal{P} = A = \frac{1}{2}F\left[\theta, \frac{3}{4}\right], \quad \mathcal{N} = \frac{1}{2}F\left[\theta, \frac{1}{4}\right], \quad B = -i\mathcal{N} = -\frac{i}{2}F\left[\theta, \frac{1}{4}\right].} \quad (2.9)$$

(b) *Consequently*

$$A' = \frac{(1+i)\pi_+}{W}, \quad B' = \frac{(1-i)\pi_-}{W}, \quad \mathfrak{h}' = \frac{2\zeta}{W}. \quad (2.10)$$

(c) *With p, q, Z as in (1.17), one has on Ω*

$$\boxed{\varpi p = \operatorname{Im} A, \quad \varpi q = \operatorname{Im} B = -\operatorname{Re} \mathcal{N}, \quad \varpi Z = \operatorname{Re} \mathfrak{h} = -\operatorname{Im} \mathfrak{h}.} \quad (2.11)$$

Each right-hand side involves only one of the three integrals of (1.11)–(1.13). Moreover

$$\varpi z = \varpi Z - \frac{\varpi}{2} = -\operatorname{Im} \mathfrak{h} - \frac{\varpi}{2}, \quad (2.12)$$

so that the third argument of (1.18), namely $K_1 z$, carries a half-period shift relative to the height.

Proof. (a) Adding (1.11) and (1.12) cancels $F[\theta, \frac{1}{4}]$ and leaves $f + g = \frac{1}{2}F[\theta, \frac{3}{4}]$; subtracting them cancels $F[\theta, \frac{3}{4}]$ and leaves $f - g = -\frac{i}{2}F[\theta, \frac{1}{4}]$, whence $\mathcal{N} = i(f - g) = \frac{1}{2}F[\theta, \frac{1}{4}]$.

(b) Differentiate (2.9) using (2.3): $A' = \frac{1}{2} \cdot \frac{2(1+i)\pi_+}{W}$ and $B' = -\frac{i}{2} \cdot \frac{2(1+i)\pi_-}{W} = \frac{(1-i)\pi_-}{W}$, since $-i(1+i) = 1-i$.

(c) By (1.15) and (1.17), $\varpi p = \varpi(x + y - 1) = \operatorname{Im}(f + g) = \operatorname{Im} A$ and $\varpi q = \varpi(x - y) = \operatorname{Im}(f - g) = \operatorname{Im} B$; and $\operatorname{Im} B = \operatorname{Im}(-i\mathcal{N}) = -\operatorname{Re} \mathcal{N}$. The third is (1.15) together with (1.16), and (2.12) is (1.17). $\square$

Remark 2.6. Part (a) is the reason for the rotation, exactly as in [P, Rem. 2.6]. Individually, x and y involve *both* $F[\theta, \frac{1}{4}]$ and $F[\theta, \frac{3}{4}]$ — integrals on two different elliptic quotients of the genus-3 curve $W^2 = 1 + 14\zeta^4 + \zeta^8$ — so no algebraic relation ties $\mathcal{T}(x)$ to ζ . The combinations (2.11) involve one integral each, and (1.2) then makes sn algebraic in ζ outright: this is Lemma 4.2.

3. THE DOMAIN Ω

The contents of this section are those of [P, §3]; the domain, its corners, its angles and its spherical area depend only on W and on the Gauss map, which the two surfaces share. Only Remark 3.4 is new, and it merely records where the corners go.

Lemma 3.1. $W^2 = (\zeta^4 + \beta^2)(\zeta^4 + \beta^{-2})$. *Hence W has eight simple branch points, $\zeta^4 = -\beta^{-2}$ (modulus $\beta^{-1/2}$) and $\zeta^4 = -\beta^2$ (modulus $\beta^{1/2}$), all with $\arg \zeta \in \frac{\pi}{4} + \frac{\pi}{2}\mathbb{Z}$.*

Proof. Expand and use $\beta^2 + \beta^{-2} = 14$, $\beta^2\beta^{-2} = 1$. $\square$

Proposition 3.2 (geometry of Ω). (a) Ω is a curvilinear square with vertices

$$\zeta_k = \beta^{-1/2} e^{i(\pi/4 + k\pi/2)}, \quad k = 0, 1, 2, 3, \quad \zeta_0 = \frac{\sqrt{3}-1}{2}(1+i), \quad (3.1)$$

so that $\max_{\Omega} |\zeta| = \beta^{-1/2} = \sqrt{2 - \sqrt{3}} = 0.5176380902\dots$ and the four vertices are exactly the four inner branch points of W ; each bounding arc meets a coordinate axis at distance $\sqrt{2} - 1$ from the origin.

(b) The interior angle of Ω at each vertex is $2\pi/3$; in the local uniformiser $\xi = (\zeta - \zeta_k)^{1/2}$ of the surface it becomes $\pi/3$, and the immersion is regular there.

(c) The four bounding arcs lie on great circles of the Riemann sphere; the spherical area of Ω is $4 \cdot \frac{2\pi}{3} - 2\pi = \frac{2\pi}{3}$, i.e., four of the 24 fundamental cells of the octahedral rotation group, and the total curvature of the patch is $-2\pi/3$.

(d) Each vertex is a flat point; the corresponding unit normals are the four directions $\frac{1}{\sqrt{3}}(\pm 1, \pm 1, +1)$.

Proof. (a) With $c = \frac{\sqrt{3}-1}{2}$ one has $c^2 = \frac{2-\sqrt{3}}{2}$, hence $|\zeta_0 + 1|^2 = (c+1)^2 + c^2 = 2c^2 + 2c + 1 = (2 - \sqrt{3}) + (\sqrt{3} - 1) + 1 = 2$ and likewise $|\zeta_0 + i|^2 = 2$, while $|\zeta_0 - 1|^2 = 2c^2 - 2c + 1 = 4 - 2\sqrt{3} < 2$. So ζ_0 is the inner intersection of the circles centered at -1 and $-i$, and $|\zeta_0|^2 = 2c^2 = \beta^{-1}$, $\arg \zeta_0 = \pi/4$; by Lemma 3.1, $\zeta_0^4 = (i\beta^{-1})^2 = -\beta^{-2}$, a branch point. The circle $|\zeta + 1| = \sqrt{2}$ meets $\mathbb{R}$ at $-1 \pm \sqrt{2}$.

(b) The radius vectors $\zeta_0 + 1$ and $\zeta_0 + i$ have arguments 15° and 75° ; near ζ_0 each disk is, to first order, a half-plane with inward normal $-(\zeta_0 - O_j)$, the two inward normals subtend 60° , and the intersection angle is $180^\circ - 60^\circ = 120^\circ$. Near a simple zero of W the integrands of (2.5) behave like $(\zeta - \zeta_k)^{-1/2}$, so $X - X(\zeta_k) = O(|\zeta - \zeta_k|^{1/2})$ and $\xi := (\zeta - \zeta_k)^{1/2}$ is a conformal chart; angles halve. The metric is $ds = \frac{1}{2} |dh| (|\zeta| + |\zeta|^{-1}) \asymp |\xi|^{-1} |\xi| |d\xi|$, finite and nonzero.

(c) A circle in $\mathbb{C}$ is a great circle iff it is stable under $\zeta \mapsto -1/\bar{\zeta}$; for $|\zeta + 1| = \sqrt{2}$ this holds since $|-(\sqrt{2} + 1) + 1| = \sqrt{2}$. The spherical excess formula gives the area. Here the boundary curves are straight lines (Proposition 6.13), which are geodesics; in [P] they are planar geodesics [P, Prop. 3.2(c)]. Gauss–Bonnet for a quadrilateral with geodesic sides and surface angles $\pi/3$ gives $\int K = 2\pi - 4(\pi - \frac{\pi}{3}) = -\frac{2\pi}{3}$.

(d) $G = \zeta$ has a simple branch point in the chart ξ , so $dG = 0$ and $K = 0$. By (2.7) with $|\zeta_0|^2 = \beta^{-1} = 2 - \sqrt{3}$ and $u = v = c = \frac{\sqrt{3}-1}{2}$,

$$N = \frac{1}{3 - \sqrt{3}} (\sqrt{3} - 1, \sqrt{3} - 1, 1 - \sqrt{3}) = \frac{1}{\sqrt{3}} (1, 1, 1).$$

The other three follow by $\zeta \mapsto i\zeta$. □

Corollary 3.3 (regularity). *f, g, h are holomorphic on Ω° and extend continuously to Ω ; in particular (1.15) defines a continuous map $X : \Omega \rightarrow \mathbb{R}^3$, real-analytic on Ω° . Near a corner ζ_k ,*

$$X(\zeta) - X(\zeta_k) = \operatorname{Re}[\mathbf{a}_k(\zeta - \zeta_k)^{1/2}] + O(|\zeta - \zeta_k|^{3/2}), \quad \mathbf{a}_k \in \mathbb{C}^3 \setminus \{0\}. \quad (3.2)$$

Proof. Holomorphy is Proposition 2.2 together with Lemma 3.1 ($W \neq 0$ on Ω° , and Ω° is simply connected, being star-shaped about 0). The only singularities of f', g', h' on Ω are the four corners, where by Lemma 3.1 the integrands of (2.5) are $O(|\zeta - \zeta_k|^{-1/2})$, which is integrable; integrating $\operatorname{const} \cdot (\zeta - \zeta_k)^{-1/2}$ gives (3.2). □

Remark 3.4. Proposition 3.2(d) identifies the vertices of Ω with flat points of D/P in the $\langle 111 \rangle$ directions, and (c) with the fact that the Gauss map of a genus-3 triply periodic minimal surface has degree 2: the translational unit consists of $2 \cdot 4\pi / (2\pi/3) = 12$ copies of the patch [8]. Since D and P share the Gauss map, the *same* four corners are flat points in both papers, with the *same* normal directions up to the sign of the third component (Remark 17.6); their images, however, differ completely — a regular tetrahedron of edge $\sqrt{2}$ here (Proposition 6.10), a unit square in the plane $z = \frac{1}{2}$ in [P, Prop. 6.4(b)]. The $(\zeta - \zeta_k)^{1/2}$ expansion halves the angle $2\pi/3$ to $\pi/3$, which is exactly the angle between two edges of a regular tetrahedron: an internal cross-check available only here.

4. LEGENDRE NORMAL FORMS: THE ALGEBRAIC DICTIONARY

Lemmas 4.1–4.4 and Theorems 4.5–4.9 are those of [P, §4] verbatim: they concern $f, g, \mathfrak{h}$ and make no reference to which real part is taken. Only §4.1, the dictionary for the function $\mathcal{T}$ of (1.19), is new.

Lemma 4.1 (the products, and the zeros on Ω). $r_+ r_- = W$, $s_+ s_- = W$ and $\pi_+ \pi_- = E = 1 + \zeta^4$. The branch points of r_+ (resp. r_-) are the four points $\zeta^2 = i\beta^{\pm 1}$ (resp. $\zeta^2 = -i\beta^{\pm 1}$). On Ω , r_+ vanishes exactly at the two corners ζ_0, ζ_2 and r_- exactly at ζ_1, ζ_3 , while $\pi_{\pm} \neq 0$, $E \neq 0$ and $s_{\pm} \neq 0$ throughout.

Proof. $(1 + 4i\zeta^2 - \zeta^4)(1 - 4i\zeta^2 - \zeta^4) = (1 - \zeta^4)^2 + 16\zeta^4 = 1 + 14\zeta^4 + \zeta^8$; the second product is (2.2). For r_+ : $\zeta^4 - 4i\zeta^2 - 1 = 0$ gives $\zeta^2 = 2i \pm i\sqrt{3} = i\beta^{\pm 1}$. On Ω we have $|\zeta|^2 \leq \beta^{-1}$ (Proposition 3.2(a)), so only $\zeta^2 = i\beta^{-1}$ is attainable, i.e., $\zeta \in \{\zeta_0, \zeta_2\}$ since $\zeta_0^2 = \zeta_2^2 = i\beta^{-1}$; conjugating gives the statement for r_- , with $\zeta_1^2 = \zeta_3^2 = -i\beta^{-1}$. Finally $\pi_{\pm} = 0$ or $E = 0$ or $s_{\pm} = 0$ would force $|\zeta| \geq 1$. $\square$

Lemma 4.2 (the dictionary at moduli $\frac{1}{4}, \frac{3}{4}$ and β^{-4}). On Ω° ,

$$(\text{sn}, \text{cn}, \text{dn})(2\mathcal{P}, \frac{3}{4}) = \left(\frac{2(1+i)\zeta}{r_+}, \frac{r_-}{r_+}, \frac{\pi_-}{r_+} \right), \quad (4.1)$$

$$(\text{sn}, \text{cn}, \text{dn})(2\mathcal{N}, \frac{1}{4}) = \left(\frac{2(1+i)\zeta}{r_+}, \frac{r_-}{r_+}, \frac{\pi_+}{r_+} \right), \quad (4.2)$$

$$(\text{sn}, \text{cn}, \text{dn})(\beta\mathfrak{h}, \beta^{-4}) = (i\beta\zeta^2, s_+, s_-). \quad (4.3)$$

Note that the argument in (4.2) is $2\mathcal{N} = +2iB$, not $-2iB$: since sn is odd, the wrong sign here produces a discrepancy of exactly -2sn in the first slot while leaving cn and dn untouched.

Proof. By (2.9), $2\mathcal{P} = F[\theta, \frac{3}{4}]$ and $2\mathcal{N} = F[\theta, \frac{1}{4}]$, so (1.2) gives $\text{sn}(2\mathcal{P}, \frac{3}{4}) = \text{sn}(2\mathcal{N}, \frac{1}{4}) = \sin \theta = s$. Then $\text{cn} = \sqrt{1 - s^2} = r_-/r_+$ and $\text{dn} = \sqrt{1 - ms^2}$ equals π_+/r_+ for $m = \frac{1}{4}$ and π_-/r_+ for $m = \frac{3}{4}$, by Lemma 2.1; all branches are $+1$ at $\zeta = 0$. Likewise $\beta\mathfrak{h} = F[\arcsin \sigma, \beta^{-4}]$ by (1.13), so $\text{sn}(\beta\mathfrak{h}, \beta^{-4}) = \sigma = i\beta\zeta^2$, and $\text{cn} = \sqrt{1 - \sigma^2} = s_+$, $\text{dn} = \sqrt{1 - \beta^{-4}\sigma^2} = s_-$ by (2.2). (All nine entries verified numerically to 16 digits: Appendix A, Cell A1.) $\square$

Lemma 4.3 (two transformations). (a) (Jacobi imaginary transformation; [2, 127.01], [19, §22.4].) For every m ,

$$\text{sn}(iu, m) = i \text{sc}(u, 1 - m), \quad \text{cn}(iu, m) = \text{nc}(u, 1 - m), \quad \text{dn}(iu, m) = \text{dc}(u, 1 - m). \quad (4.4)$$

(b) (Imaginary modulus.)

$$\text{sn}(i\sqrt{3}u, -\frac{1}{3}) = i\sqrt{3} \text{sn}(u, -3), \quad \text{equivalently} \quad \text{sn}(u, -\frac{1}{3}) = \frac{i}{\sqrt{3}} \text{sn}(-\frac{i}{\sqrt{3}}u, -3). \quad (4.5)$$

Proof. (a) is classical. For (b), substitute $\tau = \varrho/(i\sqrt{3})$ in $\text{sn}^{-1}(y, -3) = \int_0^y d\tau / \sqrt{(1 - \tau^2)(1 + 3\tau^2)}$: the integrand becomes $\frac{d\varrho/(i\sqrt{3})}{\sqrt{(1 + \varrho^2/3)(1 - \varrho^2)}}$, so $i\sqrt{3} \text{sn}^{-1}(y, -3) = \text{sn}^{-1}(i\sqrt{3}y, -\frac{1}{3})$. $\square$

Lemma 4.4 (rotation). $A(i\zeta) = -B(\zeta)$ and $B(i\zeta) = A(\zeta)$; equivalently $B(-i\zeta) = -A(\zeta)$ and $A(-i\zeta) = B(\zeta)$. Moreover $\mathfrak{h}(i\zeta) = -\mathfrak{h}(\zeta)$ and $\mathfrak{h}(i\zeta) = -\mathfrak{h}(\zeta)$.

Proof. Using $W(i\zeta) = W(\zeta)$ (the Taylor coefficients of W at 0 are a power series in ζ^4) and (2.10),

$$\frac{d}{d\zeta} A(i\zeta) = iA'(i\zeta) = \frac{i(1+i)(1+i(i\zeta)^2)}{W} = \frac{i(1+i)\pi_-}{W} = -\frac{(1-i)\pi_-}{W} = -B'(\zeta),$$

since $i(1+i) = -(1-i)$; and

$$\frac{d}{d\zeta} B(i\zeta) = iB'(i\zeta) = \frac{i(1-i)(1-i(i\zeta)^2)}{W} = \frac{i(1-i)\pi_+}{W} = \frac{(1+i)\pi_+}{W} = A'(\zeta),$$

since $i(1-i) = 1+i$. Both pairs vanish at $\zeta = 0$, so the integrated identities hold. Replacing ζ by $-i\zeta$ gives the second pair. Finally, by (2.5) and $\tau = \eta^2$,

$$h(\zeta) = \int_0^{\zeta^2} \frac{d\tau}{\sqrt{1+14\tau^2+\tau^4}} =: H(\zeta^2) \quad (4.6)$$

with H odd, so $h(i\zeta) = H(-\zeta^2) = -h(\zeta)$. $\square$

Theorem 4.5 (dictionary for A). *On Ω° , at modulus -3 ,*

$$\operatorname{sn} A = \frac{(1+i)\zeta}{\pi_-}, \quad \operatorname{cn} A = \frac{r_-}{\pi_-}, \quad \operatorname{dn} A = \frac{r_+}{\pi_-}. \quad (4.7)$$

Proof. By the Gauss transformation (Proposition 5.3, whose proof uses nothing from this section), $\operatorname{sn}(u, -3) = \frac{1}{2} \operatorname{sd}(2u, \frac{3}{4})$, $\operatorname{cn}(u, -3) = \operatorname{cd}(2u, \frac{3}{4})$, $\operatorname{dn}(u, -3) = \operatorname{nd}(2u, \frac{3}{4})$. Apply this with $u = A = \mathcal{P}$ and read off $2A$ from (4.1):

$$\operatorname{sn}(A, -3) = \frac{1}{2} \cdot \frac{2(1+i)\zeta/r_+}{\pi_-/r_+} = \frac{(1+i)\zeta}{\pi_-}, \quad \operatorname{cn}(A, -3) = \frac{r_-}{\pi_-}, \quad \operatorname{dn}(A, -3) = \frac{r_+}{\pi_-}. \quad \square$$

Theorem 4.6 (dictionary for B). *On Ω° , at modulus -3 ,*

$$\operatorname{sn} B = \frac{(1-i)\zeta}{\pi_+}, \quad \operatorname{cn} B = \frac{r_+}{\pi_+}, \quad \operatorname{dn} B = \frac{r_-}{\pi_+}. \quad (4.8)$$

Proof. By Lemma 4.4, $B(\zeta) = -A(i\zeta)$. Since sn is odd and $\operatorname{cn}, \operatorname{dn}$ even, (4.7) at $i\zeta$ gives, using $\pi_-(i\zeta) = \pi_+(\zeta)$ and $r_\pm(i\zeta) = r_\mp(\zeta)$,

$$\operatorname{sn}(B, -3) = -\frac{(1+i)(i\zeta)}{\pi_+} = \frac{(1-i)\zeta}{\pi_+}, \quad \operatorname{cn}(B, -3) = \frac{r_+}{\pi_+}, \quad \operatorname{dn}(B, -3) = \frac{r_-}{\pi_+},$$

because $-(1+i)i = 1-i$. (A second proof, from (2.9) and the imaginary transformation (4.4), is [P, Thm. 4.6]; the intermediate modulus- $\frac{3}{4}$ triple is the block `chkB34` of Appendix A, Cell A1.) $\square$

Theorem 4.7 (dictionary for the height, at modulus -3). *On Ω° ,*

$$\boxed{\operatorname{sn}(2h, -3) = \frac{2\zeta^2}{E}, \quad \operatorname{cn}(2h, -3) = \frac{1-\zeta^4}{E}, \quad \operatorname{dn}(2h, -3) = \frac{W}{E}, \quad E = 1 + \zeta^4.} \quad (4.9)$$

Equivalently $\operatorname{cd}(2h, -3) = (1-\zeta^4)/W$ and $2h = K[-3] + F[\arctan \frac{\zeta^4-1}{4\zeta^2}, -3]$.

Proof. Step 1 is (4.6).

Step 2 (the substitution $\nu = \tau - \tau^{-1}$). Then $\nu^2 = \tau^2 - 2 + \tau^{-2}$, so $1 + 14\tau^2 + \tau^4 = \tau^2(\nu^2 + 16)$, while $d\nu = \frac{\tau^2+1}{\tau^2} d\tau$; since $(\tau + \tau^{-1})^2 = \nu^2 + 4$,

$$\frac{d\tau}{\tau\sqrt{\nu^2+16}} = \frac{d\nu}{(\tau + \tau^{-1})\sqrt{\nu^2+16}} = \frac{d\nu}{\sqrt{(\nu^2+4)(\nu^2+16)}}.$$

As $\tau \rightarrow 0^+$, $\nu \rightarrow -\infty$, so with $\nu_0 := \zeta^2 - \zeta^{-2}$,

$$h = \int_{-\infty}^{\nu_0} \frac{d\nu}{\sqrt{(\nu^2+4)(\nu^2+16)}}. \quad (4.10)$$

Step 3 (Legendre form). Put $\nu = 4 \tan \varphi$. Then $\nu^2 + 16 = 16 \sec^2 \varphi$, $\nu^2 + 4 = \frac{4(1+3 \sin^2 \varphi)}{\cos^2 \varphi}$ and $d\nu = 4 \sec^2 \varphi d\varphi$, so the integrand is $\frac{d\varphi}{2\sqrt{1+3 \sin^2 \varphi}}$; since $\varphi = -\pi/2$ at $\nu = -\infty$,

$$2h = \int_{-\pi/2}^{\varphi_0} \frac{d\varphi}{\sqrt{1+3 \sin^2 \varphi}} = K[-3] + F[\varphi_0, -3], \quad \varphi_0 = \arctan \frac{\nu_0}{4}, \quad (4.11)$$

the parameter being $m = -3$ because $1 - m \sin^2 \varphi = 1 + 3 \sin^2 \varphi$.

Step 4 (quarter-period shift). By (4.11), $\operatorname{sn}(2h - K, -3) = \sin \varphi_0 = \nu_0 / \sqrt{\nu_0^2 + 16}$. Now $\nu_0 = (\zeta^4 - 1)/\zeta^2$ and $\nu_0^2 + 16 = \frac{(\zeta^4 - 1)^2 + 16\zeta^4}{\zeta^4} = \frac{W^2}{\zeta^4}$, so $\sqrt{\nu_0^2 + 16} = W/\zeta^2$. Since $\operatorname{sn}(v - K) = -\operatorname{cd} v$ [2, 122.01],

$$\operatorname{cd}(2h, -3) = -\frac{\nu_0 \zeta^2}{W} = \frac{1 - \zeta^4}{W}, \quad (4.12)$$

which is $+1$ at $\zeta = 0$, as required.

Step 5 (solving). Write S, C, D for the triple at $2h$ and $\mathcal{X} = S^2$, so $C^2 = 1 - \mathcal{X}$, $D^2 = 1 + 3\mathcal{X}$. From (4.12), $C^2/D^2 = R := (1 - \zeta^4)^2/W^2$, whence $\mathcal{X} = \frac{1-R}{1+3R}$. Now

$$W^2 - (1 - \zeta^4)^2 = 16\zeta^4, \quad W^2 + 3(1 - \zeta^4)^2 = 4(1 + \zeta^4)^2,$$

so $\mathcal{X} = \frac{16\zeta^4}{4E^2} = \frac{4\zeta^4}{E^2}$, giving the first formula of (4.9); then $C^2 = \frac{(1-\zeta^4)^2}{E^2}$ and $D^2 = \frac{W^2}{E^2}$, all branches $+1$ at $\zeta = 0$. (Verified numerically to 16 digits: Appendix A, Cell A1.) $\square$

Corollary 4.8 (the modulus- $\frac{3}{4}$ form). $\operatorname{sn}(4h, \frac{3}{4}) = \frac{4\zeta^2}{W}$, $\operatorname{cn}(4h, \frac{3}{4}) = \frac{1 - \zeta^4}{W}$, $\operatorname{dn}(4h, \frac{3}{4}) = \frac{1 + \zeta^4}{W}$.

Proof. By Proposition 5.3, $\operatorname{sn}(u, -3) = \frac{1}{2} \operatorname{sd}(2u, \frac{3}{4})$ and $\operatorname{cn}(u, -3) = \operatorname{cd}(2u, \frac{3}{4})$; solving these with (4.9) gives the claim. Directly: $W^2 - 16\zeta^4 = (1 - \zeta^4)^2$ and $W^2 - \frac{3}{4} \cdot 16\zeta^4 = (1 + \zeta^4)^2$. $\square$

Theorem 4.9 (dictionary for $2f$ and $2g$). *At modulus -3 , for all $\zeta \in \Omega^\circ$,*

$$\boxed{\operatorname{sn}(2f) = \frac{2\zeta(1 + \zeta^2)}{W}, \quad \operatorname{cn}(2f) = \frac{(1 - \zeta^2)^2}{W}, \quad \operatorname{dn}(2f) = \frac{1 + 6\zeta^2 + \zeta^4}{W}.} \quad (4.13)$$

Moreover $g(\zeta) = f(i\zeta)$, so $\operatorname{sn}(2g, -3) = \frac{2i\zeta(1 - \zeta^2)}{W}$.

Proof. $2f = A + B$ with both arguments at the same ζ ; apply the addition theorems [2, 123.01]

$$\operatorname{sn}(a+b) = \frac{S_a C_b D_b + S_b C_a D_a}{\Delta_{ab}}, \quad \operatorname{cn}(a+b) = \frac{C_a C_b - S_a S_b D_a D_b}{\Delta_{ab}}, \quad \operatorname{dn}(a+b) = \frac{D_a D_b - m S_a S_b C_a C_b}{\Delta_{ab}}, \quad (4.14)$$

$\Delta_{ab} = 1 - m S_a^2 S_b^2$, to (4.7) and (4.8). Here $S_a S_b = \frac{(1+i)(1-i)\zeta^2}{\pi_- \pi_+} = \frac{2\zeta^2}{E}$, so $\Delta_{ab} = 1 + \frac{12\zeta^4}{E^2} = \frac{W^2}{E^2}$; and $C_a C_b = D_a D_b = \frac{W}{E}$, $C_b D_b = \frac{W}{\pi_+}$, $C_a D_a = \frac{W}{\pi_-}$. Thus

$$S_a C_b D_b + S_b C_a D_a = \frac{\zeta W [(1+i)\pi_- + (1-i)\pi_+]}{E^2} = \frac{2\zeta W (1 + \zeta^2)}{E^2},$$

because $(1+i)(1 - i\zeta^2) + (1-i)(1 + i\zeta^2) = 2 + 2\zeta^2$; dividing by Δ_{ab} gives $\operatorname{sn}(2f)$. Next $\operatorname{cn}(2f) = \frac{E^2}{W^2} \cdot \frac{W}{E} [1 - \frac{2\zeta^2}{E}] = \frac{(1-\zeta^2)^2}{W}$ and $\operatorname{dn}(2f) = \frac{E^2}{W^2} \cdot \frac{W}{E} [1 + \frac{6\zeta^2}{E}] = \frac{1+6\zeta^2+\zeta^4}{W}$. Finally $df \mapsto dg$ under $\zeta \mapsto i\zeta$ (Lemma 4.4), whence $g(\zeta) = f(i\zeta)$, and $W(i\zeta) = W(\zeta)$.

Consistency checks. $\operatorname{sn}^2 + \operatorname{cn}^2 = 1$ reads $4\zeta^2(1+\zeta^2)^2 + (1-\zeta^2)^4 = W^2$, true; and $\operatorname{dn}^2 - 1 = 3\operatorname{sn}^2$ reads $(1 + 6\zeta^2 + \zeta^4)^2 - W^2 = 12\zeta^2(1 + \zeta^2)^2$, also true. $\square$

4.1. The function $\mathcal{T}$, and its two dictionaries.

Definition 4.10. For $m = -\frac{1}{3}$ put

$$T(w) := \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(w, -\tfrac{1}{3}) = -\frac{d}{dw} \log \operatorname{cn}(w, -\tfrac{1}{3}), \quad \mathcal{T}(\xi) := T(K_1 \xi), \quad K_1 = K[-\tfrac{1}{3}], \quad (4.15)$$

and, at modulus $\frac{1}{4}$,

$$D(t) := \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(2\varpi t, \tfrac{1}{4}), \quad 2\varpi = K[\tfrac{1}{4}]. \quad (4.16)$$

Thus $\mathcal{T}$ and D are the *same* Jacobi expression at the two reciprocal moduli $-\frac{1}{3}$ and $\frac{1}{4}$ of the anharmonic orbit (Remark 5.6); the relation between them is Proposition 13.3. The function $\mathcal{T}$ governs Parts I–II, D governs Part III.

Proposition 4.11 (three faces of $\mathcal{T}$). *For real ξ , and by analytic continuation for complex ξ off the poles,*

$$\mathcal{T}(\xi) = \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(K_1 \xi, -\tfrac{1}{3}) = \frac{\sqrt{3}}{2} \cdot \frac{\operatorname{sn}}{\operatorname{cn} \operatorname{dn}}(2\varpi \xi, \tfrac{1}{4}), \quad (4.17)$$

and

$$\Phi(\xi) = |\mathcal{T}(\xi)|, \quad (4.18)$$

with Φ as in (1.21).

Proof. For the second expression in (4.17): by Proposition 5.2 with $u = K_1 \xi$, the corresponding argument at modulus $\frac{1}{4}$ is $\frac{2}{\sqrt{3}}u = \frac{2K_1}{\sqrt{3}}\xi = 2\varpi \xi$ by (1.8); writing S, C, D for $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$ at $(2\varpi \xi, \frac{1}{4})$,

$$\frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(K_1 \xi, -\tfrac{1}{3}) = \frac{(\frac{\sqrt{3}}{2} \frac{S}{D}) \cdot \frac{1}{D}}{\frac{C}{D}} = \frac{\sqrt{3}}{2} \cdot \frac{S}{CD}.$$

For (4.18): the argument in (1.21) is $2K_1 \xi$, so the duplication formula [2, 124.02] in the form

$$\frac{1 - \operatorname{cn}(2w, m)}{1 + \operatorname{cn}(2w, m)} = \frac{\operatorname{sn}^2 w \operatorname{dn}^2 w}{\operatorname{cn}^2 w} \quad (4.19)$$

— proved by inserting $\operatorname{cn}(2w) = \frac{1-2\operatorname{sn}^2+m\operatorname{sn}^4}{1-m\operatorname{sn}^4}$ and simplifying $1 - \operatorname{cn}(2w) = \frac{2\operatorname{sn}^2 \operatorname{dn}^2}{1-m\operatorname{sn}^4}$, $1 + \operatorname{cn}(2w) = \frac{2\operatorname{cn}^2}{1-m\operatorname{sn}^4}$ — gives $\Phi(\xi)^2 = T(K_1 \xi)^2 = \mathcal{T}(\xi)^2$, and $\Phi \geq 0$ by (1.21). (For real ξ and $m = -\frac{1}{3} < 0$ all three Jacobi functions are real and $\operatorname{dn} \geq 1$, so the radicand in (1.21) is ≥ 0 indeed.) $\square$

Proposition 4.12 (functional equations for $\mathcal{T}$). *$\mathcal{T}$ is odd and real-meromorphic on $\mathbb{R}$, with*

$$\mathcal{T}(0) = 0, \quad \mathcal{T}(\tfrac{1}{2}) = 1, \quad \mathcal{T}(\xi + 1) = -\frac{1}{\mathcal{T}(\xi)}, \quad \mathcal{T}(\xi + 2) = \mathcal{T}(\xi), \quad \mathcal{T}'(\tfrac{1}{2}) = 4\varpi. \quad (4.20)$$

Its zeros on $\mathbb{R}$ are the even integers and its poles the odd integers; $\mathcal{T} > 0$ on $(0, 1)$ and $\mathcal{T} < 0$ on $(1, 2)$.

Proof. Oddness: sn is odd, $\operatorname{cn}, \operatorname{dn}$ even; $\mathcal{T}(0) = 0$ since $\operatorname{sn}(0) = 0$.

The shift. With $K = K_1$ and $k' = \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$, the quarter-period shifts [2, 122.01] are $\operatorname{sn}(w + K) = \operatorname{cd} w$, $\operatorname{cn}(w + K) = -k' \operatorname{sd} w$, $\operatorname{dn}(w + K) = k' \operatorname{nd} w$, whence

$$T(w + K) = \frac{\operatorname{cd} \cdot k' \operatorname{nd}}{-k' \operatorname{sd}} = -\frac{(\operatorname{cn} / \operatorname{dn})(1 / \operatorname{dn})}{\operatorname{sn} / \operatorname{dn}} = -\frac{\operatorname{cn}}{\operatorname{sn} \operatorname{dn}} = -\frac{1}{T(w)}. \quad (4.21)$$

Iterating gives period $2K_1$, i.e., the third and fourth relations.

The base value. The half-argument values [2, 124.01], valid at any modulus, are $\operatorname{sn} \frac{K}{2} = (1+k')^{-1/2}$, $\operatorname{cn} \frac{K}{2} = \left(\frac{k'}{1+k'}\right)^{1/2}$, $\operatorname{dn} \frac{K}{2} = \sqrt{k'}$; hence

$$\mathcal{T}\left(\frac{1}{2}\right) = T\left(\frac{K_1}{2}\right) = \frac{(1+k')^{-1/2}\sqrt{k'}}{(k'/(1+k'))^{1/2}} = 1.$$

Equivalently, by (4.17) at $m = \frac{1}{4}$, where $k' = \frac{\sqrt{3}}{2}$ and $\frac{S}{CD}|_{K/2} = \frac{1}{k'}$: $\mathcal{T}(\frac{1}{2}) = \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}} = 1$.

The derivative. For any modulus m ,

$$\frac{d}{dw} \frac{S}{CD} = \frac{C^2 D^2 + S^2 D^2 + m S^2 C^2}{C^2 D^2} = 1 + \frac{S^2}{C^2} + \frac{m S^2}{D^2}. \quad (4.22)$$

At $w = \frac{K}{2}$ with $m = \frac{1}{4}$, $k' = \frac{\sqrt{3}}{2}$: $S^2/C^2 = 1/k' = \frac{2}{\sqrt{3}}$ and $\frac{m S^2}{D^2} = \frac{1/4}{k'(1+k')} = \frac{1}{\sqrt{3}(2+\sqrt{3})} = 2\sqrt{3}-3$, so the bracket is $\frac{4}{\sqrt{3}}$; by (4.17) and the chain rule

$$\mathcal{T}'\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2} \cdot 2\varpi \cdot \frac{4}{\sqrt{3}} = 4\varpi.$$

Zeros, poles, signs. By (4.17) the zeros of $\mathcal{T}$ on $\mathbb{R}$ are those of $\operatorname{sn}(2\varpi\xi, \frac{1}{4})$, i.e., ξ even, and the poles those of $\operatorname{cn}(2\varpi\xi, \frac{1}{4})$, i.e., ξ odd ($\operatorname{dn} > 0$ throughout for $m < 1$). On $(0, 1)$ all of $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$ are positive, so $\mathcal{T} > 0$; then $\mathcal{T} = -1/\mathcal{T}(\cdot - 1) < 0$ on $(1, 2)$. $\square$

Corollary 4.13. $\Phi(0) = 0$, $\Phi(\frac{1}{2}) = 1$, $\Phi(\xi + 1) = 1/\Phi(\xi)$, $\Phi(-\xi) = \Phi(\xi)$, $\Phi(\xi + 2) = \Phi(\xi)$; Φ has zeros exactly at the even and poles exactly at the odd integers.

Remark 4.14 (the base value is a theorem). $\Phi(\frac{1}{2}) = 1$ is equivalent to $\operatorname{cn}(K_1, -\frac{1}{3}) = 0$, i.e., to $K_1 = K[-\frac{1}{3}]$ — which by (1.8) is the Gauss relation $K[-\frac{1}{3}] = \frac{\sqrt{3}}{2}K[\frac{1}{4}]$ of Proposition 5.5. It is used in §6 at the base point of the surface, and it is the analog of [P, Prop. 5.3].

Corollary 4.15 (half-arguments and logarithmic derivatives). With $\Lambda_m := (\log \operatorname{sn}(\cdot, m))' = \frac{\operatorname{cn} \operatorname{dn}}{\operatorname{sn}}$, one has at modulus -3

$$\operatorname{sn}^2(f, -3) = \frac{2\zeta^2}{1 + \zeta^4 + W}, \quad \Lambda_{-3}(f) = \frac{1 + \zeta^2}{\zeta}, \quad (4.23)$$

$$\operatorname{sn}^2(h, -3) = \frac{2\zeta^4}{1 + \zeta^4 + W}, \quad \Lambda_{-3}(h) = \frac{1 - \zeta^4 + W}{2\zeta^2}, \quad (4.24)$$

and in particular $\operatorname{sn}(h, -3) = \zeta \operatorname{sn}(f, -3)$.

Proof. As in [P, Cor. 4.11]: by [2, 124.01], $\operatorname{sn}^2 \frac{v}{2} = \frac{1 - \operatorname{cn} v}{1 + \operatorname{dn} v}$, so $\operatorname{sn}^2 f = \frac{W - (1 - \zeta^2)^2}{W + 1 + 6\zeta^2 + \zeta^4}$, and cross-multiplication against $\frac{2\zeta^2}{1 + \zeta^4 + W}$ reduces to $W^2 - (1 - \zeta^2)^2(1 + \zeta^4) = 2\zeta^2(1 + 6\zeta^2 + \zeta^4)$, which holds. Likewise $\operatorname{sn}^2 h = \frac{1 - \frac{1 - \zeta^4}{E}}{1 + \frac{W}{E}} = \frac{2\zeta^4}{E + W}$. The two Λ -values follow from Lemma 9.1. $\square$

Definition 4.16 (the field). $\mathbb{K}$ is the field generated over $\mathbb{C}(\zeta, \omega)$ by $r_{\pm}(\zeta)$, $r_{\pm}(\omega)$, so that $W_{\zeta} = r_{+}(\zeta)r_{-}(\zeta)$, $W_{\omega} = r_{+}(\omega)r_{-}(\omega)$ and

$$[\mathbb{K} : \mathbb{C}(\zeta, \omega)] \leq 16, \quad \text{basis } r_{+}(\zeta)^a r_{-}(\zeta)^b r_{+}(\omega)^c r_{-}(\omega)^d, \quad a, b, c, d \in \{0, 1\}. \quad (4.25)$$

Every identity in $\mathbb{K}$ is equivalent to 16 rational identities in $\mathbb{C}(\zeta, \omega)$. Note that $s_{\pm}$ do *not* occur: see Remark 4.17.

Remark 4.17 (what the quarter-period shift buys). The closed form (1.13) of the height is *unshifted*, and by (4.3) its dictionary is $(\text{sn}, \text{cn}, \text{dn})(\beta\mathfrak{h}, \beta^{-4}) = (i\beta\zeta^2, s_+, s_-)$. This is *irrational*: it introduces a new pair of radicals $s_\pm$, unrelated to $r_\pm$. In real form, applying (4.4) to $\beta\mathfrak{h} = i\beta\mathfrak{h}$ gives $\text{sc}(\beta\mathfrak{h}, 1 - \beta^{-4}) = \beta\zeta^2$, the same two radicals at the auxiliary modulus $m_3 = 1 - \beta^{-4}$.

Theorem 4.7 replaces all of this: the substitution $\nu = \tau - \tau^{-1}$ produces the *quarter-period-shifted* integral (4.11), and the shift is exactly what converts an irrational dictionary into a rational one. Consequences: the auxiliary moduli β^{-4} and m_3 , the generators $s_\pm$, and one of the two Landen steps of (5.5) disappear from the elimination, and we have $[\mathbb{K} : \mathbb{C}(\zeta, \omega)] \leq 16$, the value in [P, Def. 4.12]. *The two papers now work in one and the same field*, which is what makes the comparison of (D0) with [P]’s (P0) an identity of shape rather than an analogy (Table 12.2). The price is paid only at the half-argument $\mathfrak{h}$, where the nested radical $\sqrt{1 + \zeta^4 + W}$ appears in (4.24); the elimination of §9 never needs it.

5. THE MODULI WEB

Four parameters occur: $\frac{1}{4}$ and $\frac{3}{4}$ (in f, g), β^{-4} (in $\mathfrak{h}$), and $-\frac{1}{3}$ (in $\mathcal{T}$). Three relations tie them together, and all three are proved from scratch.

Lemma 5.1 (negative modulus). *For $\mu > 0$, $K[-\mu] = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 + \mu \sin^2 \varphi}} = \frac{1}{\sqrt{1 + \mu}} K\left[\frac{\mu}{1 + \mu}\right]$; in particular*

$$K\left[-\frac{1}{3}\right] = \frac{\sqrt{3}}{2} K\left[\frac{1}{4}\right], \quad K[-3] = \frac{1}{2} K\left[\frac{3}{4}\right]. \quad (5.1)$$

Proof. $K[-\mu] = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1+\mu s^2)}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 + \mu \sin^2 \varphi}}$; then $1 + \mu \sin^2 \varphi = (1 + \mu)(1 - \frac{\mu}{1 + \mu} \cos^2 \varphi)$ and $\varphi \mapsto \frac{\pi}{2} - \varphi$. At $\mu = \frac{1}{3}$ this gives $K[-\frac{1}{3}] = \frac{\sqrt{3}}{2} K[\frac{1}{4}]$, and at $\mu = 3$ the companion relation used in [P, §5]. $\square$

Proposition 5.2 (Gauss transformation at $\mu = \frac{1}{3}$). *Let $\mu > 0$, $m_1 = \frac{\mu}{1 + \mu}$ and $w = u\sqrt{1 + \mu}$. Then*

$$\text{sn}(u, -\mu) = \frac{1}{\sqrt{1 + \mu}} \text{sd}(w, m_1), \quad \text{cn}(u, -\mu) = \text{cd}(w, m_1), \quad \text{dn}(u, -\mu) = \text{nd}(w, m_1). \quad (5.2)$$

In particular, for $\mu = \frac{1}{3}$ (so $m_1 = \frac{1}{4}$, $\sqrt{1 + \mu} = \frac{2}{\sqrt{3}}$, $w = \frac{2u}{\sqrt{3}}$):

$$\text{sn}(u, -\frac{1}{3}) = \frac{\sqrt{3}}{2} \text{sd}\left(\frac{2u}{\sqrt{3}}, \frac{1}{4}\right), \quad \text{cn}(u, -\frac{1}{3}) = \text{cd}\left(\frac{2u}{\sqrt{3}}, \frac{1}{4}\right), \quad \text{dn}(u, -\frac{1}{3}) = \text{nd}\left(\frac{2u}{\sqrt{3}}, \frac{1}{4}\right). \quad (5.3)$$

Proof. Write S, C, D for $\text{sn}, \text{cn}, \text{dn}$ at (w, m_1) and $s = S/D = \text{sd}(w, m_1)$. By (1.2) and $\text{sn}' = \text{cn} \text{dn}$, $\text{cn}' = -\text{sn} \text{dn}$, $\text{dn}' = -m \text{sn} \text{cn}$,

$$\frac{ds}{dw} = \frac{S'D - SD'}{D^2} = \frac{CD \cdot D - S(-m_1 SC)}{D^2} = \frac{C(D^2 + m_1 S^2)}{D^2} = \frac{C}{D^2},$$

using $D^2 + m_1 S^2 = 1$. From $S = sD$ and $D^2 = 1 - m_1 S^2$,

$$S^2 = \frac{s^2}{1 + m_1 s^2}, \quad D^2 = \frac{1}{1 + m_1 s^2}, \quad C^2 = \frac{1 - (1 - m_1)s^2}{1 + m_1 s^2},$$

hence

$$\left(\frac{ds}{dw}\right)^2 = \frac{C^2}{D^4} = [1 - (1 - m_1)s^2][1 + m_1 s^2]. \quad (5.4)$$

Now set $y = s/\sqrt{1+\mu}$ and $u = w/\sqrt{1+\mu}$, so $dy/du = ds/dw$. With $m_1 = \mu/(1+\mu)$ we have $1 - m_1 = 1/(1+\mu)$ and $s^2 = (1+\mu)y^2$, so $1 - (1 - m_1)s^2 = 1 - y^2$ and $1 + m_1s^2 = 1 + \mu y^2$. Thus (5.4) becomes

$$\left(\frac{dy}{du}\right)^2 = (1 - y^2)(1 - (-\mu)y^2), \quad y(0) = 0, \quad y'(0) = 1,$$

which is the initial value problem characterising $y = \text{sn}(u, -\mu)$; uniqueness gives the first identity. Then $\text{cn}^2(u, -\mu) = 1 - y^2 = 1 - (1 - m_1)s^2 = C^2/D^2$ and $\text{dn}^2(u, -\mu) = 1 + \mu y^2 = 1 + m_1s^2 = 1/D^2$, the signs being fixed at $u = 0$. $\square$

Proposition 5.3 (Gauss transformation at $\mu = 3$). $\text{sn}(u, -3) = \frac{1}{2} \text{sd}(2u, \frac{3}{4})$,
 $\text{cn}(u, -3) = \text{cd}(2u, \frac{3}{4})$, $\text{dn}(u, -3) = \text{nd}(2u, \frac{3}{4})$.

Proof. This is Proposition 5.2 at $\mu = 3$, where $m_1 = \frac{3}{4}$ and $\sqrt{1+\mu} = 2$. (Series check: $\text{sn}(u, -3) = u + \frac{u^3}{3} + \dots$ and $\frac{1}{2} \text{sd}(2u, \frac{3}{4}) = u + \frac{u^3}{3} + \dots$.) It is used in §4, Theorems 4.5–4.6 and Corollary 4.8; the proof above is independent of §4. $\square$

Lemma 5.4 (descending Landen for $m = \frac{1}{4}$). Let $k = \frac{1}{2}$, $k' = \frac{\sqrt{3}}{2}$ and $k_1 = \frac{1-k'}{1+k'}$. Then

$$k_1 = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} = (2 - \sqrt{3})^2 = \beta^{-2}, \quad m_1 := k_1^2 = (2 - \sqrt{3})^4 = \beta^{-4} = 0.005154827\dots, \quad (5.5)$$

and

$$K[\beta^{-4}] = \frac{2}{1+k'} \cdot \frac{1}{2} K[\frac{1}{4}] = \frac{2+\sqrt{3}}{4} K[\frac{1}{4}] = \frac{2+\sqrt{3}}{2} \varpi = 1.5728283\dots \quad (5.6)$$

Moreover, with $u = (1 + k_1)v$ and $S = \text{sn}(v, m_1)$, $C = \text{cn}(v, m_1)$, $D = \text{dn}(v, m_1)$,

$$\text{sn}(u, \frac{1}{4}) = \frac{(1+k_1)S}{1+k_1S^2}, \quad \text{cn}(u, \frac{1}{4}) = \frac{CD}{1+k_1S^2}, \quad \text{dn}(u, \frac{1}{4}) = \frac{1-k_1S^2}{1+k_1S^2}, \quad (5.7)$$

and $1 + k_1 = 4/\beta$.

Proof. For (5.5): $\frac{1-\sqrt{3}/2}{1+\sqrt{3}/2} = \frac{2-\sqrt{3}}{2+\sqrt{3}} = (2 - \sqrt{3})^2$ because $(2 + \sqrt{3})^{-1} = 2 - \sqrt{3}$; squaring gives m_1 . Also $1+k_1 = 1+\beta^{-2} = \frac{\beta^2+1}{\beta^2} = \frac{2\sqrt{3}\beta+2}{\beta^2} = \frac{2(\sqrt{3}\beta+1)}{\beta^2}$; since $\sqrt{3}\beta+1 = 2\sqrt{3}+3+1 = 2(2+\sqrt{3}) = 2\beta$, this is $4/\beta$.

For (5.7): put $\psi := \frac{(1+k_1)S}{1+k_1S^2}$. Then

$$1 - \psi^2 = \frac{(1+k_1S^2)^2 - (1+k_1)^2S^2}{(1+k_1S^2)^2} = \frac{(1-S^2)(1-k_1^2S^2)}{(1+k_1S^2)^2} = \frac{C^2D^2}{(1+k_1S^2)^2},$$

because $1 + 2k_1S^2 + k_1^2S^4 - (1 + 2k_1 + k_1^2)S^2 = (1 - S^2)(1 - k_1^2S^2)$; and with $k^2 = \frac{4k_1}{(1+k_1)^2}$ — which holds, since $\frac{4\beta^{-2}}{16\beta^{-2}} = \frac{1}{4}$ —

$$1 - k^2\psi^2 = \frac{(1+k_1S^2)^2 - 4k_1S^2}{(1+k_1S^2)^2} = \frac{(1-k_1S^2)^2}{(1+k_1S^2)^2}.$$

A direct computation gives $d\psi = \frac{(1+k_1)(1-k_1S^2)}{(1+k_1S^2)^2} CD dv$, whence $d\psi[(1-\psi^2)(1-k^2\psi^2)]^{-1/2} = (1+k_1) dv$; integrating from $v = 0$ yields $F[\arcsin \psi, k^2] = (1+k_1)v = u$, i.e., $\psi = \text{sn}(u, \frac{1}{4})$, and the two displayed square roots are the cn and dn formulas.

For (5.6): $u = K[\frac{1}{4}]$ corresponds to $\psi = 1$, i.e., $S = 1$, i.e., $v = K[m_1]$; hence $K[\frac{1}{4}] = (1+k_1)K[\beta^{-4}] = \frac{4}{\beta} K[\beta^{-4}]$, i.e., $K[\beta^{-4}] = \frac{\beta}{4} K[\frac{1}{4}] = \frac{2+\sqrt{3}}{4} K[\frac{1}{4}]$, which is $\frac{2+\sqrt{3}}{2} \varpi$ by (1.7). $\square$

Proposition 5.5 (the normalizing constant).

$$\varpi = \kappa^{-1} = \frac{1}{2}K\left[\frac{1}{4}\right], \quad K_1 = K\left[-\frac{1}{3}\right] = \sqrt{3}\varpi, \quad K[\beta^{-4}] = \frac{2+\sqrt{3}}{2}\varpi, \quad (5.8)$$

and consequently $\kappa K_1 = \sqrt{3}$. Moreover

$$\int_0^{\sqrt{2}-1} \frac{(1+t^2) dt}{\sqrt{1+14t^4+t^8}} = \frac{\varpi}{2}, \quad (5.9)$$

i.e., $-ig(\sqrt{2}-1) = \varpi/2$.

Proof. The first equality of (5.8) is (1.7); the second is Lemma 5.1 with $\mu = \frac{1}{3}$, since $\frac{\sqrt{3}}{2}K[\frac{1}{4}] = \frac{\sqrt{3}}{2} \cdot 2\varpi = \sqrt{3}\varpi$; the third is (5.6). Then $\kappa K_1 = \varpi^{-1}\sqrt{3}\varpi = \sqrt{3}$.

For (5.9), divide numerator and denominator by t^2 :

$$\frac{(1+t^2) dt}{\sqrt{1+14t^4+t^8}} = \frac{(t^{-2}+1) dt}{\sqrt{t^{-4}+14+t^4}}.$$

Substitute $\varsigma = t - t^{-1}$, so $d\varsigma = (1+t^{-2})dt$, $t^2 + t^{-2} = \varsigma^2 + 2$ and $t^4 + t^{-4} = (\varsigma^2 + 2)^2 - 2 = \varsigma^4 + 4\varsigma^2 + 2$; the integrand becomes $d\varsigma/\sqrt{\varsigma^4 + 4\varsigma^2 + 16}$. As t runs over $(0, \sqrt{2}-1]$, ς increases from $-\infty$ to $(\sqrt{2}-1) - (\sqrt{2}+1) = -2$, so the integral is $\int_2^\infty d\varsigma[\varsigma^4 + 4\varsigma^2 + 16]^{-1/2}$. Scaling $\varsigma = 2\nu$ gives $\frac{1}{2} \int_1^\infty d\nu[\nu^4 + \nu^2 + 1]^{-1/2}$; and with $\nu = v + v^{-1} \in [2, \infty)$, so that $d\nu = (1 - v^{-2})dv$, $\sqrt{\nu^2 - 4} = v - v^{-1}$ and $\nu^4 + \nu^2 + 1 = v^2(\nu^2 - 1)$,

$$\int_1^\infty \frac{d\nu}{\sqrt{\nu^4 + \nu^2 + 1}} = \int_2^\infty \frac{d\nu}{\sqrt{(\nu^2 - 4)(\nu^2 - 1)}} = \frac{1}{2}K\left[\frac{1}{4}\right] = \varpi,$$

the middle step by the standard evaluation $\int_a^\infty d\nu[(\nu^2 - a^2)(\nu^2 - b^2)]^{-1/2} = a^{-1}K[b^2/a^2]$ for $a > b > 0$ (substitute $\nu = a/\sin \varphi$). Combining, the integral (5.9) is $\frac{1}{2}\varpi$. $\square$

Remark 5.6 (one curve for both surfaces). The six numbers $\frac{1}{4}, \frac{3}{4}, 4, \frac{4}{3}, -\frac{1}{3}, -3$ form a single orbit of the anharmonic group $\lambda \mapsto \{\lambda, 1-\lambda, \lambda^{-1}, (1-\lambda)^{-1}, \lambda/(\lambda-1), (\lambda-1)/\lambda\}$, with

$$j = 256 \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2} = \frac{35152}{9} = 3905.7\bar{7} \dots$$

Indeed $1 - \frac{1}{4} = \frac{3}{4}$, $1/\frac{1}{4} = 4$, $1/\frac{3}{4} = \frac{4}{3}$, $\frac{1/4}{1/4-1} = -\frac{1}{3}$, $\frac{1/4-1}{1/4} = -3$; and at $\lambda = \frac{1}{4}$ one has $1 - \lambda + \lambda^2 = \frac{13}{16}$, $\lambda^2(1-\lambda)^2 = \frac{9}{256}$, so $j = 256 \cdot \frac{2197/4096}{9/256} = \frac{35152}{9}$. Our horizontal modulus $-\frac{1}{3}$, [P]'s horizontal modulus -3 , and the two moduli $\frac{1}{4}, \frac{3}{4}$ of the closed forms (1.11)–(1.12) all lie on one and the same elliptic curve. The two papers' moduli are related by the single step $\lambda \mapsto 1/\lambda$, realized by the imaginary-modulus relation (4.5):

$$-\frac{1}{3} \xrightarrow{\lambda \mapsto 1/\lambda} -3.$$

By contrast $\beta^{-4} = (2 - \sqrt{3})^4$ is *not* in the orbit: it is 2-isogenous to $\frac{1}{4}$ (Lemma 5.4), consistent with the argument $i\beta\zeta^2$ of $\mathfrak{h}$ being quadratic in ζ . In [P] the analogous role is played by $\frac{1}{9}$, 2-isogenous to $\frac{3}{4}$ [P, Lem. 5.2]. Thus each paper carries exactly one isogeny, on the height, and the two isogenies are the two halves of the same picture.

Remark 5.7 (D versus P, moduli and scale). Here the horizontal modulus is $-\frac{1}{3} \sim \frac{1}{4}$ (Proposition 5.2) with $\kappa_D = 2/K[\frac{1}{4}]$; in [P] it is $-3 \sim \frac{3}{4}$ with $\kappa_P = 2/K[\frac{3}{4}] = 2/K'[\frac{1}{4}]$, the $K \leftrightarrow K'$ interchange demanded by conjugacy. Note that both papers use the *same* θ and hence the same two integrals $F[\theta, \frac{1}{4}]$, $F[\theta, \frac{3}{4}]$: it is the choice of real or imaginary part (Remark 2.4) that decides

TABLE 5.1. Numerical values, computed with $\mathbf{N}[\dots, 20]$ and truncated. All entries follow from the single value $K[1/4]$ through $\varpi = \frac{1}{2}K[1/4]$, $K_1 = \sqrt{3}\varpi$, $\kappa = \varpi^{-1}$ (Proposition 5.5); the relations close exactly. $K[1/9]$ and κ_P are quoted from [P, Table 5.1].

quantity	value
$K[1/4] = 2\varpi$	1.6857503548125960429
$\varpi = \kappa^{-1} = \frac{1}{2}K[1/4]$	0.8428751774062980214
$\kappa = 2/K[1/4]$	1.1864152923156540141
$K_1 = K[-\frac{1}{3}] = \sqrt{3}\varpi$	1.4599026317063392046
$K[\beta^{-4}] = \frac{2+\sqrt{3}}{2}\varpi$	1.5728283034356160518
$\beta^{-4} = (2 - \sqrt{3})^4$	0.0051548270021500098
$\beta = 2 + \sqrt{3}$	3.7320508075688772935
$\beta^{-1/2} = \sqrt{2 - \sqrt{3}}$	0.5176380902050415247
$\sqrt{2} - 1$	0.4142135623730950488
$K[1/9]$ [P]	1.6173867356247324266
$\varpi_P = K[-3] = \frac{2}{3}K[1/9]$ [P]	1.0782578237498216177
$\kappa_P = 3/(2K[1/9])$ [P]	0.9274219745722159761
$\varpi_P/\varpi_D = K'[1/4]/K[1/4]$	1.2792615711710064662 (§12.1)

which of $\frac{1}{4}, \frac{3}{4}$ governs the horizontal coordinates. The vertical modulus is β^{-4} here, reduced to $\frac{1}{4}$ by the 2-isogeny (5.5); in [P] it is $\frac{1}{9}$, reached from $\frac{3}{4}$ by the corresponding 2-isogeny. The ratio $\varpi_P/\varpi_D = \kappa_D/\kappa_P$ is the classical constant of §12.1.

6. SYMMETRIES, STRAIGHT LINES, SPECIAL POINTS

The tests of this section are logically independent of §§9–11 and of Part III. They fix the constants, they exhibit the boundary of the patch as a skew quadrilateral of straight segments, and — crucially — Proposition 6.15 supplies the inequalities $x, y \in [0, 1]$, $z \in [-1, 0]$ needed to pass between the symmetric form (1.20) and the Φ -form (1.22).

6.1. The symmetric form.

Proposition 6.1 (equivalence of the three forms). (a) *For all (x, y, z) at which the three quantities are finite, (1.18) and (1.20) are equivalent.*

(b) *Let $(x, y, z) \in [0, 1]^2 \times [-1, 0]$ and put*

$$A := \mathcal{T}(x) \in [0, +\infty], \quad B := \mathcal{T}(y) \in [0, +\infty], \quad C := \mathcal{T}(1 - z) \in [-\infty, 0]. \quad (6.1)$$

Then $\Phi(x) = A$, $\Phi(y) = B$, $\Phi(z) = -1/C$, and, all quantities being finite, (1.22) is equivalent to $ABC + A + B + C = 0$, i.e., to (1.20).

Proof. (a) By the quarter-period shift (4.21) and $\mathcal{T}(\xi + 1) = -1/\mathcal{T}(\xi)$ (Proposition 4.12),

$$\operatorname{cd}(K_1\xi, -\tfrac{1}{3}) = \frac{\operatorname{cn}}{\operatorname{dn}}(K_1\xi), \quad \operatorname{dc}(K_1\xi, -\tfrac{1}{3}) = \frac{\operatorname{dn}}{\operatorname{cn}}(K_1\xi), \quad (6.2)$$

so that (1.18) reads

$$\frac{\operatorname{cn}(K_1p)}{\operatorname{dn}(K_1p)} \cdot \frac{\operatorname{dn}(K_1q)}{\operatorname{cn}(K_1q)} \cdot \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(K_1z) = -1. \quad (6.3)$$

Now $p = x + y - 1$ and $q = x - y$, so $K_1p = K_1x + K_1y - K_1$ and $K_1q = K_1x - K_1y$; the collapse of (6.3) to the symmetric form (1.20) is Corollary 8.4 of §8, which we do not anticipate here.

(The reader who prefers to see (1.20) as the primary object may take it as the definition and read (1.18) off Corollary 8.4.)

(b) By Proposition 4.12, $\mathcal{T} \geq 0$ on $[0, 1]$, so $\Phi(x) = |\mathcal{T}(x)| = A$ and likewise $\Phi(y) = B$. For $z \in [-1, 0]$ we have $-z \in [0, 1]$, hence $\mathcal{T}(z) = -\mathcal{T}(-z) \leq 0$ and $\Phi(z) = -\mathcal{T}(z)$; also

$$C = \mathcal{T}(1 - z) = -\mathcal{T}(z - 1) = -\left(-\frac{1}{\mathcal{T}(z)}\right) = \frac{1}{\mathcal{T}(z)}, \quad (6.4)$$

so $\Phi(z) = -\mathcal{T}(z) = -1/C \geq 0$ (and $1 - z \in [1, 2]$ confirms $C \leq 0$). Now

$$\Phi(x)\Phi(y) + 1 = \Phi(z)(\Phi(x) + \Phi(y)) \iff AB + 1 = -\frac{A+B}{C} \iff ABC + A + B + C = 0. \quad \square$$

Remark 6.2 (interpretation at poles). $A = \infty$ occurs precisely for $x = 1$, $C = -\infty$ precisely for $z = 0$, and $C = 0$ precisely for $z = -1$. Dividing $ABC + A + B + C = 0$ by the offending quantity, the limiting statements are: $x = 1 \Rightarrow BC + 1 = 0$, i.e., $z = -y$ modulo 2; $y = 1 \Rightarrow AC + 1 = 0$, i.e., $z = -x$; $z = 0 \Rightarrow AB + 1 = 0$, impossible for $A, B \geq 0$ unless one is 0 and the other ∞ ; $z = -1 \Rightarrow A + B = 0$, i.e., $A = B = 0$. All of these match the boundary behaviour of $X(\Omega)$ exactly (Proposition 6.13); so (1.20) is the correct projective formulation and (1.22) its sign-resolved shadow on the box. Note that (1.18) is finite and regular at all of these points — it is the cleared-denominator form — which is why Part II works with it and not with (1.20).

Corollary 6.3 (the face-centered cubic lattice). *In the variables $(p, q, r) = (\Phi(x), \Phi(y), \Phi(z))$, equation (1.22) reads*

$$pq + 1 = r(p + q), \quad (6.5)$$

which is invariant under each of the three double inversions $(p, q) \mapsto (1/p, 1/q)$, $(p, r) \mapsto (1/p, 1/r)$, $(q, r) \mapsto (1/q, 1/r)$. Since $\Phi(\xi + 1) = 1/\Phi(\xi)$ and $\Phi(\xi + 2) = \Phi(\xi)$ (Corollary 4.13), the solution set of (1.22) is invariant under

$$(x, y, z) \mapsto (x+1, y+1, z), \quad (x, y, z) \mapsto (x+1, y, z+1), \quad (x, y, z) \mapsto (x, y+1, z+1),$$

together with $(x, y, z) \mapsto (x+2, y, z)$ and its permutations — that is, under the face-centered cubic lattice generated by $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ inside a cube of edge 2, which is precisely the translation lattice of the diamond surface.

Proof. Each substitution is checked by clearing denominators. $(p, q) \mapsto (1/p, 1/q)$: the left side becomes $\frac{1}{pq} + 1$ and the right side $r(\frac{1}{p} + \frac{1}{q}) = \frac{r(p+q)}{pq}$; multiplying by pq returns (6.5). $(p, r) \mapsto (1/p, 1/r)$: the equation becomes $\frac{q}{p} + 1 = \frac{1}{r}(\frac{1}{p} + q)$, i.e., $\frac{q+p}{p} = \frac{1+pq}{rp}$; multiplying by rp returns (6.5). The third is symmetric to the second. $\square$

Remark 6.4 (the symmetric form is not real-fcc-invariant). Corollary 6.3 is a statement about the Φ -form. The symmetric form (1.20) is *not* invariant under these translations: writing $S(A, B, C) = ABC + A + B + C$, the substitution $(x, y) \mapsto (x+1, y+1)$ sends

$$S(A, B, C) \mapsto \frac{1}{AB} [ABC - A - B + C] = \frac{S(-A, -B, C)}{AB},$$

and $A \mapsto -A$ is realized not by a real translation but by the complex one $x \mapsto x + i\tau$ with $\tau = K[\frac{3}{4}]/K[\frac{1}{4}]$. The zero set of (1.20) is thus invariant under $(x, y) \mapsto (x+1 + i\tau, y+1 + i\tau)$. The real fcc symmetry of (1.22) is therefore a genuine consequence of the absolute value in $\Phi = |\mathcal{T}|$, and the proof of Corollary 6.3 must be carried out — as it is above — in the (p, q, r) variables. *This is the one respect in which [P] is cleaner:* the relation $[P, (1.15)]$ is sign-honest, and its body-centered cubic invariance is provable directly $[P, \text{Prop. 6.3(c)}]$. See Table 12.2, row “sign structure”.

6.2. The base point, to second order.

Proposition 6.5 (the base point pins κ). *Near $\zeta = 0$ put $P := x - \frac{1}{2}$, $Q := y - \frac{1}{2}$, $Z = z + \frac{1}{2}$. Then*

$$P = \kappa v + O(|\zeta|^3), \quad Q = \kappa u + O(|\zeta|^3), \quad Z = -2\kappa uv + O(|\zeta|^4), \quad \text{so} \quad Z = -\frac{2}{\kappa}PQ + \dots, \quad (6.6)$$

while the level set (1.20) has, near $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, the second-order expansion

$$Z = -2\varpi PQ + \dots \quad (6.7)$$

These agree if and only if $\frac{2}{\kappa} = 2\varpi$, i.e., iff $\kappa = \varpi^{-1} = 2/K[\frac{1}{4}]$.

Proof. Surface. By (2.4), $W^{-1} = 1 - 7\zeta^4 + O(\zeta^8)$, hence

$$f = \zeta - \frac{\zeta^3}{3} - \frac{7\zeta^5}{5} + \dots, \quad g = i\left(\zeta + \frac{\zeta^3}{3} - \frac{7\zeta^5}{5} + \dots\right), \quad \mathfrak{h} = i\zeta^2 - \frac{7i\zeta^6}{3} + \dots \quad (6.8)$$

With $\zeta = u + iv$: $\text{Im } f = v + O(|\zeta|^3)$, $\text{Im } g = \text{Im}(i(u + iv)) = u + O(|\zeta|^3)$, and $\text{Re } \mathfrak{h} = \text{Re}(i(u^2 - v^2 + 2iuv)) = -2uv + O(|\zeta|^6)$. Then (1.15) gives (6.6), and eliminating $(u, v) = (Q/\kappa, P/\kappa)$ gives $Z = -2\kappa \cdot \frac{PQ}{\kappa^2} = -\frac{2}{\kappa}PQ$.

Level set. Write $A = \mathcal{T}(\frac{1}{2} + P) = 1 + \alpha$, $B = \mathcal{T}(\frac{1}{2} + Q) = 1 + \beta$ and, using $\mathcal{T}(\frac{3}{2} + s) = -1/\mathcal{T}(\frac{1}{2} + s)$, $C = \mathcal{T}(\frac{3}{2} - Z) = -1 + \gamma$, where by Proposition 4.12

$$\alpha = 4\varpi P + O(P^2), \quad \beta = 4\varpi Q + O(Q^2), \quad \gamma = -4\varpi Z + O(Z^2).$$

Expanding,

$$ABC + A + B + C = -(1+\alpha)(1+\beta) + \gamma(1+\alpha)(1+\beta) + 1 + \alpha + \beta + \gamma = -\alpha\beta + 2\gamma + \gamma(\alpha+\beta) + \gamma\alpha\beta.$$

To second order this is $-16\varpi^2 PQ - 8\varpi Z$, so (1.20) gives $Z = -2\varpi PQ + \dots$, which is (6.7). $\square$

Remark 6.6. The gradient of $ABC + A + B + C$ at the base point is proportional to $(0, 0, 1)$: indeed $\partial_A = BC + 1 = 0$ and $\partial_B = AC + 1 = 0$ there, while $\partial_C = AB + 1 = 2$. So the level surface has horizontal tangent plane at $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, in agreement with $G(0) = 0$ in Proposition 2.3 (vertical normal). This is why the first-order test is vacuous and the content appears at second order — exactly as in [P, Table 6.1], where $Z = \varpi pq + O(6)$ at the base point.

6.3. The two straight lines through the base point.

Proposition 6.7 (the straight lines). *Let $t \in [-(\sqrt{2} - 1), \sqrt{2} - 1]$.*

(a) *For $\zeta = t$: $x \equiv \frac{1}{2}$, $z \equiv -\frac{1}{2}$, and $y = \frac{1}{2} + \kappa \int_0^t \frac{1+\tau^2}{W} d\tau$.*

(b) *For $\zeta = it$: $y \equiv \frac{1}{2}$, $z \equiv -\frac{1}{2}$, and $x = \frac{1}{2} + \kappa \int_0^t \frac{1+\tau^2}{W} d\tau$.*

In both cases (1.20) holds identically: in case (a), $A = \mathcal{T}(\frac{1}{2}) = 1$ and $C = \mathcal{T}(\frac{3}{2}) = -1$, so

$$ABC + A + B + C = -B + 1 + B - 1 = 0$$

for every B. Both segments lie on the surface, pass through the base point, and are orthogonal.

Proof. (a) On $\zeta = t$ real, W is real and positive (Lemma 3.1 with $W(0) = 1$), and $d\zeta = dt$; hence by (2.4) and (1.15)

$$dx = \kappa \text{Im} \frac{1-t^2}{W} dt = 0, \quad dy = \kappa \text{Im} \frac{i(1+t^2)}{W} dt = \kappa \frac{1+t^2}{W} dt, \quad dz = \kappa \text{Re} \frac{2it}{W} dt = 0.$$

(b) On $\zeta = it$ one has $\zeta^2 = -t^2$, W real positive, $d\zeta = i dt$, so $f' d\zeta = \frac{i(1+t^2)}{W} dt$ (imaginary part $\frac{1+t^2}{W}$, giving dx), $g' d\zeta = \frac{i \cdot i(1-t^2)}{W} dt$ (real, so $dy = 0$), and $\mathfrak{h}' d\zeta = \frac{2i(it)i}{W} dt = \frac{-2it}{W} dt$ (imaginary, so $dz = 0$).

The final assertion uses $\mathcal{T}(\frac{1}{2}) = 1$ and $\mathcal{T}(\frac{3}{2}) = -1/\mathcal{T}(\frac{1}{2}) = -1$ from Proposition 4.12. Orthogonality of the two segments is immediate from their directions $(0, 1, 0)$ and $(1, 0, 0)$; it is also forced, since two straight lines through a point of a minimal surface are asymptotic directions, hence orthogonal. $\square$

Remark 6.8. Proposition 6.7 is the sharpest single illustration of Remark 2.4. In [P, Prop. 6.4(a)] the images of $\zeta = t$ and $\zeta = it$ are also distinguished curves, but they are *plane curves* in $y = 0$ and $x = 0$ respectively, not straight lines; conversely the images of the two *diagonals* are the straight segments there, while here they are not (Proposition 6.16). The straight lines of D and the planar geodesics of P are conjugate boundary data in the sense of Björling, which is precisely what §18 exploits.

6.4. Endpoints on the axes.

Proposition 6.9 (axis endpoints). $X(\sqrt{2} - 1) = (\frac{1}{2}, 1, -\frac{1}{2})$, $X(-(\sqrt{2} - 1)) = (\frac{1}{2}, 0, -\frac{1}{2})$, $X(i(\sqrt{2} - 1)) = (1, \frac{1}{2}, -\frac{1}{2})$ and $X(-i(\sqrt{2} - 1)) = (0, \frac{1}{2}, -\frac{1}{2})$.

Proof. By Proposition 6.7(a) and (5.9), $y(\sqrt{2} - 1) = \frac{1}{2} + \kappa \cdot \frac{\varpi}{2} = 1$; the other three follow from Proposition 6.12. $\square$

6.5. The corners: a regular tetrahedron.

Proposition 6.10 (the corner images). *Along the diagonal $\zeta = t(1 + i)$, $0 \leq t \leq t_0 := \frac{\sqrt{3}-1}{2}$, one has $\zeta^4 = -4t^4$, $W = \sqrt{1 - 56t^4 + 16t^8}$ and*

$$x = y = \frac{1}{2} + \kappa \int_0^t \frac{(1 - 2\tau^2) d\tau}{W}, \quad z = -\frac{1}{2} - 4\kappa \int_0^t \frac{\tau d\tau}{W}. \quad (6.9)$$

Moreover

$$\int_0^{t_0} \frac{(1 - 2\tau^2) d\tau}{\sqrt{1 - 56\tau^4 + 16\tau^8}} = \frac{\varpi}{2}, \quad (6.10)$$

$$\int_0^{t_0} \frac{\tau d\tau}{\sqrt{1 - 56\tau^4 + 16\tau^8}} = \frac{1}{2(2 + \sqrt{3})} K[\beta^{-4}] = \frac{\varpi}{8}. \quad (6.11)$$

Hence $X(\zeta_0) = (1, 1, -1)$, and by Proposition 6.12

$$X(\zeta_0) = (1, 1, -1), \quad X(\zeta_1) = (1, 0, 0), \quad X(\zeta_2) = (0, 0, -1), \quad X(\zeta_3) = (0, 1, 0). \quad (6.12)$$

These four points are the vertices of a regular tetrahedron of edge $\sqrt{2}$ inscribed in the cube $[0, 1]^2 \times [-1, 0]$.

Proof. (6.9). With $\zeta = t(1 + i)$ we have $\zeta^2 = 2it^2$ and $d\zeta = (1 + i) dt$, so

$$f' d\zeta = \frac{(1 - 2it^2)(1 + i)}{W} dt = \frac{(1 + 2t^2) + i(1 - 2t^2)}{W} dt,$$

$$g' d\zeta = \frac{i(1 + 2it^2)(1 + i)}{W} dt = \frac{-(1 + 2t^2) + i(1 - 2t^2)}{W} dt,$$

both with imaginary part $\frac{1-2t^2}{W}$, whence $x = y$; and $h' d\zeta = \frac{2it(1+i)^2}{W} dt = \frac{-4t}{W} dt$, real, giving the z -formula. ($W > 0$ for $0 \leq t < t_0$: indeed $W^2 = 1 - 56t^4 + 16t^8 = 16(\sigma_0^2 - t^4)(\sigma_1^2 - t^4)$ with $\sigma_0 = \frac{2-\sqrt{3}}{2} = t_0^2$ and $\sigma_1 = \frac{2+\sqrt{3}}{2} = \frac{1}{4\sigma_0}$, because $\sigma_0\sigma_1 = \frac{1}{4}$ and $\sigma_0^2 + \sigma_1^2 = \frac{56}{16}$.)

(6.11). Substituting $\sigma = \tau^2$ and using the factorization just displayed,

$$\int_0^{t_0} \frac{\tau d\tau}{W} = \frac{1}{2} \int_0^{\sigma_0} \frac{d\sigma}{4\sqrt{(\sigma_0^2 - \sigma^2)(\sigma_1^2 - \sigma^2)}} = \frac{1}{8} \cdot \frac{1}{\sigma_1} K\left[\frac{\sigma_0^2}{\sigma_1^2}\right] = \frac{1}{8} \cdot \frac{2}{2 + \sqrt{3}} K[\beta^{-4}],$$

using the standard evaluation $\int_0^a d\sigma [(a^2 - \sigma^2)(b^2 - \sigma^2)]^{-1/2} = b^{-1}K[a^2/b^2]$ for $0 < a < b$ (substitute $\sigma = a \sin \varphi$), and $\sigma_0/\sigma_1 = (2 - \sqrt{3})^2 = \beta^{-2}$. Now Lemma 5.4 gives $K[\beta^{-4}] = \frac{2+\sqrt{3}}{2}\varpi$, so the whole expression is $\frac{1}{4(2+\sqrt{3})} \cdot \frac{2+\sqrt{3}}{2}\varpi = \frac{\varpi}{8}$. Therefore $z(\zeta_0) = -\frac{1}{2} - 4\kappa \cdot \frac{\varpi}{8} = -1$.

(6.10). Divide by τ^2 and substitute $\varsigma = -\tau^{-1} - 2\tau$, so that $d\varsigma = (\tau^{-2} - 2)d\tau$ and

$$\varsigma^2 - 4 = \tau^{-2} + 4\tau^2, \quad (\varsigma^2 - 4)^2 = \tau^{-4} + 8 + 16\tau^4 \implies \tau^{-4} + 16\tau^4 - 56 = (\varsigma^2 - 4)^2 - 64.$$

Hence

$$\frac{(1 - 2\tau^2)d\tau}{\sqrt{16\tau^8 - 56\tau^4 + 1}} = \frac{(\tau^{-2} - 2)d\tau}{\sqrt{\tau^{-4} - 56 + 16\tau^4}} = \frac{d\varsigma}{\sqrt{(\varsigma^2 - 12)(\varsigma^2 + 4)}}.$$

As τ runs over $(0, t_0]$, ς increases from $-\infty$ to $-(\sqrt{3} + 1) + (\sqrt{3} - 1) = -2\sqrt{3}$. Therefore

$$\int_0^{t_0} \frac{(1 - 2\tau^2)d\tau}{W} = \int_{2\sqrt{3}}^{\infty} \frac{d\varsigma}{\sqrt{(\varsigma^2 - 12)(\varsigma^2 + 4)}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{12 + 4\sin^2 \varphi}} = \frac{1}{4}K\left[\frac{1}{4}\right] = \frac{\varpi}{2},$$

the middle step by $\varsigma = 2\sqrt{3}/\sin \varphi$ and the last using $12 + 4\sin^2 \varphi = 16 - 4\cos^2 \varphi$ together with $\varphi \mapsto \frac{\pi}{2} - \varphi$ and (1.1). Thus $x(\zeta_0) = y(\zeta_0) = \frac{1}{2} + \kappa \cdot \frac{\varpi}{2} = 1$.

The images of the other corners follow from Proposition 6.12 applied to $(1, 1, -1)$; all six mutual distances are $\sqrt{2}$, e.g., $|(1, 1, -1) - (1, 0, 0)|^2 = 2$ and $|(1, 0, 0) - (0, 0, -1)|^2 = 2$. $\square$

Remark 6.11 (why Landen is needed). Equation (6.11) is the only place in Part I where the 2-isogeny (5.5) is used, and it is indispensable: the constant $\frac{2+\sqrt{3}}{2}$ of (5.6) is exactly what sends the corner of Ω to the *lattice point* $(1, 1, -1)$ rather than to some incommensurable height. By contrast, in [P] the corner goes to $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ and the computation [P, Prop. 6.4(b)] needs no Landen step at all, because $Z \equiv 0$ on the diagonal there. This is the same $\text{Re} \leftrightarrow \text{Im}$ swap once more: here the diagonal is a curve of *varying* height, there a curve of *constant* height.

6.6. The group, and the four straight boundary segments.

Proposition 6.12 (the group). (a) *Under $\zeta \mapsto i\zeta$ one has $f \mapsto g$, $g \mapsto -f$, $\mathfrak{h} \mapsto -\mathfrak{h}$; and $f(\bar{\zeta}) = \overline{f(\zeta)}$, $g(\bar{\zeta}) = -\overline{g(\zeta)}$, $\mathfrak{h}(\bar{\zeta}) = -\overline{\mathfrak{h}(\zeta)}$. Hence*

$$\zeta \mapsto \bar{\zeta} \implies (x, y, z) \mapsto (1-x, y, -1-z), \quad \zeta \mapsto i\zeta \implies (x, y, z) \mapsto (y, 1-x, -1-z), \quad (6.13)$$

and these generate a group of order 8 acting on $X(\Omega)$. Both are rotations by π : the first about the line $\{(\frac{1}{2}, t, -\frac{1}{2})\}$, the second about the line through $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ of direction $(1, 1, 0)$.

(b) *Equation (1.22) is invariant under both.*

Proof. (a) The six identities follow by differentiating and comparing with (2.4), both sides vanishing at $\zeta = 0$; e.g., $\frac{d}{d\zeta}f(i\zeta) = if'(i\zeta) = \frac{i(1-(i\zeta)^2)}{W(i\zeta)} = \frac{i(1+\zeta^2)}{W} = g'$, using $W(i\zeta) = W(\zeta)$; and $f'(\bar{\zeta}) = (1 - \bar{\zeta}^2)/W(\bar{\zeta}) = \overline{(1 - \zeta^2)/W} = \overline{f'(\zeta)}$, because the Taylor coefficients of f' at 0 are real. Feeding these into (1.15): $\text{Im } f(\bar{\zeta}) = -\text{Im } f(\zeta)$ turns x into $1 - x$; $\text{Im } g(\bar{\zeta}) = \text{Im } (-\bar{g}) = \text{Im } g$ leaves y fixed; and $\text{Re } \mathfrak{h}(\bar{\zeta}) = -\text{Re } \mathfrak{h}$ turns z into $-1 - z$. The second line of (6.13) is the same computation with $\zeta \mapsto i\zeta$.

(b) By Corollary 4.13, $\Phi(1 - \xi) = \Phi(\xi - 1) = 1/\Phi(\xi)$ and $\Phi(-1 - \xi) = 1/\Phi(\xi)$. So the first motion of (6.13) acts on (p, q, r) by $(p, q, r) \mapsto (1/p, q, 1/r)$, which is one of the three double inversions of Corollary 6.3; the second acts by $(p, q, r) \mapsto (q, 1/p, 1/r)$, a permutation composed with a double inversion. Both preserve (6.5). $\square$

Proposition 6.13 (the boundary arcs are straight segments). *Let Γ be the arc $|\zeta + 1| = \sqrt{2}$, $\text{Re } \zeta \geq \frac{\sqrt{3}-1}{2}$ — that is, the two boundary arcs of Ω in the first and fourth quadrants. Then*

$X(\Gamma)$ is contained in the straight line

$$\ell : \left\{ \left(\frac{1}{2} + s, 1, -\frac{1}{2} - s \right) : s \in \mathbb{R} \right\},$$

and in fact $X(\Gamma)$ is exactly the segment of ℓ joining $X(\zeta_3) = (0, 1, 0)$ to $X(\zeta_0) = (1, 1, -1)$, an edge of the tetrahedron of Proposition 6.10. The analogous statement holds for the other three circles, so $X(\partial\Omega)$ is the skew quadrilateral

$$(0, 1, 0) \rightarrow (1, 1, -1) \rightarrow (1, 0, 0) \rightarrow (0, 0, -1) \rightarrow (0, 1, 0), \quad (6.14)$$

four of the six edges of that tetrahedron.

Proof. Let $\iota(\zeta) := \frac{1-\zeta}{1+\zeta} = -1 + \frac{2}{\zeta+1}$, an involution of $\widehat{\mathbb{C}}$ with fixed points $-1 \pm \sqrt{2}$, so that $R(\zeta) := \iota(\bar{\zeta})$ is the anticonformal inversion in the circle $|\zeta + 1| = \sqrt{2}$; equivalently

$$|\zeta + 1| = \sqrt{2} \iff \bar{\zeta} = \iota(\zeta), \quad (6.15)$$

since $\iota\zeta = \bar{\zeta} \iff 1 - \zeta = \bar{\zeta} + |\zeta|^2 \iff 1 - |\zeta|^2 = 2u \iff (u+1)^2 + v^2 = 2$. Thus R fixes Γ pointwise.

Step 1 (the curve is invariant). With $A_0 = (1 + \zeta)^4$, $B_0 = (1 - \zeta)^4$ one has $\iota^4 = B_0/A_0$, $A_0 + B_0 = 2(1 + 6\zeta^2 + \zeta^4)$ and $A_0B_0 = (1 - \zeta^2)^4$, hence

$$(1 + \zeta)^8 (1 + 14\iota^4 + \iota^8) = A_0^2 + 14A_0B_0 + B_0^2 = (A_0 + B_0)^2 + 12A_0B_0.$$

Writing $\sigma = \zeta^2$ and using $(1 + 6\sigma + \sigma^2)^2 = 1 + 12\sigma + 38\sigma^2 + 12\sigma^3 + \sigma^4$,

$$4(1 + 6\sigma + \sigma^2)^2 + 12(1 - \sigma)^4 = (4 + 48\sigma + 152\sigma^2 + 48\sigma^3 + 4\sigma^4) + (12 - 48\sigma + 72\sigma^2 - 48\sigma^3 + 12\sigma^4),$$

which is $16 + 224\sigma^2 + 16\sigma^4 = 16W^2(\zeta)$, the terms in σ and σ^3 cancelling. Hence

$$W(\iota\zeta) = \frac{4W(\zeta)}{(1 + \zeta)^4}, \quad (6.16)$$

the sign fixed by evaluation at $\zeta = 0$ ($\iota 0 = 1$, $W(1) = 4$).

Step 2 (the differentials are permuted). Since $\iota' = -2/(1 + \zeta)^2$, (6.16) gives $\frac{d\iota}{W(\iota)} = -\frac{(1 + \zeta)^2}{2W} d\zeta$; using $1 - \iota^2 = \frac{4\zeta}{(1 + \zeta)^2}$, $1 + \iota^2 = \frac{2(1 + \zeta^2)}{(1 + \zeta)^2}$, $2\iota = \frac{2(1 - \zeta)}{1 + \zeta}$, we obtain with $\Theta_1 = \frac{1 - \zeta^2}{W}$, $\Theta_2 = \frac{1 + \zeta^2}{W}$, $\Theta_3 = \frac{\zeta}{W}$ (all with real Taylor coefficients at 0)

$$\Theta_1(\iota) d\iota = -2\Theta_3 d\zeta, \quad \Theta_2(\iota) d\iota = -\Theta_2 d\zeta, \quad \Theta_3(\iota) d\iota = -\frac{1}{2}\Theta_1 d\zeta. \quad (6.17)$$

Step 3 (the induced motion). By (1.15) and (2.4), $dx = \kappa \operatorname{Im}(\Theta_1 d\zeta)$, $dy = \kappa \operatorname{Re}(\Theta_2 d\zeta)$, $dz = -2\kappa \operatorname{Im}(\Theta_3 d\zeta)$. Composing with $R(\zeta) = \iota(\bar{\zeta})$ and using $\Theta_j(\bar{w}) = \bar{\Theta}_j(w)$ (and likewise for $W \circ \iota$),

$$dx \circ R = -\kappa \operatorname{Im}(-2\Theta_3 d\zeta) = -dz, \quad dy \circ R = \kappa \operatorname{Re}(-\Theta_2 d\zeta) = -dy, \quad dz \circ R = 2\kappa \operatorname{Im}(-\frac{1}{2}\Theta_1 d\zeta) = -dx.$$

Hence $X \circ R = L \circ X + c$ with $L(a_1, a_2, a_3) = (-a_3, -a_2, -a_1)$ and a constant c . The eigenvalues of L are $+1$ (eigenvector $(1, 0, -1)$) and -1 twice, so $\det L = 1$ and $X \mapsto LX + c$ is a rotation by π about a line of direction $(1, 0, -1)$.

Step 4 (conclusion). R fixes Γ pointwise, so $X(\Gamma)$ lies in the fixed-point set of $X \mapsto LX + c$, a line of direction $(1, 0, -1)$. By Proposition 6.9 the point $\sqrt{2} - 1 \in \Gamma$ maps to $(\frac{1}{2}, 1, -\frac{1}{2})$, so that line is ℓ . Since X is continuous on Γ (Corollary 3.3) and maps the endpoints ζ_0, ζ_3 to $(1, 1, -1)$, $(0, 1, 0)$ (Proposition 6.10), and since x is strictly monotone along each half of Γ — $dx = \kappa \operatorname{Im}(\Theta_1 d\zeta) \neq 0$, by (6.17) and the non-degeneracy of the immersion — the image is exactly the stated segment. The other three circles follow from Proposition 6.12. $\square$

Remark 6.14 (two internal cross-checks). (i) At a corner the surface has an expansion in $(\zeta - \zeta_k)^{1/2}$ (Corollary 3.3), which halves angles; the interior angle of Ω at a corner is $2\pi/3$ (Proposition 3.2(b)), and $\frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$ — exactly the angle between two edges of a regular tetrahedron. (ii) On the edge $y = 1$ the symmetric equation (1.20) degenerates (Remark 6.2) to $AC + 1 = 0$, i.e., $\mathcal{T}(1 - z) = -1/\mathcal{T}(x) = \mathcal{T}(x + 1)$, i.e., $z = -x$ modulo 2 — which is precisely the equation of ℓ . The identity therefore passes the boundary test.

Proposition 6.15 (the convex-hull bound). $X(\Omega) \subset \text{conv}\{(1, 1, -1), (1, 0, 0), (0, 0, -1), (0, 1, 0)\} \subset [0, 1]^2 \times [-1, 0]$. In particular the hypotheses of Proposition 6.1(b) hold at every point of Ω , so (1.20) and (1.22) are equivalent there.

Proof. Each coordinate of X is harmonic on Ω° (an imaginary or real part of a holomorphic function, Proposition 2.2) and continuous on the compact set Ω (Corollary 3.3). For any $\lambda \in \mathbb{R}^3$ the function $\langle \lambda, X \rangle$ is harmonic, hence attains its maximum on $\partial\Omega$; by Proposition 6.13, $X(\partial\Omega)$ lies in the tetrahedron V with the four stated vertices, so $\max_\Omega \langle \lambda, X \rangle = \max_{\partial\Omega} \langle \lambda, X \rangle \leq \max_V \langle \lambda, \cdot \rangle$. As V is the intersection of the half-spaces determined by its faces, $X(\Omega) \subset V$; and $V \subset [0, 1]^2 \times [-1, 0]$ since all four vertices are vertices of that cube. $\square$

6.7. The diagonal mirror.

Proposition 6.16 (the diagonal). On the diagonal $\zeta = t(1 + i)$ one has $x = y$ (Proposition 6.10), so (1.20) reduces to the single-variable relation

$$\mathcal{T}(1 - z) = -\frac{2\mathcal{T}(x)}{1 + \mathcal{T}(x)^2}. \quad (6.18)$$

To leading order in t this holds if and only if $\mathcal{T}'(\frac{1}{2}) = 4\varpi$, which is Proposition 4.12. On the anti-diagonal $\zeta = t(1 - i)$ one has $x + y = 1$, i.e., $p \equiv 0$, and (1.18) reduces to $\text{dc}(K_1q, -\frac{1}{3}) \cdot \frac{\text{sn dn}}{\text{cn}}(K_1z, -\frac{1}{3}) = -1$.

Proof. With $A = B$, (1.20) reads $A^2C + 2A + C = 0$, which is (6.18). For the leading order: by (6.9), $x = \frac{1}{2} + \kappa t + O(t^3)$ and $z = -\frac{1}{2} - 2\kappa t^2 + O(t^6)$, so with $\eta = \kappa t$ and $\delta = 2\kappa t^2$ we must check $\mathcal{T}(\frac{3}{2} + \delta) = -\frac{2A}{1+A^2}$ where $A = \mathcal{T}(\frac{1}{2} + \eta)$. Since $\mathcal{T}(\frac{3}{2} + \delta) = -1/\mathcal{T}(\frac{1}{2} + \delta)$, this is $\mathcal{T}(\frac{1}{2} + \delta) = \frac{1+A^2}{2A}$. Writing $A = 1 + a$ with $a = \mathcal{T}'(\frac{1}{2})\eta + O(\eta^2)$, we get $\frac{1+A^2}{2A} = 1 + \frac{a^2}{2+2a} = 1 + \frac{1}{2}a^2 + O(a^3)$, while $\mathcal{T}(\frac{1}{2} + \delta) = 1 + \mathcal{T}'(\frac{1}{2})\delta + O(\delta^2)$. Comparing the t^2 terms: $\mathcal{T}'(\frac{1}{2}) \cdot 2\kappa = \frac{1}{2}\mathcal{T}'(\frac{1}{2})^2\kappa^2$, i.e., $\mathcal{T}'(\frac{1}{2}) = 4/\kappa = 4\varpi$.

On the anti-diagonal $\zeta = t(1 - i)$: by Lemma 4.4, $A(-i\zeta) = B(\zeta)$, and the computation of Proposition 6.10 with $(1 - i)$ in place of $(1 + i)$ gives $\text{Im}(f + g) = 0$, i.e., $p = 0$; then $\text{cd}(0) = 1$ in (1.18). $\square$

PART II. THE ELIMINATION

7. SEPARATION AND COMPLEXIFICATION

Lemma 7.1 (reality). If H is holomorphic near $(0, 0) \in \mathbb{C}^2$ and $H(\zeta, \bar{\zeta}) = 0$ for all small ζ , then $H \equiv 0$.

Proof. Write $H = \sum c_{jk} \zeta^j \omega^k$ and set $\zeta = re^{i\vartheta}$; then $\sum_{j,k} c_{jk} r^{j+k} e^{i(j-k)\vartheta} = 0$ for all small r and all ϑ . For fixed r the ϑ -Fourier coefficients must vanish, i.e., $\sum_{j-k=\nu} c_{jk} r^{j+k} = 0$ for every $\nu \in \mathbb{Z}$ and every small r ; comparing powers of r gives $c_{jk} = 0$. (This is [P, Lem. 7.1].) $\square$

TABLE 6.1. Special points. Contrast [P, Table 6.1]: there the diagonals carry $Z \equiv 0$ and the corner images are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2})$; here the *axes* carry $Z \equiv 0$ and the corner images are the tetrahedron (6.12).

ζ	(p, q, Z)	(x, y, z)	remark
0	$(0, 0, 0)$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	vertical normal; $Z = -2\varpi PQ + O(4)$
$t \in (0, \sqrt{2} - 1)$	$(y - \frac{1}{2}, \frac{1}{2} - y, 0)$	$(\frac{1}{2}, y, -\frac{1}{2})$	straight line; Prop. 6.7(a)
it	$(x - \frac{1}{2}, x - \frac{1}{2}, 0)$	$(x, \frac{1}{2}, -\frac{1}{2})$	straight line; Prop. 6.7(b)
$\sqrt{2} - 1$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(\frac{1}{2}, 1, -\frac{1}{2})$	$\mathcal{T}(y) = \infty$; Rem. 6.2
$t(1 + i)$	$(2x - 1, 0, Z)$	(x, x, z)	diagonal mirror; Prop. 6.16
ζ_0	$(1, 0, -\frac{1}{2})$	$(1, 1, -1)$	flat point, normal $\frac{(1,1,1)}{\sqrt{3}}$

Proposition 7.2 (two-variable form). *Set $\omega := \bar{\zeta}$, treated as an independent variable, and put*

$$w_p := K_1 p, \quad w_n := K_1 q, \quad w_q := K_1 z, \quad c_0 := \frac{\sqrt{3}}{2i} = -\frac{i\sqrt{3}}{2}. \quad (7.1)$$

Then $A^ = B$ and $\mathfrak{h}^* = -\mathfrak{h}$, where $*$ denotes conjugation of Taylor coefficients, and*

$$\boxed{w_p = c_0(A(\zeta) - B(\omega)), \quad w_n = c_0(B(\zeta) - A(\omega)), \quad w_q = -c_0(\mathfrak{h}(\zeta) - \mathfrak{h}(\omega)) - \frac{K_1}{2}.} \quad (7.2)$$

All three arguments are half-differences, the third carrying the constant $-K_1/2$. Consequently the swap $\sigma : \zeta \leftrightarrow \omega$ acts by

$$\sigma : \quad w_p \mapsto -w_n, \quad w_n \mapsto -w_p, \quad w_q \mapsto -w_q - K_1 \equiv K_1 - w_q \pmod{2K_1}. \quad (7.3)$$

Proof. By (2.10), $(A^*)' = \frac{(1-i)(1-i\zeta^2)}{W} = B'$, so $A^* = B$, i.e., $\overline{A(\zeta)} = B(\omega)$ and $\overline{B(\zeta)} = A(\omega)$; and the Taylor coefficients of $\mathfrak{h}$ at 0 are purely imaginary by (2.4), so $\mathfrak{h}^* = -\mathfrak{h}$, i.e., $\overline{\mathfrak{h}(\zeta)} = -\mathfrak{h}(\omega)$. Hence, by Proposition 2.5(c),

$$\varpi p = \operatorname{Im} A = \frac{A(\zeta) - B(\omega)}{2i}, \quad \varpi q = \operatorname{Im} B = \frac{B(\zeta) - A(\omega)}{2i}, \quad \varpi Z = \operatorname{Re} \mathfrak{h} = \frac{\mathfrak{h}(\zeta) - \mathfrak{h}(\omega)}{2}.$$

Multiplying by $K_1/\varpi = \sqrt{3}$ and using $z = Z - \frac{1}{2}$ gives (7.2); the third formula because $\frac{\sqrt{3}}{2} = -c_0 \cdot (-i)$ with $\mathfrak{h} = i\mathfrak{h}$. Formula (7.3) is then read off, all three being half-differences: σ exchanges $A(\zeta) - B(\omega)$ with $A(\omega) - B(\zeta) = -w_n/c_0$, and negates $\mathfrak{h}_\zeta - \mathfrak{h}_\omega$. $\square$

Remark 7.3 (the factors of i). The three formulas (7.2) are stated as *holomorphic* functions of the independent variables (ζ, ω) agreeing with (2.11) on $\omega = \bar{\zeta}$; the constant c_0 of (7.1) carries the factors of i explicitly, so no absorption into the dictionaries of §4 is needed. Note that `contPath` of Appendix A, Cell A4 returns $\mathfrak{h}$, not $\mathfrak{h} = -i\mathfrak{h}$: the third formula of (7.2) is therefore implemented as $w_q = \frac{\sqrt{3}}{2}(\mathfrak{h}_\zeta - \mathfrak{h}_\omega) - \frac{K_1}{2}$, with no further factor of i .

Remark 7.4 (D versus P, structure). In [P, Prop. 7.2] all three arguments are *half-sums*, $\varpi p = \frac{1}{2}(A_\zeta + B_\omega)$ etc., and the swap interchanges p, q while fixing Z . Here, by (7.3), the swap interchanges w_p and w_n *with a sign*, and sends w_q to $K_1 - w_q$ modulo the period. Three consequences:

- (i) The reduction of §9 is by the *sum* $\omega\partial_\zeta + \zeta\partial_\omega$, not the difference: the height-side factor $\tilde{\Theta}$ of (9.4) satisfies $\tilde{\Theta}(K_1 - w) = \tilde{\Theta}(w)$ (Lemma 9.4), while $\partial_\zeta w_q$ and $\partial_\omega w_q$ carry opposite signs (9.1).

- (ii) The exchange $w_p \leftrightarrow -w_n$ is what makes (D0) *antisymmetric*, hence vanishing on the diagonal for formal reasons, exactly as [P]’s (P0) does [P, Thm. 9.4(c)]. Both therefore amount to 8 rational identities rather than 16. Had the swap merely negated each argument separately, the reduced identity would have been symmetric and the diagonal a substantive test; it is neither.
- (iii) The diagonal carries *no* initial condition here. In [P] the diagonal $\omega = \zeta$ has $\varpi p = \varpi q$ and $\varpi Z = \mathfrak{h}(\zeta)$, and $(\star)$ restricted to it is the substantive one-variable identity [P, Thm. 8.1] that determines the right-hand side. Here $\omega = \zeta$ is the complexification of the *real axis*, on which (7.2) gives $w_p = 2c_0g(\zeta) = -w_n$ and $w_q \equiv -\frac{K_1}{2}$; so $(\star)$ holds there identically, by $\text{cd} \cdot \text{dc} = 1$ and $\Upsilon\left(\frac{K_1}{2}\right) = 1$ (Corollary 8.6(b)) — but *vacuously*, since w_q is constant and the diagonal is a *leaf* of the foliation by level sets of w_q (Theorem 9.10(d)). The initial condition must therefore come from a curve transverse to that foliation: from the arc A_1 of (10.1), where Proposition 6.13 supplies $y = 1$ and $z = -x$, so that $w_q = -K_1x$ sweeps an interval. This is the structural reason why Part II cannot close the proof, and why the complete proof is deferred to Part III (Remark 18.5).

8. THE KEY IDENTITY, AND THE MASTER IDENTITY

Everything in Part II hinges on the following elementary but, as far as we know, not commonly quoted identity. Throughout, m is an arbitrary modulus, $S_a = \text{sn}(a, m)$, $C_a = \text{cn}(a, m)$, $D_a = \text{dn}(a, m)$, and

$$\Delta_{ab} := 1 - mS_a^2S_b^2, \quad T(a) := \frac{\text{sn } a \, \text{dn } a}{\text{cn } a} = -\frac{d}{da} \log \text{cn } a. \quad (8.1)$$

We use the addition theorems (4.14) in the form

$$\text{sn}(a \pm b) = \frac{S_a C_b D_b \pm S_b C_a D_a}{\Delta_{ab}}, \quad \text{cn}(a \pm b) = \frac{C_a C_b \mp S_a S_b D_a D_b}{\Delta_{ab}}, \quad \text{dn}(a \pm b) = \frac{D_a D_b \mp m S_a S_b C_a C_b}{\Delta_{ab}}. \quad (8.2)$$

Lemma 8.1 (the twisted numerator). *For all a, b and every modulus m ,*

$$S_a D_a C_b + S_b D_b C_a = \text{sn}(a + b) \, \text{dn}(a - b) \, \Delta_{ab}, \quad (8.3)$$

and, replacing b by $-b$,

$$S_a D_a C_b - S_b D_b C_a = \text{sn}(a - b) \, \text{dn}(a + b) \, \Delta_{ab}. \quad (8.4)$$

Proof. By (8.2),

$$\text{sn}(a + b) \, \text{dn}(a - b) \, \Delta_{ab} = \frac{(S_a C_b D_b + S_b C_a D_a)(D_a D_b + m S_a S_b C_a C_b)}{\Delta_{ab}}.$$

Expand the numerator into four terms and group the first with the fourth and the second with the third:

$$S_a C_b D_b \cdot D_a D_b + S_b C_a D_a \cdot m S_a S_b C_a C_b = S_a D_a C_b (D_b^2 + m S_b^2 C_a^2),$$

$$S_a C_b D_b \cdot m S_a S_b C_a C_b + S_b C_a D_a \cdot D_a D_b = S_b D_b C_a (m S_a^2 C_b^2 + D_a^2).$$

Now, using $D^2 = 1 - mS^2$ and $C^2 = 1 - S^2$,

$$D_b^2 + m S_b^2 C_a^2 = 1 - m S_b^2 + m S_b^2 (1 - S_a^2) = 1 - m S_a^2 S_b^2 = \Delta_{ab},$$

and symmetrically $D_a^2 + m S_a^2 C_b^2 = \Delta_{ab}$. Hence the numerator equals $\Delta_{ab}(S_a D_a C_b + S_b D_b C_a)$ and (8.3) follows. Formula (8.4) is (8.3) with $b \mapsto -b$, sn being odd and cn, dn even. $\square$

Theorem 8.2 (separation of the symmetric form). *For all a, b with $\text{cn } a \text{ cn } b \neq 0$,*

$$T(a) + T(b) = \frac{\text{sn}(a+b) \text{ dn}(a-b) \Delta_{ab}}{\text{cn } a \text{ cn } b}, \quad 1 + T(a)T(b) = \frac{\text{cn}(a-b) \Delta_{ab}}{\text{cn } a \text{ cn } b}, \quad (8.5)$$

and therefore, whenever $\text{cn}(a-b) \neq 0$,

$$\boxed{\frac{T(a) + T(b)}{1 + T(a)T(b)} = \frac{\text{sn}(a+b) \text{ dn}(a-b)}{\text{cn}(a-b)}} \quad (8.6)$$

Dually, $\frac{T(a) - T(b)}{1 - T(a)T(b)} = \frac{\text{sn}(a-b) \text{ dn}(a+b)}{\text{cn}(a+b)}.$

Proof. $T(a) + T(b) = \frac{S_a D_a C_b + S_b D_b C_a}{C_a C_b}$, which is the first formula by Lemma 8.1. Also $1 + T(a)T(b) = \frac{C_a C_b + S_a S_b D_a D_b}{C_a C_b}$, and by (8.2) the numerator is $\text{cn}(a-b)\Delta_{ab}$; this is the second formula. Dividing, the factors $\Delta_{ab}/(C_a C_b)$ cancel and (8.6) results. The dual statement follows from (8.4) together with $\text{cn}(a+b)\Delta_{ab} = C_a C_b - S_a S_b D_a D_b$. $\square$

Remark 8.3 (three classical checks). $b \rightarrow 0$: both sides of (8.6) become $T(a)$. $b = a$: the left side is $\frac{2T}{1+T^2} = \frac{2SCD}{C^2+S^2D^2}$ and the right side is $\text{sn } 2a = \frac{2SCD}{1-mS^4}$; these agree because $C^2 + S^2D^2 = 1 - S^2 + S^2(1 - mS^2) = 1 - mS^4$. $m = 0$: $T = \tan$ and (8.6) reads $\frac{\tan a + \tan b}{1 + \tan a \tan b} = \frac{\sin(a+b)}{\cos(a-b)}$, which is elementary. Compare [P, Rem. 6.5]: there the analogous role is played by the *product* formula $\text{sn}(a+b)\text{sn}(a-b) = \frac{\text{sn}^2 a - \text{sn}^2 b}{1 - m \text{sn}^2 a \text{sn}^2 b}$, and the two identities are the two halves of (8.2).

Corollary 8.4 (Master Identity). *Let $m = -\frac{1}{3}$, $K_1 = K[-\frac{1}{3}] = \sqrt{3}\varpi$, and put $a = K_1x$, $b = K_1y$, $c = K_1(1-z)$. Then the symmetric form (1.20), i.e., $\mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(1-z) + \mathcal{T}(x) + \mathcal{T}(y) + \mathcal{T}(1-z) = 0$, is equivalent to*

$$\frac{\text{sn } c \text{ dn } c}{\text{cn } c} + \frac{\text{sn}(a+b) \text{ dn}(a-b)}{\text{cn}(a-b)} = 0 \quad (\text{modulus } -\frac{1}{3}), \quad (8.7)$$

i.e., to a relation involving only $a+b = K_1(x+y)$, $a-b = K_1(x-y)$ and $c = K_1(1-z)$ — that is, only the three separated coordinates (2.11). Moreover (8.7) is equivalent to (1.18), i.e., to

$$\text{cd}(K_1p, -\frac{1}{3}) \text{ dc}(K_1q, -\frac{1}{3}) \frac{\text{sn dn}}{\text{cn}}(K_1z, -\frac{1}{3}) = -1. \quad (8.8)$$

Proof. From (1.20) to (8.7). Equation (1.20) reads $\mathcal{T}(1-z)(1 + \mathcal{T}(x)\mathcal{T}(y)) = -(\mathcal{T}(x) + \mathcal{T}(y))$; divide by $1 + \mathcal{T}(x)\mathcal{T}(y)$ and apply Theorem 8.2 with $T = \mathcal{T}$ rescaled, i.e., $\mathcal{T}(\xi) = T(K_1\xi)$. (The excluded cases $\text{cn } a \text{ cn } b \text{ cn}(a-b) = 0$ and $1 + \mathcal{T}(x)\mathcal{T}(y) = 0$ are limits of (8.7) and are covered by Remark 6.2 together with continuity.)

From (8.7) to (8.8). We have $a+b = K_1(x+y) = w_p + K_1$ and $a-b = K_1q = w_n$ by (1.17) and (7.1); and $c = K_1(1-z) = K_1 - w_q$. The quarter-period shifts [2, 122.01] at $m = -\frac{1}{3}$, where $k' = \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$, read

$$\text{sn}(K_1 + v) = \text{cd } v, \quad \text{cn}(K_1 + v) = -k' \text{ sd } v, \quad \text{dn}(K_1 + v) = k' \text{ nd } v, \quad (8.9)$$

and, sn being odd and cn, dn even, $\text{sn}(K_1 - v) = \text{cd } v$, $\text{cn}(K_1 - v) = k' \text{ sd } v$, $\text{dn}(K_1 - v) = k' \text{ nd } v$. Hence

$$\frac{\text{sn}(a+b) \text{ dn}(a-b)}{\text{cn}(a-b)} = \frac{\text{cd}(w_p) \text{ dn}(w_n)}{\text{cn}(w_n)} = \text{cd}(w_p) \text{ dc}(w_n),$$

while

$$\frac{\text{sn } c \text{ dn } c}{\text{cn } c} = \frac{\text{cd}(w_q) \cdot k' \text{ nd}(w_q)}{k' \text{ sd}(w_q)} = \frac{(\text{cn} / \text{dn})(1 / \text{dn})}{\text{sn} / \text{dn}} \Big|_{w_q} = \frac{\text{cn}(w_q)}{\text{sn}(w_q) \text{ dn}(w_q)} = \frac{1}{T(w_q)}.$$

So (8.7) becomes $\text{cd}(w_p) \text{dc}(w_n) = -1/T(w_q)$, i.e., $\text{cd}(w_p) \text{dc}(w_n)T(w_q) = -1$, which is (8.8). $\square$

Remark 8.5. Corollary 8.4 is the decisive step of Part II, and the exact counterpart of [P, §8]. In [P] the corresponding collapse is achieved by the *product* formula [P, Rem. 1.3] and the diagonal identity [P, Thm. 8.1]; here it is achieved by Theorem 8.2, which is a single application of Lemma 8.1. Two features deserve note. First, (8.7) is a relation among three quantities each depending on *one* separated coordinate, which is exactly what makes the complexification of §7 usable. Second, the passage to (8.8) converts the pair $(\text{sn}(a+b), \text{dn}(a-b)/\text{cn}(a-b))$ into the pair $(\text{cd}(w_p), \text{dc}(w_n))$, which is why the two horizontal factors of $(\star)$ are of *reciprocal* type (Remark 1.5(i)) whereas [P]’s are both sn .

Corollary 8.6 (two exact checks of $(\star)$). (a) *At $\zeta = 0$: $w_p = w_n = 0$ and $w_q = -\frac{K_1}{2}$, and (8.8) holds, being equivalent to*

$$T\left(\frac{K_1}{2}\right) = 1, \quad (8.10)$$

which is Proposition 4.12.

(b) *For $\zeta = t$ real one has $w_p = -w_n$ and $w_q = -\frac{K_1}{2}$, so (8.8) reads $\text{cd}(w_p) \text{dc}(-w_p) \cdot (-1) = -1$, i.e., $\text{cd}(w_p) \text{dc}(w_p) = 1$ — true, since $\text{cd} \cdot \text{dc} = 1$ identically. The same holds for $\zeta = it$, where $w_p = w_n$ and the assertion is $\text{cd}(w_p) \text{dc}(w_p) = 1$ again.*

Proof. (a) At $\zeta = 0$ all of p, q, Z vanish, so $w_p = w_n = 0$ and $w_q = K_1 z = -\frac{K_1}{2}$; then $\text{cd}(0) = \text{dc}(0) = 1$ and (8.8) reads $T(-\frac{K_1}{2}) = -1$, i.e., (8.10) since T is odd.

(b) By Proposition 6.7(a), on $\zeta = t$ we have $x \equiv \frac{1}{2}$ and $z \equiv -\frac{1}{2}$, so $Z = 0$, $w_q = -\frac{K_1}{2}$, $p = y - \frac{1}{2}$ and $q = \frac{1}{2} - y = -p$; $T(w_q) = -1$ by (8.10). On $\zeta = it$, Proposition 6.7(b) gives $y \equiv \frac{1}{2}$, $z \equiv -\frac{1}{2}$, so $p = x - \frac{1}{2} = q$. $\square$

9. THE LOGARITHMIC DERIVATIVE, AND THE IDENTITY (D0)

The complexified arguments. Recall from Proposition 7.2 that w_p, w_n, w_q are the half-differences (7.2), with $c_0 = \sqrt{3}/2i$, and that the swap acts by (7.3). Differentiating (7.2) with (2.10),

$$\begin{aligned} \partial_\zeta w_p &= \frac{c_0(1+i)\pi_+(\zeta)}{W_\zeta}, & \partial_\zeta w_n &= \frac{c_0(1-i)\pi_-(\zeta)}{W_\zeta}, & \partial_\zeta w_q &= -\frac{2c_0\zeta}{W_\zeta}, \\ \partial_\omega w_p &= -\frac{c_0(1-i)\pi_-(\omega)}{W_\omega}, & \partial_\omega w_n &= -\frac{c_0(1+i)\pi_+(\omega)}{W_\omega}, & \partial_\omega w_q &= +\frac{2c_0\omega}{W_\omega}. \end{aligned} \quad (9.1)$$

Note the opposite signs in the last column: this, and not any property of the height function, is what determines the eliminating combination of §9.1.

Lemma 9.1 (two half-argument formulas). (a) *Let $\Lambda_m := (\log \text{sn}(\cdot, m))' = \frac{\text{cn} \text{dn}}{\text{sn}}$. Then, for every m ,*

$$\Lambda_m\left(\frac{v}{2}\right) = \frac{\text{cn}(v, m) + \text{dn}(v, m)}{\text{sn}(v, m)}. \quad (9.2)$$

(b) *Let $\Xi_m := (\log \text{dc}(\cdot, m))'$. Then, for every m ,*

$$\Xi_m\left(\frac{v}{2}\right) = \frac{\text{sn}(v, m) [(1-m) + \text{dn}(v, m) - m \text{cn}(v, m)]}{(1 + \text{cn}(v, m))(1 + \text{dn}(v, m))}. \quad (9.3)$$

Both right-hand sides are rational in the triple $(\text{sn}, \text{cn}, \text{dn})(v, m)$: no square root is extracted.

Proof. (a) Write S, C, D for the values at v . By [2, 124.01], $\operatorname{sn}^2 \frac{v}{2} = \frac{1-C}{1+D}$ and $\operatorname{cn}^2 \frac{v}{2} = \frac{C+D}{1+D}$, hence $\operatorname{dn}^2 \frac{v}{2} = 1 - m \frac{1-C}{1+D} = \frac{(1-m)+D+mC}{1+D}$, so

$$\Lambda_m^2\left(\frac{v}{2}\right) = \frac{(C+D)[(1-m)+D+mC]}{(1+D)(1-C)},$$

while the square of the right side of (9.2) is $\frac{(C+D)^2}{(1-C)(1+C)}$. These agree iff $[(1-m)+D+mC](1+C) = (C+D)(1+D)$; the left side is $1-m+D+C+DC+mC^2$ and the right side is $C+D+CD+D^2 = C+D+CD+1-m(1-C^2)$, the same. Both sides of (9.2) are $\sim 2/v$ as $v \rightarrow 0$, fixing the sign.

(b) By the same two formulas, $\operatorname{dc}^2 \frac{v}{2} = \frac{\operatorname{dn}^2 \frac{v}{2}}{\operatorname{cn}^2 \frac{v}{2}} = \frac{1+D}{1+C}$, so $\log \operatorname{dc} \frac{v}{2} = \frac{1}{2} [\log(1+D) - \log(1+C)]$ and, differentiating in v with $C' = -SD$, $D' = -mSC$,

$$\frac{1}{2} \Xi_m\left(\frac{v}{2}\right) = \frac{1}{2} \left[\frac{-mSC}{1+D} - \frac{-SD}{1+C} \right] = \frac{1}{2} S \frac{D(1+D) - mC(1+C)}{(1+C)(1+D)}.$$

Now $D(1+D) - mC(1+C) = D + D^2 - mC - mC^2 = D + (1 - mS^2) - mC - m(1 - S^2) = (1-m) + D - mC$, using $D^2 = 1 - mS^2$ and $C^2 = 1 - S^2$. This is (9.3). $\square$

Remark 9.2. Part (a) is modulus-free and is [P, Lem. 9.1] verbatim; it is used in Corollary 4.15. Part (b) is the device that makes §9 work here, and it is the exact structural counterpart of (a): in [P] the quantity to be halved is $\Lambda_m = (\log \operatorname{sn})'$, because the arguments there are half-sums of sn-data; here it is $\Xi_m = (\log \operatorname{dc})'$, because $(\star)$ carries cd and dc rather than sn (Remark 1.5(i)). Together with Theorem 4.7 this is what keeps $\mathbb{K}$ of degree ≤ 16 (Remark 4.17): wherever a half-argument logarithmic derivative is needed, (9.2) or (9.3) supplies it rationally.

Definition 9.3. At modulus $m = -\frac{1}{3}$ put

$$\Upsilon := \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}, \quad \Xi := \Upsilon - \frac{m \operatorname{sn} \operatorname{cn}}{\operatorname{dn}} = \frac{4}{3} \cdot \frac{\operatorname{sn}}{\operatorname{cn} \operatorname{dn}}, \quad \tilde{\Theta} := \frac{\operatorname{dn}^2 - m \operatorname{sn}^2 \operatorname{cn}^2}{\operatorname{sn} \operatorname{cn} \operatorname{dn}}. \quad (9.4)$$

Then

$$(\log \operatorname{cd})' = -\Xi, \quad (\log \operatorname{dc})' = +\Xi, \quad (\log \Upsilon)' = \tilde{\Theta}, \quad (9.5)$$

and $\Upsilon, \Xi, \tilde{\Theta}$ are all odd, with $\Xi(0) = 0$ and

$$\Xi\left(\frac{K_1}{2}\right) = \frac{2}{\sqrt{3}}, \quad \tilde{\Theta}\left(\frac{K_1}{2}\right) = \frac{4}{\sqrt{3}}, \quad \tilde{\Theta} \geq \frac{1}{\operatorname{sn} \operatorname{cn} \operatorname{dn}} > 0 \text{ on } (0, K_1). \quad (9.6)$$

Verification. The closed form of Ξ . At $m = -\frac{1}{3}$ one has $\operatorname{dn}^2 = 1 + \frac{1}{3} \operatorname{sn}^2$ and $\operatorname{cn}^2 = 1 - \operatorname{sn}^2$, so $3 \operatorname{dn}^2 + \operatorname{cn}^2 = 4$ identically; hence

$$\Xi = \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}} + \frac{1}{3} \frac{\operatorname{sn} \operatorname{cn}}{\operatorname{dn}} = \frac{\operatorname{sn}(3 \operatorname{dn}^2 + \operatorname{cn}^2)}{3 \operatorname{cn} \operatorname{dn}} = \frac{4}{3} \cdot \frac{\operatorname{sn}}{\operatorname{cn} \operatorname{dn}}.$$

The three logarithmic derivatives. $\log \operatorname{cd} = \log \operatorname{cn} - \log \operatorname{dn}$ with $(\log \operatorname{cn})' = -\operatorname{sn} \operatorname{dn} / \operatorname{cn} = -\Upsilon$ and $(\log \operatorname{dn})' = -m \operatorname{sn} \operatorname{cn} / \operatorname{dn}$, so $(\log \operatorname{cd})' = -\Upsilon + m \operatorname{sn} \operatorname{cn} / \operatorname{dn} = -\Xi$; and $\operatorname{dc} = 1 / \operatorname{cd}$. For Υ : $\log \Upsilon = \log \operatorname{sn} + \log \operatorname{dn} - \log \operatorname{cn}$, so

$$(\log \Upsilon)' = \frac{\operatorname{cn} \operatorname{dn}}{\operatorname{sn}} - \frac{m \operatorname{sn} \operatorname{cn}}{\operatorname{dn}} + \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}} = \frac{\operatorname{cn}^2 \operatorname{dn}^2 + \operatorname{sn}^2 \operatorname{dn}^2 - m \operatorname{sn}^2 \operatorname{cn}^2}{\operatorname{sn} \operatorname{cn} \operatorname{dn}} = \frac{\operatorname{dn}^2 - m \operatorname{sn}^2 \operatorname{cn}^2}{\operatorname{sn} \operatorname{cn} \operatorname{dn}} = \tilde{\Theta},$$

using $\operatorname{sn}^2 + \operatorname{cn}^2 = 1$.

The values (9.6). By (13.3) with $k' = \sqrt{1-m} = \frac{2}{\sqrt{3}}$ one has $\operatorname{sn} \operatorname{cn} \operatorname{dn} \big|_{K_1/2} = \frac{k'}{1+k'}$, $\operatorname{dn}^2 = k'$, $\operatorname{sn}^2 \operatorname{cn}^2 = \frac{k'}{(1+k')^2}$ and $\frac{\operatorname{sn}}{\operatorname{cn} \operatorname{dn}} = \frac{1}{k'}$; hence $\Xi = \frac{4}{3} \cdot \frac{1}{k'} = \frac{4}{3} \cdot \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}$, and

$$\tilde{\Theta}\left(\frac{K_1}{2}\right) = \frac{k' + \frac{1}{3} \frac{k'}{(1+k')^2}}{\frac{k'}{1+k'}} = (1+k') + \frac{1}{3(1+k')} = \frac{3+2\sqrt{3}}{3} + \frac{2\sqrt{3}-3}{3} = \frac{4\sqrt{3}}{3} = \frac{4}{\sqrt{3}},$$

using $1+k' = \frac{3+2\sqrt{3}}{3}$ and $(\sqrt{3}+2)^{-1} = 2-\sqrt{3}$. Finally $\operatorname{dn} \geq 1$ and $-m > 0$ give $\operatorname{dn}^2 - m \operatorname{sn}^2 \operatorname{cn}^2 \geq 1$. (All four values verified to 20 digits: Appendix A, Cell A5, test 4.) $\square$

Lemma 9.4 (shift invariance). *For all w ,*

$$\Upsilon(K_1 - w) = \frac{1}{\Upsilon(w)}, \quad \frac{\operatorname{sn} \operatorname{cn}}{\operatorname{dn}}(K_1 - w) = \frac{\operatorname{sn} \operatorname{cn}}{\operatorname{dn}}(w), \quad \tilde{\Theta}(K_1 - w) = \tilde{\Theta}(w), \quad (9.7)$$

and consequently

$$\Xi(w) + \Xi(K_1 - w) = \tilde{\Theta}(w). \quad (9.8)$$

Proof. By (8.9) the triple $(\operatorname{sn}, \operatorname{cn}, \operatorname{dn})$ at $K_1 - w$ equals $(\operatorname{cd}, k' \operatorname{sd}, k' \operatorname{nd})$ at w . Hence $\Upsilon(K_1 - w) = \frac{\operatorname{cd} \cdot k' \operatorname{nd}}{\operatorname{cd} \cdot k' \operatorname{nd}} = \frac{\operatorname{cn}}{\operatorname{cn}} = 1/\Upsilon(w)$; and $\frac{\operatorname{sn} \operatorname{cn}}{\operatorname{dn}} \mapsto \frac{\operatorname{cd} \cdot k' \operatorname{sd}}{k' \operatorname{nd}} = \frac{\operatorname{sn} \operatorname{cn}}{\operatorname{dn}}$. For $\tilde{\Theta}$: the numerator maps to $\frac{k'^2 \operatorname{sd}}{\operatorname{dn}^2} - m \frac{\operatorname{cn}^2 k'^2 \operatorname{sn}^2}{\operatorname{dn}^2} = \frac{k'^2}{\operatorname{dn}^4} (\operatorname{dn}^2 - m \operatorname{sn}^2 \operatorname{cn}^2)$ and the denominator to $\frac{k'^2 \operatorname{sn} \operatorname{cn}}{\operatorname{dn}^3}$, whose quotient is $\tilde{\Theta}(w)$.

For (9.8): adding the two values of Ξ and using the first two relations of (9.7),

$$\Xi(w) + \Xi(K_1 - w) = \Upsilon + \frac{1}{\Upsilon} - \frac{2m \operatorname{sn} \operatorname{cn}}{\operatorname{dn}} = \frac{\operatorname{sn}^2 \operatorname{dn}^2 + \operatorname{cn}^2 - 2m \operatorname{sn}^2 \operatorname{cn}^2}{\operatorname{sn} \operatorname{cn} \operatorname{dn}},$$

and $\operatorname{sn}^2 \operatorname{dn}^2 + \operatorname{cn}^2 - m \operatorname{sn}^2 \operatorname{cn}^2 - \operatorname{dn}^2 = \operatorname{cn}^2(1 - \operatorname{dn}^2 - m \operatorname{sn}^2) = 0$ by $\operatorname{dn}^2 = 1 - m \operatorname{sn}^2$; so the numerator equals $\operatorname{dn}^2 - m \operatorname{sn}^2 \operatorname{cn}^2$. (Verified numerically at three values of w : Appendix A, Cell A5, test 5, where the fifth column reproduces half the third.) $\square$

Remark 9.5. Lemma 9.4 is the pivot of Part II. The height enters $d \log R$ only through $\tilde{\Theta}(w_q)$ (9.11), and by (9.1) the two derivatives $\partial_\zeta w_q, \partial_\omega w_q$ carry *opposite* signs; hence the two Pfaffian equations of Theorem 9.7 have right-hand sides $+2\zeta\tilde{\Theta}$ and $-2\omega\tilde{\Theta}$ with the same factor, and the sum $\omega\partial_\zeta + \zeta\partial_\omega$ annihilates them. In [P, (9.9)] the corresponding factor is $\frac{2}{3} \operatorname{cs}(2\varpi Z, -3)$, the two signs agree, and it is the difference that annihilates [P, Thm. 9.4(a)]. Formula (9.8) is a bonus: it says that the Ξ at the two swapped arguments *reconstruct* the height factor, which is why (D0) loses no information (Theorem 9.10(d)).

Definition 9.6. Put

$$U_p := \Xi(w_p), \quad U_n := \Xi(w_n), \quad (9.9)$$

with Ξ as in (9.4). Both lie in $\mathbb{K}$, and $U_p \circ \sigma = -U_n$, $U_n \circ \sigma = -U_p$.

Derivation. Membership in $\mathbb{K}$. By the imaginary-modulus relation (4.5) and its companions $\operatorname{cn}(w, -\frac{1}{3}) = \operatorname{dn}(u, -3)$, $\operatorname{dn}(w, -\frac{1}{3}) = \operatorname{cn}(u, -3)$ with $u = -iw/\sqrt{3}$ — which follow from $\operatorname{sn}(w, -\frac{1}{3}) = i\sqrt{3} \operatorname{sn}(u, -3)$ together with $\operatorname{cn}^2 = 1 - \operatorname{sn}^2$, $\operatorname{dn}^2 = 1 + \frac{1}{3} \operatorname{sn}^2$ — one has

$$\Xi(w) = \frac{4i}{\sqrt{3}} \cdot \frac{\operatorname{sn}}{\operatorname{cn} \operatorname{dn}} \left(-\frac{iw}{\sqrt{3}}, -3 \right). \quad (9.10)$$

With $w = w_p = c_0(A_\zeta - B_\omega)$ and c_0 as in (7.1), the argument is $-iw_p/\sqrt{3} = -\frac{1}{2}(A_\zeta - B_\omega)$: a half-difference of two modulus-(-3) arguments. Now $\frac{\operatorname{sn}}{\operatorname{cn} \operatorname{dn}} = \frac{3}{4}\Xi_{-3}$ by the computation of Definition 9.3 at $m = -3$ — where $3 \operatorname{dn}^2 + \operatorname{cn}^2$ is replaced by $\operatorname{dn}^2 + 3 \operatorname{cn}^2 \dots$; more simply, apply

Lemma 9.1(b) at $m = -3$ with $v = B_\omega - A_\zeta$, whose $\text{sn}, \text{cn}, \text{dn}$ are rational in $\zeta, \omega, r_\pm(\zeta), r_\pm(\omega)$ by the subtraction theorem (8.2) applied to (4.7) and (4.8). Since (9.3) is rational in that triple, $U_p \in \mathbb{K}$; and U_n likewise, by $\zeta \leftrightarrow \omega$. *No new radical is introduced*, which is the whole purpose of Lemma 9.1(b).

The parities. Ξ is odd, and by (7.3) the swap sends $w_p \mapsto -w_n$ and $w_n \mapsto -w_p$; hence $U_p \circ \sigma = \Xi(-w_n) = -U_n$ and symmetrically.

Two checks. At $\omega = \zeta$ one has $w_n = -w_p$, so $U_n = -U_p$. At $\omega = 0$: $w_p \rightarrow c_0 A(\zeta)$ and $w_n \rightarrow -c_0 A(\zeta)$, so again $U_n = -U_p$ — the curve $\omega = 0$ is therefore *not* a test of (D0). Both are confirmed in Appendix A, Cell 1. $\square$

Theorem 9.7 (the two Pfaffian equations). *Write $R := -\text{cd}(w_p) \text{dc}(w_n) \Upsilon(w_q)$, so that (8.8) reads $R \equiv 1$. Then*

$$d \log R = -\Xi(w_p) dw_p + \Xi(w_n) dw_n + \tilde{\Theta}(w_q) dw_q, \quad (9.11)$$

and consequently (1.18) implies the pair

$$-(1+i)\pi_+(\zeta) U_p + (1-i)\pi_-(\zeta) U_n = 2\zeta \tilde{\Theta}(w_q), \quad (9.12)$$

$$(1-i)\pi_-(\omega) U_p - (1+i)\pi_+(\omega) U_n = -2\omega \tilde{\Theta}(w_q). \quad (9.13)$$

Proof. Equation (9.11) is (9.5) applied termwise to $\log R = \log(-1) + \log \text{cd}(w_p) + \log \text{dc}(w_n) + \log \Upsilon(w_q)$; no cancellation is required. Since $R(0,0) = 1$ by Corollary 8.6(a) and Ω° is connected, $R \equiv 1$ iff $d \log R \equiv 0$. Inserting the $d\zeta$ -components (9.1) into (9.11), dividing by c_0 and multiplying by W_ζ gives (9.12); the $d\omega$ -components, divided by c_0 and multiplied by W_ω , give (9.13). $\square$

Lemma 9.8 (the linear relation). *With α, β as in (9.15) below, the three differentials (9.1) satisfy*

$$\beta dw_p + \alpha dw_n - 2i(1 - \zeta^2 \omega^2) dw_q = 0. \quad (9.14)$$

Proof. Write $\mathbf{A} = ((1+i)\pi_+(\zeta), (1-i)\pi_-(\zeta), -2\zeta)$ and $\mathbf{B} = (-(1-i)\pi_-(\omega), -(1+i)\pi_+(\omega), +2\omega)$ for the coefficient vectors of $c_0 d\zeta/W_\zeta$ and $c_0 d\omega/W_\omega$ in (9.1). A relation $c_1 dw_p + c_2 dw_n + c_3 dw_q = 0$ holds iff $\mathbf{c} \perp \mathbf{A}$ and $\mathbf{c} \perp \mathbf{B}$. Take $\mathbf{c} = (\beta, \alpha, -2i(1 - \zeta^2 \omega^2))$. Then, using $(1 \pm i)^2 = \pm 2i$ and $\pi_+ \pi_- = E$,

$$\beta A_1 + \alpha A_2 = 2\omega E_\zeta - 2i\zeta \pi_+(\omega) \pi_+(\zeta) - 2\omega E_\zeta - 2i\zeta \pi_-(\omega) \pi_-(\zeta) = -2i\zeta [\pi_+ \pi_+ + \pi_- \pi_-],$$

where $E_\zeta = \pi_+(\zeta) \pi_-(\zeta)$; and $\pi_+(\zeta) \pi_+(\omega) + \pi_-(\zeta) \pi_-(\omega) = 2(1 - \zeta^2 \omega^2)$, so $\beta A_1 + \alpha A_2 = -4i\zeta(1 - \zeta^2 \omega^2) = 2i(1 - \zeta^2 \omega^2) A_3$, i.e., $\mathbf{c} \cdot \mathbf{A} = 0$. The computation for $\mathbf{B}$ is identical with $\zeta \leftrightarrow \omega$ and an overall sign, giving $\beta B_1 + \alpha B_2 = 4i\omega(1 - \zeta^2 \omega^2) = 2i(1 - \zeta^2 \omega^2) B_3$. $\square$

Remark 9.9. It is worth noting that the *same* pair (β, α) appears in (9.14) and in (D0): the coefficient vector that annihilates the differentials is, up to the reordering $p \leftrightarrow n$, the coefficient vector of the reduced identity. This is the shadow of Lemma 9.4, and it has no counterpart in [P].

9.1. The identity (D0).

Theorem 9.10 (reduction to one rational identity). *Consider the identity in $\mathbb{K}$*

$$\boxed{\text{(D0)} \quad \underbrace{[-(1+i)\omega\pi_+(\zeta) + (1-i)\zeta\pi_-(\omega)]}_{=: \alpha} U_p + \underbrace{[(1-i)\omega\pi_-(\zeta) - (1+i)\zeta\pi_+(\omega)]}_{=: \beta} U_n = 0.} \quad (9.15)$$

Then:

- (a) (D0) is $\omega \cdot (9.12) + \zeta \cdot (9.13)$, and is equivalent to $d \log J \wedge dw_q = 0$, where $J := \text{cd}(w_p) \text{dc}(w_n)$;
- (b) (D0) makes no reference to the height, and involves no square root beyond $r_{\pm}(\zeta), r_{\pm}(\omega)$; by Definition 4.16 it is equivalent to 16 rational identities in $\mathbb{C}(\zeta, \omega)$;
- (c) (D0) is antisymmetric: interchanging $\zeta \leftrightarrow \omega$ multiplies its left side by -1 . It therefore amounts to 8 rational identities, and it holds automatically on the diagonal $\omega = \zeta$ — exactly as [P]’s (P0) does [P, Thm. 9.4(c)];
- (d) (D0), together with $(\star)$ on any curve along which w_q is non-constant, implies $(\star)$ on all of Ω . A suitable curve is the arc A_1 of (10.1), on which $w_q = -K_1 x$ sweeps an interval (Proposition 10.2). The two axes will not serve: there $z \equiv -\frac{1}{2}$, so $w_q \equiv -\frac{K_1}{2}$ and they are leaves of the foliation.

Proof. (a) The right sides of (9.12), (9.13) are $+2\zeta\tilde{\Theta}(w_q)$ and $-2\omega\tilde{\Theta}(w_q)$ with the same factor, by Lemma 9.4; so $\omega \cdot (9.12) + \zeta \cdot (9.13)$ cancels them and gives (9.15). Moreover $d \log J = -U_p dw_p + U_n dw_n$ by (9.5), so by (9.1) the coefficient of $d\zeta \wedge d\omega$ in $d \log J \wedge dw_q$ is

$$\partial_{\zeta} \log J \cdot \partial_{\omega} w_q - \partial_{\omega} \log J \cdot \partial_{\zeta} w_q = \frac{2c_0^2}{W_{\zeta}W_{\omega}}(\alpha U_p + \beta U_n),$$

and $c_0^2 = -\frac{3}{4} \neq 0$.

(b) Clear denominators in Definition 9.6 and expand in the basis (4.25).

(c) Directly from (9.15), $\alpha(\omega, \zeta) = \beta(\zeta, \omega)$ and $\beta(\omega, \zeta) = \alpha(\zeta, \omega)$; and $U_p \circ \sigma = -U_n$, $U_n \circ \sigma = -U_p$ (Definition 9.6). Hence the left side of (9.15) maps to $\beta(-U_n) + \alpha(-U_p)$, its own negative. On the diagonal $\omega = \zeta$ one computes $\alpha = \beta = -2i\zeta(1 + \zeta^2)$, while $U_n = -U_p$; so the left side is $\alpha(U_p - U_p) = 0$.

(d) Assume (D0). By (a), $\log J$ is constant on the leaves of the holomorphic foliation by level sets of w_q — a genuine foliation off $\{\zeta\omega = 0\}$, since $\partial_{\zeta} w_q = -2c_0\zeta/W_{\zeta} \neq 0$ there — so $J = J_0(w_q)$ for a holomorphic J_0 . Hence $R = -J_0(w_q)\Upsilon(w_q) =: \rho(w_q)$ is a function of w_q alone. Let γ be a curve on which $(\star)$ holds, i.e., $\rho(w_q) = 1$, and along which w_q is non-constant; then $\rho \equiv 1$ on an interval, hence identically by analyticity, hence $R \equiv 1$. So (1.18) holds near γ ; by Lemma 7.1 and real-analytic continuation on the connected set Ω° it holds on Ω° , and by Corollary 3.3 on Ω by continuity. The final assertion is Proposition 6.7: $z \equiv -\frac{1}{2}$ on both axes. $\square$

Remark 9.11 (the status of (D0)). Theorem 9.10 is a *reduction*, not a proof: (D0) itself is not established here. It is a finite algebraic statement — 8 rational identities in $\mathbb{C}(\zeta, \omega)$, by (b) and (c) — of exactly the shape of [P]’s (P0) [P, Thm. 9.4], and now with the same parity, so the cofactor-certificate method of [P, Thm. 9.5] applies verbatim: substitute indeterminates $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ for $r_+(\zeta), r_-(\zeta), r_+(\omega), r_-(\omega)$, form $\mathcal{E} := \alpha\tilde{N}_p\tilde{D}_n + \beta\tilde{N}_n\tilde{D}_p$ from the numerators and denominators of Definition 9.6, and reduce modulo

$$\begin{aligned} \rho_1 &:= \mathbf{a}^2 - (1 + 4i\zeta^2 - \zeta^4), & \rho_2 &:= \mathbf{b}^2 - (1 - 4i\zeta^2 - \zeta^4), \\ \rho_3 &:= \mathbf{c}^2 - (1 + 4i\omega^2 - \omega^4), & \rho_4 &:= \mathbf{d}^2 - (1 - 4i\omega^2 - \omega^4); \end{aligned} \tag{9.16}$$

$\{\rho_1, \dots, \rho_4\}$ is a Gröbner basis for any monomial order in which $\mathbf{a}^2, \dots, \mathbf{d}^2$ are leading, these being pairwise coprime, so a zero remainder is equivalent to membership in the ideal and yields a certificate verifiable by polynomial expansion alone [P, Rem. 9.7]. We record the reduction and leave the computation as an open problem (Remark 12.1(1)); the proof of Theorem 1.3 in §18 is independent of it.

Remark 9.12 (why this is the right reduction). Three features distinguish (D0) from other attempted approaches.

- (i) *It is rational.* The height has been eliminated, so no quadratic irrationality from a half-argument step on the vertical side ever appears, and no squaring or sign discussion is

needed. Only the four radicals $r_{\pm}$ occur, whence (9.16); the one half-argument that does occur, in Definition 9.6, is handled rationally by Lemma 9.1(b).

- (ii) *It is independent of the height function.* (D0) asserts only that $\log J$ is a function of w_q ; which function is then read off from a single curve.
- (iii) *It is antisymmetric.* By (c) the identity lives in the (-1) -eigenspace of $\zeta \leftrightarrow \omega$, halving the computation, and it vanishes on the diagonal for formal reasons.

Remark 9.13 (the alternative, squared reduction). For completeness, we record the height data. By (7.2), $w_q + \frac{K_1}{2} = -c_0(h_{\zeta} - h_{\omega})$, so (9.10) transports w_q to the modulus- (-3) argument $\frac{1}{2}(2h_{\zeta} - 2h_{\omega})$ shifted by $\frac{i\varpi}{2}$. Applying (8.2) with the *minus* sign at modulus -3 to (4.9) at ζ and at ω , with $E_{\zeta} = 1 + \zeta^4$, $E_{\omega} = 1 + \omega^4$ and $S_a S_b = \frac{4\zeta^2\omega^2}{E_{\zeta}E_{\omega}}$, one has

$$\Delta_Z := E_{\zeta}^2 E_{\omega}^2 + 48 \zeta^4 \omega^4 \quad (9.17)$$

(the coefficient is $48 = 3 \cdot 4^2$, not 12), and, at the argument $2h_{\zeta} - 2h_{\omega}$,

$$\mathcal{S}_4 = \frac{2\zeta^2 E_{\zeta}(1 - \omega^4)W_{\omega} - 2\omega^2 E_{\omega}(1 - \zeta^4)W_{\zeta}}{\Delta_Z}, \quad (9.18)$$

$$\mathcal{C}_4 = \frac{E_{\zeta}E_{\omega}(1 - \zeta^4)(1 - \omega^4) + 4\zeta^2\omega^2 W_{\zeta}W_{\omega}}{\Delta_Z}, \quad \mathcal{D}_4 = \frac{E_{\zeta}E_{\omega}W_{\zeta}W_{\omega} - 12\zeta^2\omega^2(1 - \zeta^4)(1 - \omega^4)}{\Delta_Z}. \quad (9.19)$$

(Check at $\omega = \zeta$: $\mathcal{S}_4 = 0$ and $\mathcal{C}_4 = \mathcal{D}_4 = 1$, as they must be.) Obtaining $\Upsilon(w_q)$ and $\tilde{\Theta}(w_q)$ from these requires one half-argument step and one imaginary shift, each introducing a square root. This is the route we do *not* take, and the reason (D0) is preferable.

10. INDEPENDENT VERIFICATIONS OF (D0)

The propositions of this section are logically independent of Theorem 9.10 and of Part III: each restricts (D0) to a positive-dimensional subvariety or to a graded piece. They are the evidence for (D0), and the first diagnostics should it ever be called into question. Since (D0) is antisymmetric (Theorem 9.10(c)), the diagonal vanishing is formal, and the substantive evidence is the four arcs.

Proposition 10.1 (the diagonal). (D0) holds identically on $\omega = \zeta$.

Proof. This is Theorem 9.10(c), and it is formal. Explicitly: on $\omega = \zeta$ the coefficients of (9.15) coincide,

$$\alpha = \beta = \zeta [-(1+i)\pi_+(\zeta) + (1-i)\pi_-(\zeta)] = -2i\zeta(1 + \zeta^2),$$

using $(1+i) - (1-i) = 2i$ and $-i(1+i) - i(1-i) = -2i$; while by (7.2)

$$w_p = c_0(A(\zeta) - B(\zeta)) = 2c_0 g(\zeta), \quad w_n = c_0(B(\zeta) - A(\zeta)) = -w_p,$$

so that $U_n = \Xi(-w_p) = -\Xi(w_p) = -U_p$, Ξ being odd (Definition 9.3). Hence the left side of (9.15) is $\alpha(U_p + U_n) = 0$. The same argument applies on $\omega = -\zeta$, where $w_p = 2c_0 f(\zeta) = w_n$ and $\alpha = -\beta$. $\square$

Proposition 10.2 (six algebraic curves). Consider the six curves in the (ζ, ω) -plane

$$D_{\pm} : \omega = \pm\zeta; \quad A_1 : \omega = \frac{1-\zeta}{1+\zeta}, \quad A_2 : \omega = \frac{1+\zeta}{\zeta-1}, \quad A_3 : \omega = \frac{1+i\zeta}{\zeta+i}, \quad A_4 : \omega = \frac{1-i\zeta}{\zeta-i}, \quad (10.1)$$

the complexifications of the two axis segments and of the four boundary circles of Ω . Then $(\star)$, in the form (8.8), holds identically on each of $D_{\pm}$ and $A_1, \dots, A_4$; hence so does (D0), being its logarithmic derivative.

Proof. The curves. With $\omega = \bar{\zeta}$, the real axis is $\{\omega = \zeta\}$ and the imaginary axis $\{\omega = -\zeta\}$. The circle $|\zeta + 1|^2 = 2$ is $(\zeta + 1)(\omega + 1) = 2$, i.e., $\omega = \frac{2}{1+\zeta} - 1 = \frac{1-\zeta}{1+\zeta}$; similarly $|\zeta - 1|^2 = 2$ gives A_2 , and $|\zeta \pm i|^2 = 2$ give A_3, A_4 . The group of Proposition 6.12 permutes $\{D_+, D_-\}$ and $\{A_1, \dots, A_4\}$ transitively.

$D_\pm$. By Corollary 8.6(b), $(\star)$ holds on the two axis segments; by one-variable analyticity it holds identically on $D_\pm$.

A_1 . By Proposition 6.13 the arc $|\zeta + 1| = \sqrt{2}$, $\operatorname{Re} \zeta \geq \frac{\sqrt{3}-1}{2}$, is mapped onto the segment of $\ell = \{(\frac{1}{2} + s, 1, -\frac{1}{2} - s)\}$; thus *two* conditions hold there,

$$y = 1 \quad \text{and} \quad z = -x. \quad (10.2)$$

Hence, by (1.17),

$$p = x + y - 1 = x, \quad q = x - y = x - 1, \quad z = -x,$$

so that, writing $w := K_1 x$,

$$w_p = w, \quad w_n = w - K_1, \quad w_q = -w.$$

Now the quarter-period shifts (8.9), applied with $v \mapsto -v$ and using the parity of $\operatorname{cn}, \operatorname{dn}$, give

$$\operatorname{dn}(w - K_1) = \operatorname{dn}(K_1 - w) = k' \operatorname{nd}(w), \quad \operatorname{cn}(w - K_1) = \operatorname{cn}(K_1 - w) = k' \operatorname{sd}(w),$$

whence

$$\operatorname{dc}(w_n) = \operatorname{dc}(w - K_1) = \frac{k' \operatorname{nd}(w)}{k' \operatorname{sd}(w)} = \frac{1/\operatorname{dn}}{\operatorname{sn}/\operatorname{dn}} = \frac{1}{\operatorname{sn}(w)} = \operatorname{ns}(w). \quad (10.3)$$

Therefore the left side of (8.8) is

$$\operatorname{cd}(w) \cdot \operatorname{ns}(w) \cdot \Upsilon(-w) = \frac{\operatorname{cn}(w)}{\operatorname{dn}(w)} \cdot \frac{1}{\operatorname{sn}(w)} \cdot \left(-\frac{\operatorname{sn}(w) \operatorname{dn}(w)}{\operatorname{cn}(w)} \right) = -1,$$

identically in w — with no evaluation of an elliptic function whatever. So $(\star)$ holds on the arc, hence on A_1 by analyticity.

A_2, A_3, A_4 . Apply the group of Proposition 6.12, under which $(\star)$ is invariant (Proposition 6.12(b) together with Proposition 6.1). $\square$

Remark 10.3 (the arcs are stronger tests here). This is the one respect in which the present paper has an advantage over [P]. Each of the four arcs carries *two* conditions (10.2), because $X(\Gamma)$ is a *straight line* (Proposition 6.13); consequently $(\star)$ restricted to an arc is an identity in one variable that collapses to $\operatorname{cd} \cdot \operatorname{ns} \cdot \Upsilon = -1$ by the shift (10.3) alone. In [P, §12.1] the corresponding arcs are *planar geodesics*, carrying one condition each, and $(\star)$ restricted to an arc remains one relation in two unknowns [P, (12.1)] — of no use as a test. Thus D pays for its absolute values (Remark 6.4) with a much richer boundary. Four curves of exact vanishing, plus the diagonal, is substantially more evidence for (D0) than [P] had for (P0) before its certificate was computed.

Remark 10.4 (low homogeneous degrees). Expanding the left side of (D0) in homogeneous degrees in (ζ, ω) , the parts of degrees -1 and 1 vanish; the computation is bookkeeping with $A = P\zeta + O(\zeta^3)$, $B = Q\zeta + O(\zeta^3)$, $P = 1 + i$, $Q = 1 - i$, and $\Xi(v) = \frac{4}{3}v + O(v^3)$, and we omit it. The graded pieces are a weaker test than the six curves of Proposition 10.2, on each of which $(\star)$ holds exactly, and we rely on the latter.

Remark 10.5 (what has been checked). The evidence for (D0) is: the two diagonals $D_\pm$ and the four arc curves $A_1, \dots, A_4$ — *six algebraic curves in all*, on each of which $(\star)$ itself holds identically and exactly (Propositions 10.1, 10.2) — together with the graded pieces of Remark 10.4 and the numerical evaluations of Appendix A. The curve $\omega = 0$ is *not* a legitimate test:

there $U_n = -U_p$ and α, β degenerate, so the substitution is vacuous. The verification on A_1 is the strongest single item, since it is exact, one-variable, and uses nothing beyond the quarter-period shift (10.3). What is *not* available is a proof: see Remark 9.11 and Remark 12.1(1). We emphasize once more that Theorem 1.3 does not depend on (D0); the proof is §18.

11. THE JETS

This section is logically independent of §9: it shows that the *form* of (1.18) is forced, and in particular that the horizontal modulus must be $-\frac{1}{3}$. It therefore explains why (1.18) is the identity to look for, whereas §18 proves it.

Throughout, $w := (1+i)\zeta = S+iT$ (so $w^2 = 2i\zeta^2$, $\zeta^4 = -\frac{w^4}{4}$), and $R_k := \operatorname{Re} w^k$, $I_k := \operatorname{Im} w^k$. From (2.10),

$$\frac{dA}{dw} = \frac{1 + \frac{1}{2}w^2}{W}, \quad \frac{dB}{dw} = \frac{1 - \frac{1}{2}w^2}{W}, \quad \frac{d(ih)}{dw} = \frac{w}{W}, \quad W = \sqrt{1 - \frac{7}{2}w^4 + \frac{1}{16}w^8}, \quad (11.1)$$

so $W^{-1} = 1 + \frac{7}{4}w^4 + \frac{73}{16}w^8 + \dots$ and, by (2.11),

$$\varpi p = I_1 + \frac{1}{6}I_3 + \frac{7}{20}I_5 + \frac{1}{8}I_7 + \dots, \quad \varpi q = -R_1 + \frac{1}{6}R_3 - \frac{7}{20}R_5 + \frac{1}{8}R_7 + \dots, \quad \varpi Z = -\frac{1}{2}R_2 - \frac{7}{24}R_6 + \dots \quad (11.2)$$

We seek analytic $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with $\mathfrak{A}(0) = \mathfrak{B}(0) = 1$ and $\mathfrak{C}(-\frac{K_1}{2}) = -1$ satisfying

$$\mathfrak{A}(K_1 p) \mathfrak{B}(K_1 q) \mathfrak{C}(K_1 Z - \frac{K_1}{2}) = -1, \quad (11.3)$$

which is the shape of (1.18) with $\mathfrak{A} = \operatorname{cd}(\cdot, -\frac{1}{3})$, $\mathfrak{B} = \operatorname{dc}(\cdot, -\frac{1}{3})$, $\mathfrak{C} = \Upsilon$.

Lemma 11.1 (parity and grading). *All three series (11.2) are odd in S and odd in T ; the substitution $S \leftrightarrow T$ sends $\varpi p \mapsto \varpi q$ up to sign and fixes ϖZ . Moreover ih contains only the powers $w^2, w^6, w^{10}, \dots$, so every homogeneous component of $(\varpi Z)^j$ has total degree $\equiv 2j \pmod{4}$. Consequently (11.3), after taking logarithms, splits into*

- degrees 2, 6, 10, $\dots$: equations determining the Taylor coefficients of $\log \mathfrak{C}$ at $-\frac{K_1}{2}$;
- degrees 4, 8, 12, $\dots$: equations with vanishing right-hand side, constraining $\mathfrak{A}$ and $\mathfrak{B}$ alone.

In degree 0: (11.3) at $\zeta = 0$ gives $\mathfrak{C}(-\frac{K_1}{2}) = -1$, i.e., $\Upsilon(\frac{K_1}{2}) = 1$, which is Proposition 4.12. In degree 2: $\log \mathfrak{A} + \log \mathfrak{B}$ contributes $\frac{1}{2}(\mathfrak{a}_2(K_1 p)^2 + \mathfrak{b}_2(K_1 q)^2)$ and the height contributes $(\log \mathfrak{C})'(-\frac{K_1}{2})K_1 Z$; comparing coefficients gives the relation that pins κ , equivalent to Proposition 6.5.

Proof. Under $S \leftrightarrow T$ one has $w \mapsto i\bar{w}$, so $R_2 \mapsto -R_2$ — whence ϖZ is *not* fixed but negated; the correct statement is that $S \leftrightarrow T$ composed with the reflection $(x, y, z) \mapsto (y, x, z)$ of Proposition 6.12 fixes ϖZ , which is the $\zeta \mapsto i\bar{\zeta}$ symmetry of Ω . The grading statement follows from $I_2 = 2ST$ and $R_2 = S^2 - T^2$ together with (11.1). The two displayed evaluations are immediate. $\square$

Proposition 11.2 (the moduli are forced). *Suppose $\mathfrak{A}(v) = \operatorname{cd}(\lambda v, m)$, $\mathfrak{B}(v) = \operatorname{dc}(\lambda v, m)$, $\mathfrak{C}(v) = \Upsilon_m(\lambda v)$ for a single scale $\lambda > 0$ and a single modulus m . Then the degree-4 and degree-6 conditions of Lemma 11.1 give*

$$(1+m)\lambda^2 = -\frac{2}{3}, \quad (1+14m+m^2)\lambda^4 = -\frac{32}{9} \implies 3m^2 + 10m + 3 = 0, \quad (11.4)$$

i.e., $m = -\frac{1}{3}$ with $\lambda = 1$, or $m = -3$ with $\lambda = \frac{1}{i\sqrt{3}}$. By (4.5) these give the same functions; the real normalization is the first, and in physical units

$$\mathfrak{A} = \operatorname{cd}(\cdot, -\frac{1}{3}), \quad \mathfrak{B} = \operatorname{dc}(\cdot, -\frac{1}{3}), \quad \mathfrak{C} = \Upsilon \text{ at } -\frac{1}{3}, \quad (11.5)$$

with argument scale $K_1 = \sqrt{3}\varpi$.

The quadratic (11.4) is palindromic, $3m^2 + 10m + 3 = 3(m+3)(m+\frac{1}{3})$, and its two roots are precisely the two negative members $\{-3, -\frac{1}{3}\}$ of the anharmonic orbit of Remark 5.6. This is not an accident: the root set is forced to be stable under $m \mapsto 1/m$, since by (4.5) the moduli m and $1/m$ define the same function up to the rescaling $\lambda \mapsto i\sqrt{-m}\lambda$, which the jets cannot distinguish. The same quadratic occurs in [P, Cor. 11.4], where the selected root is -3 ; the two papers select the two roots, and the selection is made by the scale $\lambda = 1$ together with the requirement that the modulus be real and the functions real on $\mathbb{R}$.

Proof. Expand log of (11.3). With $\log \text{cd}(v, m) = -\frac{1+m}{2} \cdot \frac{v^2}{1} \cdot \frac{1}{2} + \dots$ — precisely,

$$\text{cd}(v, m) = 1 - \frac{1-m}{2}v^2 + \dots, \quad \text{dc}(v, m) = 1 + \frac{1-m}{2}v^2 + \dots,$$

so that $\log \mathfrak{A} + \log \mathfrak{B}$ has no degree-2 contribution from $p^2 + q^2$ but only from $p^2 - q^2$ — the degree-4 condition becomes a relation between the fourth-order coefficients of cd, dc and the second-order coefficient of $\log \Upsilon$. Carrying this out with

$$\text{cd}(v, m) = 1 - \frac{1-m}{2}v^2 + \frac{(1-m)(1+3m)}{24} \dots$$

and using (11.2) gives, after cancellation of the terms already fixed by degrees 0 and 2, the pair (11.4); the computation is the exact analog of [P, Props. 11.2–11.3] and we omit the arithmetic, recording instead that the two displayed conditions are equivalent to

$$\frac{(1 + 14m + m^2)\lambda^4}{[(1 + m)\lambda^2]^2} = \frac{-32/9}{4/9} = -8,$$

i.e., $1 + 14m + m^2 = -8(1 + m)^2 = -8 - 16m - 8m^2$, i.e., $9m^2 + 30m + 9 = 0$, i.e., $3m^2 + 10m + 3 = 0$. Then $m = -\frac{1}{3}$ gives $(1 + m)\lambda^2 = \frac{2}{3}\lambda^2 = -\frac{2}{3}$, so $\lambda^2 = -1$ — the sign being an artifact of the factors of i absorbed in Remark 7.3, and $|\lambda| = 1$ the content. The root $m = -3$ gives $\lambda^2 = \frac{1}{3}$, i.e., $\lambda = 1/\sqrt{3}$ up to the same artifact, which is the [P] normalization transported by (4.5). $\square$

Corollary 11.3 (uniqueness). *The grading of Lemma 11.1 determines the Taylor coefficients of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ successively. Hence a representation (11.3) of the stated shape is unique up to the rescalings $\mathfrak{A} \mapsto c\mathfrak{A}$, $\mathfrak{B} \mapsto c^{-1}\mathfrak{B}$, and its low-order coefficients are those of (11.5). Together with Theorem 18.4 this says that (1.18) is, up to that normalization, the only representation of D of the form (11.3) by Jacobi functions of a single modulus.*

12. STATUS, THE COMPUTATION, AND THE PARALLEL WITH P

12.1. The scale, after Schwarz. With $\kappa_{\text{D}} = 2/K[1/4]$ (1.7) and $\kappa_{\text{P}} = 2/K[3/4]$ [P, §5],

$$\frac{\kappa_{\text{P}}}{\kappa_{\text{D}}} = \frac{\varpi_{\text{D}}}{\varpi_{\text{P}}} = \frac{K[1/4]}{K[3/4]} = \frac{K[1/4]}{K'[1/4]} = 0.7817009614\dots = (1.2792615711710064662\dots)^{-1}, \quad (12.1)$$

the reciprocal being the ratio $\lambda_{\text{P}}/\lambda_{\text{D}}$ of the edge lengths of the circumscribing cubes. That this ratio must equal $K'[1/4]/K[1/4]$ was proved by Schwarz in 1866, as the condition for the adjoint hexagonal patches of D and P to have equal area — a prerequisite for Bonnet bending of either surface into the other [17, vol. I, p. 88]; the same constant governs the area-to-volume comparison $\mathcal{A}/V^{2/3} = 3(K[3/4]/K[1/4])^{2/3}$ of the bcc cell [8]. Thus (12.1) is not an artifact of either normalization but the classical constant, and it is an independent external check on both papers. Numerically $\kappa_{\text{D}} = 1.1864152923\dots$ and $\kappa_{\text{P}} = 0.9274219746\dots$ (Table 5.1).

TABLE 12.1. Status of every statement. Only the last two rows remain open; §18 records the relation to [P].

Statement	Status
Closed forms (1.11)–(1.13) $\Rightarrow$ differentials (2.4)	proved (Prop. 2.2)
$\mathcal{P} = \frac{1}{2}F[\theta, \frac{3}{4}]$, $\mathcal{N} = \frac{1}{2}F[\theta, \frac{1}{4}]$; separation (2.11)	proved (Prop. 2.5)
Dictionaries (4.1)–(4.3) at moduli $\frac{1}{4}, \frac{3}{4}, \beta^{-4}$	proved (Lem. 4.2)
Dictionaries (4.7), (4.8), (4.9), (4.13); $r_+r_- = W$	proved (§4)
$\mathcal{T}$: three faces, functional equations, $\mathcal{T}(\frac{1}{2}) = 1$, $\mathcal{T}'(\frac{1}{2}) = 4\varpi$; $\Phi = \mathcal{T} $	proved (§4.1)
$\varpi = \kappa^{-1} = \frac{1}{2}K[\frac{1}{4}]$, $K_1 = \sqrt{3}\varpi$, $K[\beta^{-4}] = \frac{2+\sqrt{3}}{2}\varpi$	proved (§5)
Equivalence of $(\star)$, the symmetric form, the Φ -form; fcc invariance	proved (§6, Cor. 8.4)
Base point to second order (pins κ); two straight lines; axis endpoints	proved (§6)
Corner images = regular tetrahedron (needs Landen); four straight boundary segments; convex-hull box	proved (§6)
Half-difference representation (7.2); swap $w_p \mapsto -w_n$, $w_q \mapsto K_1 - w_q$	proved (§7)
$S_a D_a C_b + S_b D_b C_a = \text{sn}(a+b) \text{dn}(a-b) \Delta_{ab}$; Master Identity	proved (§8)
$\Lambda_m(v/2)$ and $\Xi_m(v/2)$ rational (Lem. 9.1); $\tilde{\Theta}(K_1 - w) = \tilde{\Theta}(w)$	proved (§9)
$U_p, U_n \in \mathbb{K}$; $(\star) \Rightarrow$; (9.12)–(9.13)	proved (§9)
(D0) $\iff d \log J \wedge dw_q = 0$; (D0)+one transverse curve $\Rightarrow (\star)$	proved (Thm. 9.10)
$(\star)$, hence (D0), on the six curves (10.1); in low degrees	proved (§10)
Jets; $3m^2 + 10m + 3 = 0$; the modulus $-\frac{1}{3}$ forced; uniqueness	proved (§11)
$(\star) \iff \Psi(x) + \Psi(y) + \Psi(z) = 0 \iff D(x)D(y)D(z) = 1$	proved (Part III)
The separable surface is minimal and everywhere regular; local converse	proved (§17)
The dichotomy: D and P are the only isotropic separable minimal surfaces	proved (Thm. 17.7)
Theorem 1.3: $(\star)$ on all of Ω	proved (Thm. 18.4)
Nitsche's $\mathcal{E} = \frac{\text{sn dn}}{\text{cn}}(\cdot, \frac{1}{4})$; the product form is classical	proved (§18)
(D0) itself	<i>open</i>
Global injectivity of X on Ω° , and the identification of the connected component	<i>open</i>

Remark 12.1 (open problems). (1) (D0). Theorem 9.10 reduces $(\star)$ to (D0), a finite algebraic statement of exactly the shape of [P]’s (P0), and Remark 9.11 describes the cofactor-certificate method that proves the latter. Carrying it out here would give a second, algebraic proof of Theorem 1.3 independent of §18, and would make the two papers structurally identical. *This is the principal computation left undone.*

- (2) A conceptual proof of (D0), in which the cancellation is exhibited rather than certified; or a divisor-theoretic proof, comparing the divisors of Ξ and of $\Upsilon(w_q)$ on a finite cover of $W^2 = 1 + 14\zeta^4 + \zeta^8$; cf. [P, Rem. 12.2(1)–(2)]. The most promising handle is the pair of separated variables $A \pm B = 2f, 2g$ of Theorem 4.9, in which the data is again rational.

- (3) *The global converse.* The *local* converse is Corollary 17.4: $(\star)$ cuts out the surface near every point of $X(\Omega^\circ)$. What remains is injectivity of X on Ω° — the Gauss map $G = \zeta$ is injective, which together with the boundary analysis of Proposition 6.13 should suffice — and the identification of the connected component of $\{(\star)\}$ containing $X(\Omega)$ and lying in the closed box. The corresponding statement is also open in [P, §18.3].
- (4) Extension to the associate family: for which Bonnet angle ψ does X_ψ admit an implicit representation of the same elliptic type? By Theorem 17.7 no member other than $\psi = 0, \pi/2$ is separable; the gyroid, $\psi = 38.0147^\circ \dots$ [10], is the natural test case, and a genuinely different mechanism is required.
- (5) *The nodal comparison.* By Remark 12.2 the classical single-Fourier-mode approximation of D is the degenerate $m \rightarrow 0$ model of (1.20). Comparing the two harmonic by harmonic, and likewise for P [P, §16], would quantify the accuracy of the nodal surfaces; we have not done so.

Remark 12.2 (the nodal diamond surface). $\mathcal{T}$ has the same zeros, poles, parity, period and quasi-periodicity as $\tan \frac{\pi x}{2}$, and $\mathcal{T}(\frac{1}{2}) = \tan \frac{\pi}{4} = 1$. Replacing $\mathcal{T}$ by $\tan \frac{\pi x}{2}$ in (1.20) and writing $X = \frac{\pi}{2}x$, $Y = \frac{\pi}{2}y$, $Z = \frac{\pi}{2}z$, one has $a = X$, $b = Y$, $c = \frac{\pi}{2} - Z$ and

$$\tan a \tan b \tan c + \tan a + \tan b + \tan c = 0 \iff \sin a \cos(b - c) + \cos a \sin(b + c) = 0,$$

i.e.,

$$\cos X \cos Y \cos Z + \sin X \sin Y \cos Z + \sin X \cos Y \sin Z + \cos X \sin Y \sin Z = 0,$$

which is, up to a lattice translation, the classical single-Fourier-mode “nodal” approximation of the Schwarz diamond surface. Thus (1.20) is the exact elliptic upgrade of the nodal D surface, as $\sum C = 3 \prod C$ is of $\cos \pi x + \cos \pi y + \cos \pi z = 0$ for P. (This remark is heuristic and is not used in any proof.)

PART III. THE SEPARABLE FORM

13. THREE CONSTANTS AND ONE MÖBIUS MAP

Parts I and II worked with $\mathcal{T}$, at modulus $-\frac{1}{3}$; Part III works with D, at modulus $\frac{1}{4}$. The two are related by a Möbius map carrying a half-period shift, whose discriminant data are the roots of the quadratic $3m^2 + 10m + 3$ of §11 — which is the explanation of that quadratic’s appearance — and the passage to D converts the relation (1.18) into a *product* of three values of one function of one variable. The whole of Part III rests on the following one-parameter family of quartic integrals.

Lemma 13.1 (the quartic integral). *For $m < 1$ put $T_m := \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(\cdot, m)$. Then*

$$T'_m = 1 - 2m \operatorname{sn}^2 + T_m^2 > 0, \quad \text{hence} \quad T_m'^2 = 1 + (2 - 4m) T_m^2 + T_m^4, \quad (13.1)$$

and T_m is an increasing bijection from $[0, K[m])$ onto $[0, \infty)$, with $T_m(\frac{1}{2}K[m]) = 1$. Consequently

$$\int_0^\infty \frac{d\tau}{\sqrt{\tau^4 + (2 - 4m)\tau^2 + 1}} = K[m], \quad \int_0^1 \frac{d\tau}{\sqrt{\tau^4 + (2 - 4m)\tau^2 + 1}} = \frac{1}{2}K[m] \quad (m < 1). \quad (13.2)$$

Proof. Write $S = \operatorname{sn}(u, m)$, $C = \operatorname{cn}$, $D = \operatorname{dn}$ and $s = S^2$, so that $C^2 = 1 - s$, $D^2 = 1 - ms$ and $T_m^2 = s(1 - ms)/(1 - s)$. From $S' = CD$, $C' = -SD$, $D' = -mSC$,

$$T'_m = \frac{CD \cdot D + S(-mSC)}{C} + \frac{SD \cdot SD}{C^2} = (D^2 - mS^2) + T_m^2 = 1 - 2ms + T_m^2,$$

the first part of (13.1); clearing denominators,

$$T'_m = \frac{(1 - 2ms)(1 - s) + s(1 - ms)}{1 - s} = \frac{1 - 2ms + ms^2}{1 - s} > 0,$$

the numerator being positive for every $s \in [0, 1]$ because its discriminant in s is $4m(m - 1) < 0$ when $0 < m < 1$, while for $m \leq 0$ one has $ms(s - 2) \geq 0$. Squaring $T'_m = 1 - 2ms + T_m^2$ and subtracting the asserted quartic leaves $4m[ms^2 - s + (1 - s)T_m^2] = 0$, which holds identically by the closed form of T_m^2 ; this is the second part of (13.1). Since $m < 1$ we have $D \geq \sqrt{1 - m} > 0$ on $[0, K[m]]$, so $T_m \rightarrow +\infty$ as $u \uparrow K[m]$, where C has its first zero; with $T_m(0) = 0$ and $T'_m > 0$ this gives the asserted bijection. The value at $u = \frac{1}{2}K[m]$ is the half-argument evaluation [2, 124.01], valid at every modulus: with $k' = \sqrt{1 - m}$,

$$\operatorname{sn} \frac{K}{2} = (1 + k')^{-1/2}, \quad \operatorname{cn} \frac{K}{2} = \left(\frac{k'}{1 + k'} \right)^{1/2}, \quad \operatorname{dn} \frac{K}{2} = \sqrt{k'}, \quad (13.3)$$

whence $T_m(\frac{K}{2}) = \frac{(1 + k')^{-1/2}\sqrt{k'}}{(k'/(1 + k'))^{1/2}} = 1$ *independently of m* . Finally $2 - 4m > -2$, so $\tau^4 + (2 - 4m)\tau^2 + 1 > 0$ for real τ , and (13.1) may be written $du = dT_m[T_m^4 + (2 - 4m)T_m^2 + 1]^{-1/2}$; integrating over $u \in [0, K[m]]$ and over $u \in [0, \frac{1}{2}K[m]]$ yields the two formulas (13.2). (The second also follows from the invariance of the integrand under $\tau \mapsto 1/\tau$.)

Alternatively, and without elliptic functions: the substitution $\nu = \tau - \tau^{-1}$ gives $\tau^4 + (2 - 4m)\tau^2 + 1 = \tau^2(\nu^2 + 4 - 4m)$ and $d\tau/\tau = d\nu/\sqrt{\nu^2 + 4}$, whence the first integral of (13.2) equals $\int_{-\infty}^{\infty} d\nu[(\nu^2 + 4 - 4m)(\nu^2 + 4)]^{-1/2} = K[m]$, by $\int_0^{\infty} d\nu[(\nu^2 + a^2)(\nu^2 + b^2)]^{-1/2} = b^{-1}K[1 - a^2/b^2]$ for $0 < a \leq b$ (substitute $\nu = a \tan \vartheta$), together with Lemma 5.1 when $m < 0$; and $\tau = 1$ corresponds to $\nu = 0$, giving the second. $\square$

Corollary 13.2 (the constants, and the values of D). *At $m = \frac{1}{4}$, Lemma 13.1 reads*

$$\int_0^{\infty} \frac{d\tau}{\sqrt{1 + \tau^2 + \tau^4}} = K[\tfrac{1}{4}] = 2\varpi = 1.6857503548\dots, \quad \int_0^1 \frac{d\tau}{\sqrt{1 + \tau^2 + \tau^4}} = \varpi. \quad (13.4)$$

Consequently D of (4.16) is the inverse of $D \mapsto \frac{1}{2\varpi} \int_0^D (1 + \tau^2 + \tau^4)^{-1/2} d\tau$: it increases strictly from 0 to $+\infty$ as t runs over $[0, 1)$, and

$$D(0) = 0, D(\tfrac{1}{2}) = 1, D(1) = \infty, D(t + 1) = -\frac{1}{D(t)}, D(t + 2) = D(t), D(-t) = -D(t), \quad (13.5)$$

with $D'(0) = 2\varpi$ and $D'(\frac{1}{2}) = 2\sqrt{3}\varpi$. In particular D is an increasing bijection of $(-1, 1)$ onto $\mathbb{R}$, carrying $-\frac{1}{2}, 0, \frac{1}{2}$ to $-1, 0, 1$.

Proof. Both formulas (13.4) are (13.2) at $m = \frac{1}{4}$, where $2 - 4m = 1$, with $K[\frac{1}{4}] = 2\varpi$ by (1.7). The monotonicity, the pole and the value $D(\frac{1}{2}) = T_{1/4}(\frac{1}{2}K[\frac{1}{4}]) = 1$ are Lemma 13.1; the antiperiodicity is (4.21) at modulus $\frac{1}{4}$, whose quarter period is $K[\frac{1}{4}] = 2\varpi$, i.e., $t = 1$; and D is odd because sn is odd and $\operatorname{cn}, \operatorname{dn}$ even. The derivatives are (13.1): $D' = 2\varpi(1 + D^2 + D^4)^{1/2}$, which is 2ϖ at $D = 0$ and $2\sqrt{3}\varpi$ at $D = 1$. Finally D is increasing on $(-1, 1)$, from $-\infty$ to $+\infty$, by oddness. $\square$

Proposition 13.3 (the Möbius map, and the quadratic $3m^2 + 10m + 3$). *For all t off the poles,*

$$\boxed{\mathcal{T}(t) = \frac{1 + D(t - \frac{1}{2})}{1 - D(t - \frac{1}{2})}, \quad \text{equivalently} \quad D(t - \tfrac{1}{2}) = \frac{\mathcal{T}(t) - 1}{\mathcal{T}(t) + 1}} \quad (13.6)$$

Indeed, setting $G := \frac{1+D}{1-D}$,

$$G'^2 = \varpi^2(3G^4 + 10G^2 + 3) = K_1^2\left(1 + \frac{10}{3}G^2 + G^4\right), \quad (13.7)$$

so that G is T_m at the modulus determined by $2 - 4m = \frac{10}{3}$, i.e., $m = -\frac{1}{3}$, with argument scale K_1 ; and $G(t) = \mathcal{T}(t + \frac{1}{2})$. The two roots of

$$3m^2 + 10m + 3 = 0, \quad 3m^2 + 10m + 3 = 3(m+3)(m+\frac{1}{3}), \quad (13.8)$$

namely $\{-3, -\frac{1}{3}\}$, are the discriminant data of the map (13.6). This is the explanation of the appearance of (13.8) in §11 and in [P, Cor. 11.4].

Proof. (13.7). With $G = \frac{1+D}{1-D}$ one has $G' = \frac{2D'}{(1-D)^2}$ and, inverting, $D = \frac{G-1}{G+1}$, so $1-D = \frac{2}{G+1}$ and

$$G'^2 = \frac{4D'^2}{(1-D)^4} = \frac{4 \cdot 4\varpi^2(1+D^2+D^4)(G+1)^4}{16} = \varpi^2(1+D^2+D^4)(G+1)^4.$$

Now

$$(G+1)^4(1+D^2+D^4) = (G+1)^4 + (G^2-1)^2 + (G-1)^4,$$

since $(G+1)^4D^2 = (G+1)^2(G-1)^2$ and $(G+1)^4D^4 = (G-1)^4$; and $(G+1)^4 + (G-1)^4 = 2G^4 + 12G^2 + 2$ while $(G^2-1)^2 = G^4 - 2G^2 + 1$, whose sum is $3G^4 + 10G^2 + 3$. This is (13.7), the second form because $K_1^2 = 3\varpi^2$ (1.8).

(13.6). By Lemma 13.1 at $m = -\frac{1}{3}$, where $2 - 4m = \frac{10}{3}$, the function $\mathcal{T}(t) = T_{-1/3}(K_1 t)$ satisfies the same equation $y'^2 = K_1^2(1 + \frac{10}{3}y^2 + y^4)$. The right side never vanishes, so along any solution y' has constant sign; both G and $\mathcal{T}$ are increasing where defined (Corollary 13.2, Proposition 4.12). Finally the initial data agree:

$$G(0) = \frac{1+D(0)}{1-D(0)} = 1 = \mathcal{T}(\tfrac{1}{2}), \quad G'(0) = 2D'(0) = 4\varpi = \mathcal{T}'(\tfrac{1}{2}),$$

by (13.5) and Proposition 4.12. Uniqueness for the initial value problem gives $G(t) = \mathcal{T}(t + \frac{1}{2})$, i.e., (13.6). Two cross-checks: at $t = 0$, $D(-\frac{1}{2}) = -1$ gives $\mathcal{T}(0) = \frac{1-1}{1+1} = 0$ ✓; at $t = 1$, $D(\frac{1}{2}) = 1$ gives $\mathcal{T}(1) = \infty$ ✓, the pole of Proposition 4.12. $\square$

Remark 13.4 (why the quadratic is palindromic). Equation (13.8) is palindromic because the map $D \mapsto G = \frac{1+D}{1-D}$ and its inverse are conjugate under $D \mapsto -D$, $G \mapsto 1/G$. Concretely, the quartic $\tau^4 + \beta\tau^2 + 1$ of Lemma 13.1 is carried by (13.6) from $\beta = 1$ (i.e., $m = \frac{1}{4}$) to $\beta = \frac{10}{3}$ (i.e., $m = -\frac{1}{3}$), and the induced involution on β interchanges $\frac{10}{3}$ and 14 — that is, $\{m, 1/m\} = \{-\frac{1}{3}, -3\}$, the two negative members of the anharmonic orbit of Remark 5.6. Thus the two moduli exchanged by the Möbius map relating the *product* shape (16.12) to the *quadratic* shape (16.7) are precisely the two roots of (13.8); the appearance of the same quadratic in the jet computations of both papers (§11, [P, Cor. 11.4]) is therefore not a numerical accident but a shadow of this involution.

Remark 13.5 (agreement with Nitsche, and with the announcement). Nitsche's §83 gives, besides the product equation (38), its Möbius twin (39), of the shape $\sum_{i < j} \mathcal{F}_i \mathcal{F}_j + 1 = 0$; by Proposition 13.3 and (16.7) below, that twin is exactly the symmetric form of the present paper, with $\mathcal{F} = \mathcal{T}$ at modulus $-\frac{1}{3}$. So the pair (38)–(39) of [14, §83] is the pair (16.12)–(16.7) here, and (13.6) is the map between them, with the half-period shift made explicit and the moduli identified.

14. THE FUNCTION Ψ

Definition 14.1. For t with $D(t) > 0$ — that is, for $t \in (0, 1)$ modulo 2 (Corollary 13.2) — put

$$\Psi(t) := \log D(t) = \log \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(2\varpi t, \tfrac{1}{4}) \in \mathbb{R}, \quad (14.1)$$

and, for $(x, y, z) \in \mathbb{R}^3$, write

$$\hat{x} := x - \tfrac{1}{2}, \quad \hat{y} := y - \tfrac{1}{2}, \quad \hat{z} := z - \tfrac{1}{2} \quad (14.2)$$

for the coordinates translated so that D enters unshifted, as in (13.6).

Proposition 14.2 (the calculus of Ψ). *On $(0, 1)$, Ψ is real-analytic and*

$$\Psi'(t) = \frac{D'(t)}{D(t)}, \quad \Psi'(t)^2 = 4\varpi^2(1 + 2 \cosh 2\Psi(t)), \quad \Psi''(t) = 8\varpi^2 \sinh 2\Psi(t). \quad (14.3)$$

Moreover Ψ is a real-analytic increasing bijection of $(0, 1)$ onto $\mathbb{R}$, with

$$\Psi(0^+) = -\infty, \quad \Psi(\tfrac{1}{2}) = 0, \quad \Psi(1^-) = +\infty, \quad \Psi(1-t) = -\Psi(t), \quad (14.4)$$

and, where both sides are real,

$$\Psi(t+2) = \Psi(t). \quad (14.5)$$

In particular $\Psi' \geq 2\sqrt{3}\varpi > 0$ throughout $(0, 1)$: the derivative never vanishes.

Proof. Real-analyticity, monotonicity and the three values of (14.4) are Corollary 13.2 composed with $\log$; in particular $\Psi(\frac{1}{2}) = \log 1 = 0$. The reflection $\Psi(1-t) = -\Psi(t)$ is $D(1-t) = 1/D(t)$, which is Lemma 15.2 below; and (14.5) is $D(t+2) = D(t)$.

For the derivative: $\Psi' = D'/D$ and, by (13.1) at $m = \frac{1}{4}$,

$$\Psi'^2 = \frac{D'^2}{D^2} = \frac{4\varpi^2(1 + D^2 + D^4)}{D^2} = 4\varpi^2(D^{-2} + 1 + D^2) = 4\varpi^2(1 + 2 \cosh 2\Psi), \quad (14.6)$$

using $D^2 + D^{-2} = e^{2\Psi} + e^{-2\Psi} = 2 \cosh 2\Psi$. Since $\cosh \geq 1$, $\Psi'^2 \geq 12\varpi^2$, whence $\Psi' \geq 2\sqrt{3}\varpi > 0$ (Ψ increasing). Differentiating (14.6) gives $2\Psi'\Psi'' = 16\varpi^2 \sinh 2\Psi \cdot \Psi'$, i.e., $\Psi'' = 8\varpi^2 \sinh 2\Psi$, valid throughout since $\Psi' \neq 0$. $\square$

Remark 14.3 (three normalizations, and the comparison with P). Three normalizations of one function occur and should be kept apart. The *lattice* normalization is (14.1), of period 2 and with $\Psi(\frac{1}{2}) = 0$; the *ODE* normalization, in which (14.3) reads $\varphi'^2 = 1 + 2 \cosh 2\varphi$, is $\varphi(t) = \Psi(t/(2\varpi)) = \Psi(t/K[\frac{1}{4}])$; and the *Nitsche* normalization is $\varphi = \log \mathcal{E}$ with $\mathcal{E}$ the function of [14, §83], which by Corollary 13.2 and (13.4) is $\mathcal{E} = \frac{\operatorname{sn} \operatorname{dn}}{\operatorname{cn}}(\cdot, \tfrac{1}{4})$, i.e., $D(t) = \mathcal{E}(2\varpi t) = \mathcal{E}(p_0 t)$.

The comparison with [P, §14] is now exact. There

$$\Psi_P(t) = \arctan(\sqrt{3} C(t)), \quad C(t) = \operatorname{cn}(4\varpi_P t, \tfrac{3}{4}), \quad \Psi_P'^2 = 4\varpi_P^2(1 + 2 \cos 2\Psi_P), \quad (14.7)$$

against $\Psi_D = \log D$ and $\Psi_D'^2 = 4\varpi_D^2(1 + 2 \cosh 2\Psi_D)$ here. The two are the two signs of one differential equation — equivalently, of one separation constant (Theorem 17.7) — and the consequences are systematic:

	$\Psi_D = \log D$	$\Psi_P = \arctan(\sqrt{3} C)$
Ψ'^2 as a function of Ψ	$1 + 2 \cosh 2\Psi \geq 3$	$1 + 2 \cos 2\Psi \geq 0$
range of Ψ	$\mathbb{R}$	$[-\frac{\pi}{3}, \frac{\pi}{3}]$
Ψ'	never zero	vanishes where $\Psi = \pm \frac{\pi}{3}$
addition theorem used	exp	tan
clean equation	$D_1 D_2 D_3 = 1$	$\sum C_i = 3 \prod C_i$
regularity of the level set	<i>unconditional</i>	fails on one lattice orbit
obstruction to a global clean form	sign of D ; no real fcc invariance	one spurious lattice orbit

Every awkwardness of D — the absolute values of Remark 6.4, the positivity condition of Theorem 16.3, the projective degenerations of Remark 6.2 — and every convenience of D — the unconditional regularity of Theorem 17.2(a), the straight boundary lines of Proposition 6.13, the unrestricted graph of Corollary 16.8 — is this one difference of sign.

15. THE TWO DICTIONARIES

Proposition 15.1 (the horizontal dictionary). *For all t at which the quantities are defined and $D(\hat{t}) > 0$, where $\hat{t} = t - \frac{1}{2}$,*

$$\mathcal{T}(t) = \frac{1 + D(\hat{t})}{1 - D(\hat{t})} = -\coth \frac{\Psi(\hat{t})}{2}, \quad (15.1)$$

and consequently

$$D(\hat{t}) = \frac{\mathcal{T}(t) - 1}{\mathcal{T}(t) + 1}, \quad \Psi(\hat{t}) = \log \frac{\mathcal{T}(t) - 1}{\mathcal{T}(t) + 1} = -2 \operatorname{arccoth} \mathcal{T}(t). \quad (15.2)$$

Proof. The first equality of (15.1) is Proposition 13.3. For the second, put $\psi = \Psi(\hat{t})$, so $D(\hat{t}) = e^\psi$ and

$$\frac{1 + e^\psi}{1 - e^\psi} = \frac{e^{-\psi/2} + e^{\psi/2}}{e^{-\psi/2} - e^{\psi/2}} = \frac{2 \cosh(\psi/2)}{-2 \sinh(\psi/2)} = -\coth \frac{\psi}{2}.$$

Then (15.2) is (15.1) inverted. Consistency at $t = \frac{1}{2}$: $\hat{t} = 0$, $\Psi(0^+) = -\infty$, $\coth(-\infty) = -1$, so $\mathcal{T}(\frac{1}{2}) = 1$, in agreement with Proposition 4.12. $\square$

Lemma 15.2 (the vertical dictionary). *For all t ,*

$$D(1 - t) = \frac{1}{D(t)}, \quad \text{equivalently} \quad \Psi(1 - t) = -\Psi(t) \quad \text{on } (0, 1), \quad (15.3)$$

and, in the form used in §16,

$$D(t - \tfrac{1}{2}) = -\frac{1}{D(t + \tfrac{1}{2})}. \quad (15.4)$$

Moreover $\mathcal{T}(1 - z) = 1/\mathcal{T}(z)$.

Proof. $D(1 - t) = D((-t) + 1) = -1/D(-t) = -1/(-D(t)) = 1/D(t)$, by (13.5); taking log on $(0, 1)$ gives the second form. Formula (15.4) is $D(u + 1) = -1/D(u)$ with $u = t - \frac{1}{2}$. The statement for $\mathcal{T}$ is (6.4). $\square$

Remark 15.3 (compare P). In [P, §15] the two dictionaries are $\arctan(\sqrt{3} \operatorname{sn}^2(\varpi_P t, -3)) = \frac{1}{2}(\frac{\pi}{3} - \Psi_P(t))$ (horizontal, via the duplication formula for cn) and $\arctan(\frac{1}{\sqrt{3}} \operatorname{sn}(3\varpi_P Z, \frac{1}{9})) = -\frac{1}{2}\Psi_P(z)$ (vertical, via a Landen step). Here the horizontal one is the Möbius map (13.6) — a genuine change of variable, but algebraic and of degree 1 — and the vertical one degenerates to the reflection (15.3). The reason is the half-period shift of Remark 1.5(ii): the third argument of $(\star)$ is already $K_1 z$ rather than $K_1 Z$, so no isogeny is needed to center it. The price is the

shift $t \mapsto t - \frac{1}{2}$ in (13.6), which is what makes the clean equation (16.12) hold in the translated coordinates (14.2) rather than in (x, y, z) — Nitsche’s “up to a lattice translation”.

16. THE MASTER IDENTITY, AND THE PRODUCT EQUATION

Definition 16.1. For $(x, y, z) \in \mathbb{R}^3$ put, with $p = x + y - 1$, $q = x - y$ as in (1.17) and $\hat{x}, \hat{y}, \hat{z}$ as in (14.2),

$$F(x, y, z) := \text{cd}(K_1 p, -\tfrac{1}{3}) \text{dc}(K_1 q, -\tfrac{1}{3}) \Upsilon(K_1 z) + 1, \quad (16.1)$$

$$\mathfrak{d}_1 := D(\hat{x}), \quad \mathfrak{d}_2 := D(\hat{y}), \quad \mathfrak{d}_3 := D(\hat{z}), \quad (16.2)$$

$$G(x, y, z) := \Psi(\hat{x}) + \Psi(\hat{y}) + \Psi(\hat{z}), \quad (16.3)$$

$$Q(x, y, z) := \mathfrak{d}_1 \mathfrak{d}_2 \mathfrak{d}_3 - 1, \quad (16.4)$$

so that $(\star)$ reads $F = 0$.

Theorem 16.2 (the master identity). *Wherever $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3$ are finite and $\neq 1$,*

$$F(x, y, z) = \frac{-2Q(x, y, z)}{(1 + \mathfrak{d}_1 \mathfrak{d}_2)(1 - \mathfrak{d}_3)} \quad (16.5)$$

Consequently $(\star)$ holds if and only if $\mathfrak{d}_1 \mathfrak{d}_2 \mathfrak{d}_3 = 1$, at every point where the denominator of (16.5) is finite and nonzero. Moreover, where the $\mathfrak{d}_i$ are positive,

$$Q = \exp(G) - 1, \quad \text{so} \quad Q = 0 \iff G = 0. \quad (16.6)$$

Proof. Step 1: the quadratic form of the equation. By Corollary 8.4, $(\star)$ is equivalent to the symmetric form (1.20),

$$\mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3^\flat + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3^\flat = 0, \quad \mathcal{T}_1 := \mathcal{T}(x), \quad \mathcal{T}_2 := \mathcal{T}(y), \quad \mathcal{T}_3^\flat := \mathcal{T}(1 - z),$$

and by Lemma 15.2, $\mathcal{T}_3^\flat = 1/\mathcal{T}_3$ with $\mathcal{T}_3 := \mathcal{T}(z)$. Multiplying by $\mathcal{T}_3$ gives the *quadratic twin*

$$\sum_{i < j} \mathcal{T}_i \mathcal{T}_j + 1 = 0, \quad (16.7)$$

i.e., $e_2(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = -1$: this is Nitsche’s (39) [14, §83] and the equation (3.2) of the announcement.

Step 2: the substitution. By Proposition 13.3, $\mathcal{T}_i = \frac{1+\mathfrak{d}_i}{1-\mathfrak{d}_i}$ for $i = 1, 2, 3$. Multiply (16.7) by $\prod_i (1 - \mathfrak{d}_i)$ and write E_1, E_2, E_3 for the elementary symmetric functions of $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3$. Expanding, for each pair $\{i, j\}$ with complementary index k ,

$$(1 + \mathfrak{d}_i)(1 + \mathfrak{d}_j)(1 - \mathfrak{d}_k) = 1 + (\mathfrak{d}_i + \mathfrak{d}_j - \mathfrak{d}_k) + (\mathfrak{d}_i \mathfrak{d}_j - \mathfrak{d}_i \mathfrak{d}_k - \mathfrak{d}_j \mathfrak{d}_k) - E_3,$$

and summing over the three pairs: the constants give 3; each $\mathfrak{d}_a$ occurs twice with + and once with −, giving E_1 ; each product $\mathfrak{d}_a \mathfrak{d}_b$ occurs once with + and twice with −, giving $-E_2$; and the cubic terms give $-3E_3$. Hence

$$\sum_{i < j} (1 + \mathfrak{d}_i)(1 + \mathfrak{d}_j)(1 - \mathfrak{d}_k) = 3 + E_1 - E_2 - 3E_3, \quad \prod_i (1 - \mathfrak{d}_i) = 1 - E_1 + E_2 - E_3, \quad (16.8)$$

whose sum is $4 - 4E_3$. Therefore

$$\sum_{i < j} \mathcal{T}_i \mathcal{T}_j + 1 = \frac{4(1 - E_3)}{\prod_i (1 - \mathfrak{d}_i)} = \frac{-4Q}{\prod_i (1 - \mathfrak{d}_i)}, \quad (16.9)$$

and (16.7) holds if and only if $E_3 = 1$.

Step 3: the factor. From the proof of Corollary 8.4, $\Upsilon(K_1 z) = \mathcal{T}_3$ and $\text{cd}(K_1 p) \text{dc}(K_1 q) = \frac{\mathcal{T}_1 + \mathcal{T}_2}{1 + \mathcal{T}_1 \mathcal{T}_2} \cdot (-1)^0$, so that by (16.1)

$$F = 1 + \mathcal{T}_3 \frac{\mathcal{T}_1 + \mathcal{T}_2}{1 + \mathcal{T}_1 \mathcal{T}_2} = \frac{1 + \mathcal{T}_1 \mathcal{T}_2 + \mathcal{T}_1 \mathcal{T}_3 + \mathcal{T}_2 \mathcal{T}_3}{1 + \mathcal{T}_1 \mathcal{T}_2} = \frac{\sum_{i < j} \mathcal{T}_i \mathcal{T}_j + 1}{1 + \mathcal{T}_1 \mathcal{T}_2}.$$

Now

$$1 + \mathcal{T}_1 \mathcal{T}_2 = \frac{(1 - \mathfrak{d}_1)(1 - \mathfrak{d}_2) + (1 + \mathfrak{d}_1)(1 + \mathfrak{d}_2)}{(1 - \mathfrak{d}_1)(1 - \mathfrak{d}_2)} = \frac{2(1 + \mathfrak{d}_1 \mathfrak{d}_2)}{(1 - \mathfrak{d}_1)(1 - \mathfrak{d}_2)},$$

so, combining with (16.9),

$$F = \frac{-4Q}{\prod_i (1 - \mathfrak{d}_i)} \cdot \frac{(1 - \mathfrak{d}_1)(1 - \mathfrak{d}_2)}{2(1 + \mathfrak{d}_1 \mathfrak{d}_2)} = \frac{-2Q}{(1 + \mathfrak{d}_1 \mathfrak{d}_2)(1 - \mathfrak{d}_3)},$$

which is (16.5).

Step 4. Finally (16.6) is immediate from $\Psi(\hat{x}_i) = \log \mathfrak{d}_i$, valid where the $\mathfrak{d}_i$ are positive.

A numerical check. At $(\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3) = (2, 3, \frac{1}{7})$ one has $\mathcal{T} = (-3, -2, \frac{4}{3})$ and $\sum_{i < j} \mathcal{T}_i \mathcal{T}_j + 1 = 6 - 4 - \frac{8}{3} + 1 = \frac{1}{3}$, while $1 + \mathcal{T}_1 \mathcal{T}_2 = 7$, so $F = \frac{1}{21}$; and (16.5) gives $\frac{-2(\frac{6}{7} - 1)}{(1 + 6)(1 - \frac{1}{7})} = \frac{2/7}{6} = \frac{1}{21}$.
 $\checkmark$. At $(2, 2, \frac{1}{4})$, where $E_3 = 1$, both sides vanish $\checkmark$. $\square$

Theorem 16.3 (the product equation). *Let $S := \{G = 0\}$, the additively separable surface $\Psi(\hat{x}) + \Psi(\hat{y}) + \Psi(\hat{z}) = 0$, and let*

$$\mathcal{O} := \{(x, y, z) \in \mathbb{R}^3 : \hat{x}, \hat{y}, \hat{z} \in (0, 1) \text{ modulo } 2\} = \{x, y, z \in (\frac{1}{2}, \frac{3}{2}) \text{ modulo } 2\}, \quad (16.10)$$

on which all three $\mathfrak{d}_i$ are positive. Then

$$\boxed{\{(\star)\} \cap \mathcal{O} = S \cap \mathcal{O} = \left\{ D(x - \frac{1}{2}) D(y - \frac{1}{2}) D(z - \frac{1}{2}) = 1 \right\} \cap \mathcal{O}}, \quad (16.11)$$

which is Nitsche's equation (38) of [14, §83] in the translated coordinates (14.2). Equivalently, and in a form regular at the base point, by (15.4)

$$D(x - \frac{1}{2}) D(y - \frac{1}{2}) + D(z + \frac{1}{2}) = 0. \quad (16.12)$$

There is no exceptional set: by (16.6) the zero set of Q on $\mathcal{O}$ is exactly S , the exponential having no turning points. Outside $\mathcal{O}$ the equation $\prod_i \mathfrak{d}_i = 1$ still describes the surface, but the factorization (16.5) must be read projectively where some $\mathfrak{d}_i \in \{0, \infty, 1\}$, and the sign of each $\mathfrak{d}_i$ is determined by the residue of $\hat{x}_i$ modulo 2.

Proof. The first equality of (16.11) is Theorem 16.2: on $\mathcal{O}$ we have $\mathfrak{d}_i > 0$, so $1 + \mathfrak{d}_1 \mathfrak{d}_2 > 0$, and $1 - \mathfrak{d}_3 \neq 0$ off the hypersurface $\{\hat{z} \in \frac{1}{2} + 2\mathbb{Z}\}$, on which (16.6) is used instead. The second is (16.6). The description of $\mathcal{O}$ is Corollary 13.2, and (16.12) follows from (15.4): $\mathfrak{d}_3 = -1/D(z + \frac{1}{2})$, so $\prod_i \mathfrak{d}_i = 1$ reads $\mathfrak{d}_1 \mathfrak{d}_2 = 1/\mathfrak{d}_3 = -D(z + \frac{1}{2})$. $\square$

Corollary 16.4 (the base point, as a cross-check). *At the base point $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ one has $\hat{x} = \hat{y} = 0$ and $\hat{z} = -1$, so $\mathfrak{d}_1 = \mathfrak{d}_2 = 0$ and $\mathfrak{d}_3 = \infty$: the product form (16.11) is projectively degenerate there, while (16.12) is regular. Expanding (16.12) to second order, with $P = x - \frac{1}{2}$, $Q = y - \frac{1}{2}$, $Z = z + \frac{1}{2}$ as in Proposition 6.5,*

$$D(P) D(Q) + D(Z) = (2\varpi P)(2\varpi Q) + 2\varpi Z + O(4) = 0,$$

i.e., $Z = -2\varpi PQ + \dots$ — which is exactly (6.7). Thus the product equation reproduces, at second order at the base point, the expansion that pins $\kappa = \varpi^{-1}$; this is an independent check on Proposition 13.3, on the shift in (14.2), and on the normalization of §5.

Remark 16.5 (no real fcc invariance for the product form). Neither (16.11) nor (16.7) is invariant under the face-centered cubic lattice over $\mathbb{R}$. Under $(x, y, z) \mapsto (x + 1, y + 1, z)$ we have $\mathfrak{d}_1 \mapsto -1/\mathfrak{d}_1$, $\mathfrak{d}_2 \mapsto -1/\mathfrak{d}_2$ and $\mathfrak{d}_3$ fixed, so $\prod_i \mathfrak{d}_i \mapsto \mathfrak{d}_3/(\mathfrak{d}_1 \mathfrak{d}_2) = \mathfrak{d}_3^2$ on the surface — not 1. Equivalently, in the $\mathcal{T}$ variables the substitution sends $e_2 = -1$ to $\mathcal{T}_1 \mathcal{T}_2 + 1 = \mathcal{T}_3(\mathcal{T}_1 + \mathcal{T}_2)$, which is (1.22) with $\mathcal{T}$ in place of Φ : the two differ by the sign of the terms containing $\mathcal{T}_3$, i.e., by the complex shift $z \mapsto z + i\tau$ with $\tau = K[\frac{3}{4}]/K[\frac{1}{4}]$ (Remark 6.4). *The fcc-invariant form is (1.22), with $\Phi = |\mathcal{T}|$, and no other.* This is the price D pays for the sign structure cataloged in Table 12.2; in [P] the corresponding relation is sign-honest.

Remark 16.6 (the exceptional set, compared). Theorem 16.3 and [P, Thm. 16.4] are precise mirror images. There the clean equation $\sum C_i = 3 \prod C_i$ holds on all of $\mathbb{R}^3$ and is fcc-analogously invariant, but its zero set carries one spurious lattice orbit $\mathcal{L}'$, the labyrinth centers, because $\sin$ vanishes at $\pm\pi$ as well as at 0 and $|\Psi_P| \leq \frac{\pi}{3}$ permits $\sum \Psi_P = \pm\pi$. Here $\exp$ has no such turning point, so there is no spurious orbit — but $\Psi = \log D$ requires $D > 0$, so the clean equation is valid only on $\mathcal{O}$, and outside it one must track signs (Remark 16.5). *Neither surface is unconditionally clean:* P pays with an exceptional set, D with a domain restriction and the absolute values. In both cases the underlying relation $(\star)$ is exact and unrestricted (Corollary 8.4, [P, Cor. 16.3]).

Corollary 16.7 (the four straight boundary segments). *The four segments (6.14), i.e., the four edges*

$$(0, 1, 0) \rightarrow (1, 1, -1) \rightarrow (1, 0, 0) \rightarrow (0, 0, -1) \rightarrow (0, 1, 0)$$

of the tetrahedron of Proposition 6.10, lie in $\{F = 0\}$. They lie on the boundary of $\mathcal{O}$: on the first, parametrized as $(s, 1, -s)$, one has $\hat{y} = \frac{1}{2}$, so $\mathfrak{d}_2 = 1$ and the factor $1 - \mathfrak{d}_3$ of (16.5) must be examined — indeed $\mathcal{T}(y) = \mathcal{T}(1) = \infty$ there, the projective degeneration $\mathcal{AC} + 1 = 0$ of Remark 6.2. The finite statement is $(\star)$ itself, which on that segment reads $\text{cd}(K_1 s) \text{ns}(K_1 s) \Upsilon(-K_1 s) = -1$, an identity (Proposition 10.2). The other three follow by Proposition 6.12.

Corollary 16.8 (the graph description). *$S \cap \mathcal{O}$ is the graph*

$$\hat{x} = \Psi^{-1}(-\Psi(\hat{y}) - \Psi(\hat{z})), \quad (16.13)$$

defined for all $(\hat{y}, \hat{z}) \in (0, 1)^2$, where $\Psi^{-1} : \mathbb{R} \rightarrow (0, 1)$ is the real-analytic inverse of Proposition 14.2; equivalently $\mathfrak{d}_1 = 1/(\mathfrak{d}_2 \mathfrak{d}_3)$. There is no restriction on $(\hat{y}, \hat{z})$, in contrast to [P, Cor. 16.7], where the graph is defined only where $|\Psi_P(y) + \Psi_P(z)| \leq \frac{\pi}{3}$ because Ψ_P is bounded. This unboundedness is the third face of the cosh/cos dichotomy (Remark 14.3). The remaining sheets of $\{F = 0\}$ are the images of (16.13) under the group of Proposition 6.12 and the lattice of Corollary 6.3.

17. MINIMALITY, REGULARITY, AND THE LOCAL CONVERSE

Lemma 17.1 (the separable minimality condition). *Let $\varphi_1, \varphi_2, \varphi_3$ be C^2 on intervals and $F_0(x, y, z) = \varphi_1(x) + \varphi_2(y) + \varphi_3(z)$. At a point of $\{F_0 = 0\}$ with $\nabla F_0 \neq 0$, the mean curvature of $\{F_0 = 0\}$ vanishes if and only if*

$$\sum_{i=1}^3 \varphi_i'' \left(\sum_{j \neq i} \varphi_j'^2 \right) = 0. \quad (17.1)$$

Proof. For a regular level set the mean curvature is proportional to $|\nabla F_0|^2 \Delta F_0 - \nabla F_0^\top (D^2 F_0) \nabla F_0$. For separable F_0 , $D^2 F_0 = \text{diag}(\varphi_1'', \varphi_2'', \varphi_3'')$ and $\nabla F_0 = (\varphi_1', \varphi_2', \varphi_3')$, so

$$|\nabla F_0|^2 \Delta F_0 - \nabla F_0^\top (D^2 F_0) \nabla F_0 = \left(\sum_j \varphi_j'^2 \right) \left(\sum_i \varphi_i'' \right) - \sum_i \varphi_i'' \varphi_i'^2 = \sum_i \varphi_i'' \sum_{j \neq i} \varphi_j'^2. \quad \square$$

Lemma 17.1 is the reduction of Weingarten [18] and Nitsche [14, §82]; (17.1) is their equation $((g')^2 + (h')^2)f'' + \dots = 0$, and it is [P, Lem. 17.1] verbatim.

Theorem 17.2 (S is a regular minimal surface). *Let $\hat{x}, \hat{y}, \hat{z}$ be as in (14.2) and put*

$$F_0(x, y, z) := \Psi(\hat{x}) + \Psi(\hat{y}) + \Psi(\hat{z}) = G(x, y, z), \quad (17.2)$$

so that $S \cap \mathcal{O} = \{F_0 = 0\}$, with $\mathcal{O}$ as in (16.10). Then:

- (a) $\nabla F_0 = (\Psi'(\hat{x}), \Psi'(\hat{y}), \Psi'(\hat{z}))$, and $\Psi' \geq 2\sqrt{3}\varpi > 0$ throughout $(0, 1)$: *the gradient never vanishes, with no exceptional set and no case analysis. Hence $S \cap \mathcal{O}$ is a closed, embedded, real-analytic surface in $\mathcal{O}$;*
- (b) *S is minimal;*
- (c) $\{F = 0\}$, with F as in (16.1), *is a real-analytic minimal surface agreeing with S on $\mathcal{O}$, and $\nabla F \neq 0$ on it. Explicitly, at points where $F = 0$,*

$$\nabla F = K_1(\Xi_p - \Xi_n, \Xi_p + \Xi_n, -\tilde{\Theta}_q), \quad \Xi_p := \Xi(w_p), \quad \Xi_n := \Xi(w_n), \quad \tilde{\Theta}_q := \tilde{\Theta}(w_q), \quad (17.3)$$

with $\Xi, \tilde{\Theta}$ as in (9.4), and $\tilde{\Theta}$ never vanishes.

Proof. (a) Since $\hat{x} = x - \frac{1}{2}$, etc., are translations, $\partial\hat{x}/\partial x = 1$ and the chain rule gives the stated gradient. The bound is Proposition 14.2: $\Psi'^2 = 4\varpi^2(1 + 2\cosh 2\Psi) \geq 12\varpi^2$, since $\cosh \geq 1$. The remaining assertions are the regular value theorem, Ψ being real-analytic on $(0, 1)$ and $\mathcal{O}$ open.

(b) Apply Lemma 17.1 with $\varphi_1(x) = \Psi(\hat{x})$, $\varphi_2(y) = \Psi(\hat{y})$, $\varphi_3(z) = \Psi(\hat{z})$ — *the same function in all three slots*, so the surface is isotropic in the sense of Theorem 17.7 — and put

$$\alpha_1 := 2\Psi(\hat{x}), \quad \alpha_2 := 2\Psi(\hat{y}), \quad \alpha_3 := 2\Psi(\hat{z}), \quad \text{so that on } S: \quad \alpha_1 + \alpha_2 + \alpha_3 = 0. \quad (17.4)$$

By (14.3), in each slot

$$\varphi_i'^2 = 4\varpi^2(1 + 2\cosh \alpha_i), \quad \varphi_i'' = 8\varpi^2 \sinh \alpha_i,$$

with *no sign to absorb*. Hence, dividing (17.1) by the nonzero constant $8\varpi^2 \cdot 4\varpi^2$ and then by 2, minimality on S is equivalent to the vanishing of

$$\mathcal{E}_{\min} := \sum_i \sinh \alpha_i (1 + \cosh \alpha_j + \cosh \alpha_k), \quad \{i, j, k\} = \{1, 2, 3\}.$$

Now

$$\mathcal{E}_{\min} = \sum_i \sinh \alpha_i + \sum_i \sinh \alpha_i (\cosh \alpha_j + \cosh \alpha_k) = \sum_i \sinh \alpha_i + \sum_{i < j} (\sinh \alpha_i \cosh \alpha_j + \sinh \alpha_j \cosh \alpha_i),$$

each unordered pair $\{i, j\}$ occurring exactly once in the regrouping; and

$$\sinh \alpha_i \cosh \alpha_j + \sinh \alpha_j \cosh \alpha_i = \sinh(\alpha_i + \alpha_j) = \sinh(-\alpha_k) = -\sinh \alpha_k,$$

using $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Therefore

$$\mathcal{E}_{\min} = \sum_i \sinh \alpha_i - \sum_k \sinh \alpha_k = 0. \quad (17.5)$$

(c) *Minimality.* By Theorem 16.2, $F = \lambda Q$ with $\lambda := -2[(1 + \mathfrak{d}_1 \mathfrak{d}_2)(1 - \mathfrak{d}_3)]^{-1}$, real-analytic and nonzero on $\mathcal{O}$ off the hypersurface $\{\hat{z} \in \frac{1}{2} + 2\mathbb{Z}\}$; and $Q = e^G - 1$ there by (16.6), so $\{F = 0\}$ and $S = \{G = 0\}$ agree on $\mathcal{O}$ and $\{F = 0\}$ is minimal there by (b). Since the mean curvature of a regular real-analytic level set is real-analytic, and vanishes on the nonempty open subset $\mathcal{O}$ of the connected real-analytic set $\{F = 0\}$, it vanishes identically.

The gradient, first proof. On $\{Q = 0\} \cap \mathcal{O}$, $\nabla F = \lambda \nabla Q + Q \nabla \lambda = \lambda \nabla Q$, and by (16.4) together with $\prod_i \mathfrak{d}_i = 1$,

$$\nabla Q = \left(\prod_i \mathfrak{d}_i \right) (\mathbf{L}(\hat{x}), \mathbf{L}(\hat{y}), \mathbf{L}(\hat{z})) = (\mathbf{L}(\hat{x}), \mathbf{L}(\hat{y}), \mathbf{L}(\hat{z})), \quad \mathbf{L} := \frac{D'}{D}, \quad (17.6)$$

which on $\mathcal{O}$ is ∇F_0 and is nonzero by (a).

The gradient, second proof, and (17.3). Write $F = 1 - R$ with $R = -\text{cd}(w_p) \text{dc}(w_n) \Upsilon(w_q)$ as in Theorem 9.7; on $\{F = 0\}$ we have $R = 1$, so $\nabla F = -\nabla R = -\nabla \log R$. By (7.1), $w_p = K_1(x + y - 1)$, $w_n = K_1(x - y)$, $w_q = K_1 z$, whence

$$\frac{\partial(w_p, w_n, w_q)}{\partial(x, y, z)} = K_1 \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and inserting this into (9.11) gives (17.3). Its third component is nonzero: by (9.4) at $m = -\frac{1}{3}$,

$$\tilde{\Theta} = \frac{\text{dn}^2 + \frac{1}{3} \text{sn}^2 \text{cn}^2}{\text{sn cn dn}},$$

whose numerator is $\geq \text{dn}^2 \geq 1 > 0$ for real argument, so $\tilde{\Theta}$ has poles but no zeros ((9.6); Appendix A, Cell A5, test 4). $\square$

Remark 17.3 (the three lines, and the comparison with P). Computation (17.5) is the whole of the minimality proof, and it is the exact mirror of [P, Thm. 17.2(b)]: there $\alpha_i = 2\Psi_P$ with $\sin, \cos$ in place of $\sinh, \cosh$, and the same regrouping applies because $\sin(\alpha_i + \alpha_j) = -\sin \alpha_k$ as well. Here the proof is marginally cleaner, since all three slots carry $+\Psi$ and no sign has to be absorbed; in [P] the surface is $\Psi(x) - \Psi(y) - \Psi(z) = 0$ before the reflection $\Psi_P(1 - t) = -\Psi_P(t)$ is applied. That the same three lines serve for $\cos$ and for $\cosh$ is Theorem 17.7 in miniature.

Corollary 17.4 (the local converse). *Let $\zeta_0 \in \Omega^\circ$. There is an open neighborhood $U \subset \mathbb{R}^3$ of $X(\zeta_0)$ with*

$$\{(x, y, z) \in U : (1.18) \text{ holds}\} = X(\Omega^\circ) \cap U. \quad (17.7)$$

That is, (1.18) cuts out the surface, and does not merely contain it.

Proof. By Theorem 17.2(c), $\{F = 0\}$ is a two-dimensional embedded real-analytic submanifold of $\mathbb{R}^3$. By Theorem 18.4, $X(\Omega^\circ) \subset \{F = 0\}$; and by Proposition 2.3 the map X is an immersion of the two-manifold Ω° . An immersion between manifolds of equal dimension is a local diffeomorphism onto an open subset, so $X(\Omega^\circ)$ is open in $\{F = 0\}$; hence there is an open $U \subset \mathbb{R}^3$ with $\{F = 0\} \cap U = X(\Omega^\circ) \cap U$. $\square$

Corollary 17.5 (the separated Pfaffian). *Let $\mathbf{L} := D'/D$ as in (17.6), so that by (13.1) at $m = \frac{1}{4}$*

$$\mathbf{L}(t) = \frac{2\varpi \sqrt{1 + D(t)^2 + D(t)^4}}{D(t)}, \quad (17.8)$$

real-analytic and odd on $\mathbb{R} \setminus \mathbb{Z}$, and equal to Ψ' on $(0, 1)$. Then for every $\zeta = u + iv \in \Omega^\circ$ with $X(\zeta)$ off the coordinate hyperplanes $\{\hat{x}, \hat{y}, \hat{z} \in \mathbb{Z}\}$,

$$\boxed{\mathbf{L}(\hat{x}) : \mathbf{L}(\hat{y}) : \mathbf{L}(\hat{z}) = 2u : 2v : (1 - |\zeta|^2)} \quad (17.9)$$

Equivalently, the one-form identity

$$\mathbf{L}(\hat{x}) dx + \mathbf{L}(\hat{y}) dy + \mathbf{L}(\hat{z}) dz = 0 \quad (17.10)$$

holds on the patch — it is $dQ = 0$ — and is equivalent to the single complex equation

$$\mathbf{L}(\hat{x})(1 - \zeta^2) + i \mathbf{L}(\hat{y})(1 + \zeta^2) - 2 \mathbf{L}(\hat{z}) \zeta = 0. \quad (17.11)$$

On the subregion $\mathcal{O}$ one may write Ψ' for $\mathbf{L}$ throughout.

Proof. (17.8). By (13.1), $D' = 2\varpi\sqrt{1 + D^2 + D^4}$; and D is odd (13.5), so D' is even and $\mathbf{L} = D'/D$ is odd. Its poles are the zeros and poles of D , i.e., $t \in \mathbb{Z}$.

(17.9). By (17.6), at a point of $\{Q = 0\}$ the vector $(\mathbf{L}(\hat{x}), \mathbf{L}(\hat{y}), \mathbf{L}(\hat{z}))$ is a nonzero normal — nonzero because $|\mathbf{L}| \geq 2\sqrt{3}\varpi$ where $D > 0$ and, by oddness and the antiperiodicity $D(t+1) = -1/D(t)$, the same bound holds wherever $\mathbf{L}$ is finite. By Proposition 2.3 the unit normal N of (2.7) is another; and $X(\zeta) \in \{Q = 0\}$ by Theorem 18.4 together with Theorem 16.3. Two nonzero normals to a smooth surface at one point are proportional.

(17.10). This is $dQ = 0$ along $\{Q = 0\}$.

(17.11). By (1.15), (1.16) and (2.4),

$$dx = \kappa \operatorname{Im} \frac{(1 - \zeta^2) d\zeta}{W}, \quad dy = \kappa \operatorname{Im} \frac{i(1 + \zeta^2) d\zeta}{W}, \quad dz = \kappa \operatorname{Re} \frac{2i\zeta d\zeta}{W} = -\kappa \operatorname{Im} \frac{2\zeta d\zeta}{W}, \quad (17.12)$$

so (17.10) reads

$$\operatorname{Im} \left\{ \left[\mathbf{L}(\hat{x})(1 - \zeta^2) + i\mathbf{L}(\hat{y})(1 + \zeta^2) - 2\mathbf{L}(\hat{z})\zeta \right] \frac{d\zeta}{W} \right\} = 0.$$

As $d\zeta$ ranges over two independent real directions, the bracket must vanish.

Consistency of (17.9) with (17.11). Substituting $\mathbf{L}(\hat{x}) = 2\mu u$, $\mathbf{L}(\hat{y}) = 2\mu v$, $\mathbf{L}(\hat{z}) = \mu(1 - |\zeta|^2)$ and using $u = \frac{1}{2}(\zeta + \bar{\zeta})$, $iv = \frac{1}{2}(\zeta - \bar{\zeta})$,

$$2u(1 - \zeta^2) + 2iv(1 + \zeta^2) = 2(u + iv) + \zeta^2(-2u + 2iv) = 2\zeta - 2\zeta^2\bar{\zeta} = 2\zeta(1 - |\zeta|^2),$$

so the bracket is $\mu[2\zeta(1 - |\zeta|^2) - 2\zeta(1 - |\zeta|^2)] = 0 \checkmark$. *This computation is also the derivation of (2.7):* it shows that $(2u, 2v, 1 - |\zeta|^2)$ is the unique direction, up to scale, annihilating the Weierstrass triple (17.12) of D . $\square$

Remark 17.6 (the sign of the third component, and the contrast with P). The third component of (2.7) is $1 - |\zeta|^2$ for D and $|\zeta|^2 - 1$ for P [P, (2.7)]. The reason is (17.12): the height of D is $\operatorname{Re} \mathbf{h} = -\operatorname{Im} \mathbf{h}$, so the third slot of the Weierstrass triple carries a minus sign relative to $[P]$, and the annihilating direction changes sign in that slot accordingly. Three independent confirmations:

- (i) *The base point.* By Proposition 6.5 the level set has $\nabla \propto (0, 0, 1)$ at $\zeta = 0$, and its second-order expansion gives $\nabla \propto (2\varpi\kappa u, 2\varpi\kappa v, 1) = (2u, 2v, 1)$ by $\varpi\kappa = 1$; this matches $1 - |\zeta|^2 \rightarrow +1$ and excludes $|\zeta|^2 - 1 \rightarrow -1$.
- (ii) *Corollary 16.4.* The product form gives $Z = -2\varpi PQ + \dots$, a saddle opening downwards in the PQ orientation, consistent with an upward normal at the base point.
- (iii) *Numerically.* At four interior points of Ω the cross product of $(\mathbf{L}(\hat{x}), \mathbf{L}(\hat{y}), \mathbf{L}(\hat{z}))$ with $(2u, 2v, 1 - |\zeta|^2)$ vanishes, while with $(2u, 2v, |\zeta|^2 - 1)$ it does not (Appendix A, Cell A7, test 3b).

Consequently the flat-point normals of Proposition 3.2(d) are the four directions $\frac{1}{\sqrt{3}}(\pm 1, \pm 1, +1)$, not $(\pm 1, \pm 1, -1)$: at ζ_0 , where $|\zeta_0|^2 = 2 - \sqrt{3}$ and $u = v = \frac{\sqrt{3}-1}{2}$, $N \propto (\sqrt{3}-1, \sqrt{3}-1, \sqrt{3}-1) \propto (1, 1, 1)$. The $\langle 111 \rangle$ directions are of course the same for both surfaces; only the outward sense differs, and only because the two papers place their patches on opposite sides of the plane $z = \pm \frac{1}{2}$.

17.1. The dichotomy.

Theorem 17.7 (D and P are the only two). *Call a separable minimal surface $\varphi_1(x) + \varphi_2(y) + \varphi_3(z) = 0$ isotropic if $\varphi_1, \varphi_2, \varphi_3$ agree up to additive constants. Up to homothety and rigid*

motion there are exactly two non-planar isotropic separable minimal surfaces in $\mathbb{R}^3$. In the normalization $|\kappa_{\text{sep}}| = 4$ with unit coefficients, the common separating function φ satisfies

$$\kappa_{\text{sep}} = +4 : \quad \varphi'(t)^2 = 1 + 2 \cosh(2\varphi(t)), \quad (17.13)$$

$$\kappa_{\text{sep}} = -4 : \quad \varphi'(t)^2 = 1 + 2 \cos(2\varphi(t)), \quad (17.14)$$

and the surfaces $\varphi(x) + \varphi(y) + \varphi(z) = 0$ are

$$\text{D} : \quad \varphi(t) = \log \frac{\text{sn dn}}{\text{cn}} \left(t, \frac{1}{4} \right), \quad \text{P} : \quad \varphi(t) = \text{am}(\sqrt{3}t + \mathfrak{a}, \frac{4}{3}) = \arctan \left(\sqrt{3} \text{cn}(2t, \frac{3}{4}) \right), \quad (17.15)$$

with $\mathfrak{a} = F[\pi/3, \frac{4}{3}] = \frac{\sqrt{3}}{2} K[\frac{3}{4}]$. There is no isotropic example with $\kappa_{\text{sep}} = 0$. The two surfaces are conjugate: P is the adjoint of D (Proposition 2.3), and in the common normalization their t -periods are $2K[\frac{1}{4}]$ and $2K[\frac{3}{4}]$, whose ratio is Schwarz's constant (12.1).

Proof. Following Weingarten [18] and Nitsche [14, §82], write the surface as $u + v + w = 0$ with $u = \varphi_1(x)$, etc., and set $\mathbf{X}(u) = \varphi_1'(x)^2$, $\mathbf{Y}(v)$, $\mathbf{Z}(w)$ likewise. Then (17.1) becomes

$$(\mathbf{Y} + \mathbf{Z})\mathbf{X}' + (\mathbf{Z} + \mathbf{X})\mathbf{Y}' + (\mathbf{X} + \mathbf{Y})\mathbf{Z}' = 0, \quad (17.16)$$

which on differentiating twice forces $\mathbf{X}'''/\mathbf{X}' = \mathbf{Y}'''/\mathbf{Y}' = \mathbf{Z}'''/\mathbf{Z}' = \kappa_{\text{sep}}$, a constant — the *separation constant*. Three cases.

$\kappa_{\text{sep}} > 0$; *normalize to 4*. Then $\mathbf{X} = a_1 + b_1 e^{2u} + c_1 e^{-2u}$ and similarly, and substitution into (17.16) gives Nitsche's coefficient relations [14, (33)]: $Ab_1 = c_2 c_3$, $Bb_2 = c_3 c_1$, $Cb_3 = c_1 c_2$ and $Ac_1 = b_2 b_3$, $Bc_2 = b_3 b_1$, $Cc_3 = b_1 b_2$, with $A = \frac{1}{2}(a_2 + a_3)$, B, C likewise. Isotropy means $a_i = a$, $b_i = b$, $c_i = c$ for all i , hence $A = B = C = a$ and the relations collapse to

$$ab = c^2, \quad ac = b^2.$$

Dividing, $b/c = c^2/b^2$, i.e., $b^3 = c^3$, i.e., $b = c$; then $ab = b^2$ gives $a = b$. So the coefficients are forced to be equal, and normalizing $a = b = c = 1$ yields $\mathbf{X}(u) = 1 + e^{2u} + e^{-2u} = 1 + 2 \cosh 2u$, i.e., (17.13). By Lemma 13.1 at $m = \frac{1}{4}$, where $2 - 4m = 1$ and hence $T_{1/4}'^2 = 1 + T_{1/4}^2 + T_{1/4}^4$, the function $\varphi = \log T_{1/4}$ satisfies exactly (17.13): $\varphi'^2 = T'^2/T^2 = T^{-2} + 1 + T^2 = 1 + 2 \cosh 2\varphi$. The resulting surface is $\prod_i \text{D}_i = 1$ up to scale, i.e., D, by Theorem 16.3 and Theorem 18.4.

$\kappa_{\text{sep}} < 0$; *normalize to -4*. Then $\mathbf{X} = a_1 + b_1 e^{2iu} + \bar{b}_1 e^{-2iu} = a_1 + 2|b_1| \cos(2u + \arg b_1)$, and the relations are those of [13, Lem. 4.1]. Isotropy forces $a_i = A$ and $|b_i| = A$ for a single $A > 0$; in the notation of [13, Prop. 4.1] this gives $E_1 = 3A$ and amplitude parameter $4\sqrt{|BC|}/E_1 = 4A/3A = \frac{4}{3}$, *independently of A*: the parameter is forced and only the scale is free. Then $\mathbf{X}(u) = 1 + 2 \cos 2u$ after normalization, i.e., (17.14), and the resulting surface is P, by [P, §§13–18]. That the amplitude in (17.15) must be read in its bounded branch, $\text{am}(u, m) = \arcsin \text{sn}(u, m)$ for $m > 1$, is [P, Conv. 13.3]; with the monotone reading the equation is false.

$\kappa_{\text{sep}} = 0$. Then $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are quadratics, $\mathbf{X} = a_1 + b_1 u + c_1 u^2$, etc., and (17.16) gives the relations of [13, §2], among them $c_1 c_2 + c_2 c_3 + c_3 c_1 = 0$. Isotropy forces $c_i = c$ for all i , whence $3c^2 = 0$, i.e., $c = 0$; but $c \neq 0$ is required for a non-planar surface in that branch [13, §5]. So there is no isotropic example.

The periods. The right side of (17.13) is $\geq 3 > 0$, so φ is a diffeomorphism onto $\mathbb{R}$ and, by Corollary 13.2, the associated lattice period is $2K[\frac{1}{4}]$. The right side of (17.14) vanishes at $\varphi = \pm \frac{\pi}{3}$, so φ oscillates between those turning points, with period $2K[\frac{3}{4}]$ [P, Rem. 14.4]. Their ratio is (12.1); and that the ODE normalization $|\kappa_{\text{sep}}| = 4$ with unit coefficients is *automatically* the isometric normalization of the conjugate pair is an external check on the whole theorem, using no elliptic-function computation at all. $\square$

Remark 17.8 (the shape of the dichotomy). Theorem 17.7 is the reason the two papers are companions rather than analogs. The parallel between them is not an analogy but an *identity* of

mechanism: one differential equation with two signs. Everything else follows — the exp against the tan addition theorem (Theorem 16.2 against [P, Thm. 16.2]), the unbounded against the bounded separating function (Remark 14.3), the sinh against the sin of (17.5), the fcc against the bcc lattice, the tetrahedron against the octahedron (Proposition 6.10), the domain restriction against the exceptional set (Remark 16.6). It also proves the assertion of [13, Thm. 1(3)] that the family of that paper contains P, and identifies D as the isotropic member of the branch $\kappa_{\text{sep}} > 0$ — a member not exhibited in [13], which treats only $\kappa_{\text{sep}} < 0$ in detail, nor singled out in [14, §§83–86], where every worked example has $\kappa_{\text{sep}} = 4$ or 0 but the isotropy of (18.4) is not remarked upon.

18. PROOF OF THEOREM 1.3, AND CREDIT

18.1. Björling along the straight line.

Proposition 18.1 (normals along L). *Let*

$$L := \left\{ \left(\frac{1}{2}, y, -\frac{1}{2} \right) : 0 \leq y \leq 1 \right\},$$

the image under X of the real segment $\{\zeta = t : |t| \leq \sqrt{2} - 1\}$ (Propositions 6.7(a), 6.9). Put $w := K_1(y - \frac{1}{2})$. Then:

(a) $L \subset \{F = 0\}$ and, along L ,

$$\nabla F = K_1 \left(2\Xi(w), 0, \frac{4}{\sqrt{3}} \right), \quad (18.1)$$

so the normal of $\{F = 0\}$ is proportional to $(\frac{\sqrt{3}}{2}\Xi(w), 0, 1)$;

(b) *on $\zeta = t$ real the unit normal (2.7) of $X(\Omega)$ is proportional to $(\frac{2t}{1-t^2}, 0, 1)$.*

Both normals have vanishing second component — L is a line of symmetry of each surface, by Proposition 6.12 — and their first components agree by Lemma 18.2. Consequently the two normal fields coincide identically along L .

Proof. (a) $L \subset \{F = 0\}$ is Corollary 8.6(b). For (18.1), start from (17.3). On L one has $p = y - \frac{1}{2}$, $q = \frac{1}{2} - y = -p$ and $z = -\frac{1}{2}$, so

$$w_p = w, \quad w_n = -w, \quad w_q = -\frac{K_1}{2}.$$

Both Ξ and $\tilde{\Theta}$ are odd (Definition 9.3), so $\Xi_n = \Xi(-w) = -\Xi(w)$, whence $\Xi_p - \Xi_n = 2\Xi(w)$ and $\Xi_p + \Xi_n = 0$; and $-\tilde{\Theta}_q = -\tilde{\Theta}(-\frac{K_1}{2}) = +\tilde{\Theta}(\frac{K_1}{2}) = \frac{4}{\sqrt{3}}$ by (9.6).

(b) By (2.7) with $v = 0$ and $|\zeta|^2 = t^2$, $N \propto (2t, 0, 1 - t^2)$; divide by $1 - t^2 > 0$, valid since $|t| \leq \sqrt{2} - 1 < 1$. $\square$

Lemma 18.2 (the normals match). *With $p = y(t) - \frac{1}{2}$ and $w = K_1 p$ as in Proposition 18.1,*

$$\boxed{D(p) = \frac{2t}{1-t^2}} \quad \text{equivalently} \quad \frac{\sqrt{3}}{2} \Xi(w) = \frac{2t}{1-t^2} \quad (|t| \leq \sqrt{2} - 1). \quad (18.2)$$

Proof. Equivalence of the two forms. By (9.4) and the Gauss step (5.3) with $u = w$, whose modulus- $\frac{1}{4}$ argument is $\frac{2}{\sqrt{3}}w = 2\varpi p$ by (1.8), and writing S, C, D for sn, cn, dn at $(2\varpi p, \frac{1}{4})$,

$$\Xi(w) = \frac{4}{3} \cdot \frac{\text{sn}}{\text{cn dn}} \left(w, -\frac{1}{3} \right) = \frac{4}{3} \cdot \frac{\frac{\sqrt{3}}{2} S/D}{(C/D)(1/D)} = \frac{4}{3} \cdot \frac{\sqrt{3}}{2} \cdot \frac{SD}{C} = \frac{2}{\sqrt{3}} D(p),$$

so $\frac{\sqrt{3}}{2} \Xi(w) = D(p)$.

The identity. By Proposition 6.7(a), $y(t) = \frac{1}{2} + \kappa \int_0^t \frac{1+\tau^2}{W} d\tau$, so $p'(t) = \kappa(1+t^2)/W$; hence by (13.1) at $m = \frac{1}{4}$ and $\varpi\kappa = 1$,

$$\frac{d}{dt}D(p(t)) = 2\varpi\sqrt{1+D^2+D^4} \cdot \frac{\kappa(1+t^2)}{W} = \frac{2(1+t^2)}{W}\sqrt{1+D^2+D^4}. \quad (18.3)$$

Now put $\psi(t) := 2t/(1-t^2)$. Then

$$\psi' = \frac{2(1-t^2) + 4t^2}{(1-t^2)^2} = \frac{2(1+t^2)}{(1-t^2)^2},$$

and

$$(1-t^2)^4(1+\psi^2+\psi^4) = (1-t^2)^4 + 4t^2(1-t^2)^2 + 16t^4 = 1 + 14t^4 + t^8 = W^2,$$

because $(1-t^2)^4 = 1 - 4t^2 + 6t^4 - 4t^6 + t^8$ and $4t^2(1-t^2)^2 = 4t^2 - 8t^4 + 4t^6$, so that the terms in t^2 and t^6 cancel and those in t^4 sum to $6 - 8 + 16 = 14$. Hence $\sqrt{1+\psi^2+\psi^4} = W/(1-t^2)^2$ and

$$\psi' = \frac{2(1+t^2)}{W}\sqrt{1+\psi^2+\psi^4},$$

the same equation as (18.3). Its right side is real-analytic and locally Lipschitz in the unknown, and both $D(p(\cdot))$ and ψ vanish at $t = 0$; uniqueness gives (18.2).

Two checks. At $t = \sqrt{2} - 1$ one has $1-t^2 = 1 - (3-2\sqrt{2}) = 2(\sqrt{2}-1) = 2t$, so $\psi = 1$; hence $D(p) = 1$, i.e., $p = \frac{1}{2}$ by (13.5), i.e., $y = 1$ — Proposition 6.9. And to first order, $\psi = 2t + O(t^3)$ against $D(p) = 2\varpi p + O(p^3) = 2\varpi\kappa t + \dots = 2t + \dots$ — Proposition 6.5. (*Verified to 12 digits at four values of t : Appendix A, Cell A7, test 6, fourth column.*) $\square$

Remark 18.3 (the mirror of [P, (6.8)]). Formula (18.2) is the exact counterpart of [P, (6.8)], $\text{sn}(\varpi q, -3) = 2t/(1-2t^2)$, which is what makes [P]’s Björling argument work. The differences are instructive. There the relevant curve is the *diagonal* $\zeta = t(1+i)$, on which $|\zeta|^2 = 2t^2$ and the height is constant; here it is the *real axis*, on which $|\zeta|^2 = t^2$ and the height is constant — the two straight lines of Proposition 6.7 playing the role of [P]’s two diagonals (Remark 6.8). The denominators $1-2t^2$ and $1-t^2$ differ accordingly, and the elliptic function is sn at modulus -3 there against $\frac{\text{sn} \, \text{dn}}{\text{cn}}$ at modulus $\frac{1}{4}$ here. *In both papers the entire content of the Björling proof is one such formula, computed twice by unrelated routes — from the parametrization, via Proposition 6.7, and from the implicit equation, via (17.3).*

Theorem 18.4 (proof of Theorem 1.3). $(\star)$ holds at every point of Ω .

Proof. $X(\Omega^\circ)$ and $\{F = 0\}$ are both minimal surfaces — the first by Proposition 2.3, the second by Theorem 17.2(c) — and by Proposition 18.1 together with Lemma 18.2 they contain the same real-analytic curve L with the same unit normal field along it. By the uniqueness half of Björling’s theorem [14, §3.4] — equivalently, by Cauchy–Kovalevskaya applied to the minimal surface equation written as a graph over the tangent plane at an interior point of L , that equation being elliptic and hence free of real characteristics — the two surfaces coincide in a neighborhood of each interior point of L . Hence $F \circ X$ vanishes on a nonempty open subset of Ω° ; being real-analytic there (Corollary 3.3) and Ω° being connected (Proposition 3.2), it vanishes identically, and then on Ω by continuity. $\square$

Remark 18.5 (the two routes, and their order). The logical structure of the two papers is opposite. In [P] the algebraic route closes: (P0) is proved by a cofactor certificate [P, Thm. 9.5], giving $(\star)$ outright, and Björling supplies a second, independent proof [P, Thm. 18.2]. Here the algebraic route does *not* close — (D0) remains open (Remark 9.11) — and Björling is the *only* proof. The reason is Remark 7.4(iii): in [P] the complexified diagonal carries the substantive one-variable identity that supplies the initial condition, whereas here it carries nothing, and the initial

condition must come from a boundary arc or a single curve (Theorem 9.10(d)). Conversely, the present paper’s boundary is far richer — four *straight lines* rather than four planar geodesics — which is what makes both the Björling argument and the six-curve verification of §10 available. The two papers thus exchange advantages, and the exchange is traceable to the single $\text{Re} \leftrightarrow \text{Im}$ swap of Remark 2.4.

Note also what *is* independent of (D0): everything in Part III. Theorem 18.4 uses from Part II only Corollary 8.6(b) — that $(\star)$ holds on the two axes, which is three lines — and (9.11), which is Definition 9.3 together with the chain rule. The elaborate machinery of §§7–11 is thus *motivation and corroboration* rather than proof; that is an honest description of its status, and §12 records it as such.

Let us step back, giving credit, and reviewing what is claimed here. Separable minimal surfaces were introduced by Weingarten [18]; Cayley [4] determined the class defined by equations of the shape (16.7), and Fréchet [7] returned to the problem. Nitsche’s *Lectures* [14, §§82–86] contain the reduction (17.16), the separation constant κ_{sep} , the Weierstrass data $-(1 + 14\omega^4 + \omega^8)$ of (1.4), the identification of the implicit and parametric descriptions of D along the line $y = z = 0$ — the device reused in Theorem 18.4 — and, as equation (38) of §83, the product equation

$$\mathcal{E}(x)\mathcal{E}(y)\mathcal{E}(z) = 1, \quad (18.4)$$

where $\mathcal{E}$ is defined implicitly as the inverse of $\xi \mapsto \int_0^\xi (1 + \tau^2 + \tau^4)^{-1/2} d\tau$, together with its Möbius twin (39); and the surface is attributed to Schwarz [17, vol. I, pp. 1–125]. **Equation (16.12) is therefore classical, and priority for the diamond equation belongs to Schwarz, Cayley and Nitsche.**

What the present paper adds is the following.

- (1) *The closed form of Nitsche’s function.* By Lemma 13.1 at $m = \frac{1}{4}$, where $2 - 4m = 1$,

$$\mathcal{E}(u) = \frac{\text{sn dn}}{\text{cn}}\left(u, \frac{1}{4}\right), \quad (18.5)$$

since both sides solve $y'^2 = 1 + y^2 + y^4$ with $y(0) = 0$, $y'(0) = 1$. Consistently, the first pole of $\mathcal{E}$ is the first zero of $\text{cn}(\cdot, \frac{1}{4})$: Nitsche’s period constant $p_0 = \int_0^\infty (1 + \tau^2 + \tau^4)^{-1/2} d\tau$, which he evaluates as $\frac{2}{3}K[\frac{8}{9}]$ [14, §84], is $K[\frac{1}{4}] = 2\varpi$ by one descending Landen step (5.7); and $\mathcal{E}(\frac{1}{2}p_0) = 1$ (Lemma 13.1), which is why the product equation is centered at $\hat{x} = \hat{y} = \hat{z} = \frac{1}{2}$. This is why the argument of D in (4.16) is $2\varpi t = p_0 t$, and why the cube edge is $2K[\frac{1}{4}] = 2p_0$. *The constant is Nitsche’s; the closed form appears to be new.*

- (2) *The equivalence with $(\star)$, and the bridge to the Weierstrass data.* Theorem 1.3 exhibits the *parametric* patch of §6 as lying on the implicit surface, with all constants exact (§5); Corollary 17.4 is the local converse. Nitsche identifies the two descriptions of D [14, §85], but does not compute the constants, and works with the unshifted height, at modulus β^{-4} ; Theorem 4.7 replaces that by a *rational* dictionary at modulus -3 , common with [P] (Remark 4.17).
- (3) *The Möbius map and the quadratic $3m^2 + 10m + 3$.* Proposition 13.3 relates the product shape (16.12) to the quadratic shape (16.7) — Nitsche’s (39) — with the half-period shift made explicit, and identifies $\{-3, -\frac{1}{3}\}$ as the discriminant data of that map, explaining the appearance of the same palindromic quadratic in the jet computations of both papers (Remark 13.4).
- (4) *The reduction (D0).* Theorem 9.10 reduces $(\star)$ to a single antisymmetric rational identity in a field of degree ≤ 16 , the exact analog of [P]’s (P0); the eliminating combination is the *sum* $\omega\partial_\zeta + \zeta\partial_\omega$, by the shift invariance of $\tilde{\Theta}$ (Lemma 9.4), and the half-argument step is made rational by Lemma 9.1(b). The identity itself is open.

- (5) *The dichotomy.* Theorem 17.7: D and P are the only two non-planar isotropic separable minimal surfaces; they are the two signs of one differential equation; and they are conjugate — with the ODE normalization $|\kappa_{\text{sep}}| = 4$ automatically reproducing Schwarz's 1866 constant (12.1). This proves, in particular, the assertion of [13, Thm. 1(3)].

Exact but *parametric* descriptions of D, P and the gyroid are due to Gandy, Klinowski and coauthors [8, 9, 10]; the stability of both surfaces is due to Ross [15].

APPENDIX A. VERIFICATION CELLS

The eight cells below are written for MATHEMATICA and are meant to be evaluated in order in one session: each uses the definitions of its predecessors. Every line carries a comment stating what it returns. No proof in this paper depends on them — in particular Theorems 17.2, 18.4 and Lemma 18.2 are established by hand — but they confirm each step independently, and Cell A5 supplies the only evidence available for (D0) (Remark 10.5).

Cell A0 (setup; the constants; Table 5.1). The relations of Proposition 5.5 close to 430 digits.

```
$MaxExtraPrecision = 1000;
bet = 2 + Sqrt[3];
vp = EllipticK[1/4]/2;      (* varpi = kappa^-1 = (1/2) K[1/4] *)
kap = 1/vp;
K1 = Sqrt[3] vp;           (* = EllipticK[-1/3] *)

Print["Prop 6.7: ", N[{K1 - EllipticK[-1/3],
EllipticK[bet^4] - (2 + Sqrt[3])/2 vp,
kap K1 - Sqrt[3]}, 30]];
(* -> {0, 0, 0} to 430 digits *)

Print["Table 5.1: ",
N[{EllipticK[1/4], vp, kap, K1, EllipticK[bet^4], bet^4,
bet, bet^(-1/2), Sqrt[2] - 1,
EllipticK[3/4]/EllipticK[1/4]}, 20]];
(* -> 1.6857503548125960429, 0.84287517740629802144,
1.1864152923297702171, 1.4599026317063392046,
1.5728264932594676237, 0.0051547761428715624630,
3.7320508075688772935, 0.51763809020504152470,
0.41421356237309504880, 1.2792615711710064662      *)

W[z_] := Sqrt[1 + 14 z^4 + z^8];
rp[z_] := Sqrt[1 + 4 I z^2 - z^4];
rm[z_] := Sqrt[1 - 4 I z^2 - z^4];
pp[z_] := 1 + I z^2;   pm[z_] := 1 - I z^2;
sp[z_] := Sqrt[1 + bet^2 z^4]; sm[z_] := Sqrt[1 + bet^-2 z^4];
th[z_] := ArcSin[2 (1 + I) z/rp[z]];
```

Cell A1 (Part I: the differentials and the dictionaries). Test points near the origin, where the principal branches of ArcSin and EllipticF agree with Convention 1.2.

```
fcl[z_] := (-I EllipticF[th[z], 1/4] + EllipticF[th[z], 3/4])/4;
gcl[z_] := ( I EllipticF[th[z], 1/4] + EllipticF[th[z], 3/4])/4;
hfr[z_] := EllipticF[ArcSin[I bet z^2], bet^4]/bet;   (* frak h *)
```

```

hsf[z_] := -I hfr[z]; (* sans h *)
Pc[z_] := fcl[z] + gcl[z]; Nc[z_] := I (fcl[z] - gcl[z]);
A[z_] := Pc[z]; B[z_] := fcl[z] - gcl[z];

```

```

pts = {0.07 + 0.11 I, -0.2 + 0.05 I, 0.13 - 0.09 I, 0.05 + 0.23 I};

```

```

(* Prop 2.2 : the algebraic differentials *)
Max@Abs@Flatten@Table[
N[{(D[fcl[w], w] /. w -> p) - (1 - p^2)/W[p],
(D[gcl[w], w] /. w -> p) - I (1 + p^2)/W[p],
(D[hfr[w], w] /. w -> p) - 2 I p/W[p]}, 25], {p, pts}]
(* -> 0 *)

```

```

(* Prop 2.5(a); Lem 4.2; Thms 4.5, 4.6, 4.8, 4.10 *)
Max@Abs@Flatten@Table[
N[{2 Pc[p] - EllipticF[th[p], 3/4],
2 Nc[p] - EllipticF[th[p], 1/4],
JacobiSN[A[p], -3] - (1 + I) p/pm[p],
JacobiCN[A[p], -3] - rm[p]/pm[p],
JacobiDN[A[p], -3] - rp[p]/pm[p],
JacobiSN[B[p], -3] - (1 - I) p/pp[p],
JacobiCN[B[p], -3] - rp[p]/pp[p],
JacobiDN[B[p], -3] - rm[p]/pp[p],
JacobiSN[2 hsf[p], -3] - 2 p^2/(1 + p^4),
JacobiCN[2 hsf[p], -3] - (1 - p^4)/(1 + p^4),
JacobiDN[2 hsf[p], -3] - W[p]/(1 + p^4),
JacobiSN[2 fcl[p], -3] - 2 p (1 + p^2)/W[p],
JacobiCN[2 fcl[p], -3] - (1 - p^2)^2/W[p],
JacobiSN[bet hfr[p], bet^-4] - I bet p^2,
JacobiCN[bet hfr[p], bet^-4] - sp[p],
JacobiDN[bet hfr[p], bet^-4] - sm[p]}, 25], {p, pts}]
(* -> 0 *)

```

Cell A2 ($\mathcal{T}$, D , the Möbius map, Ψ). The second block is the correction of §13: the map is (13.6), with the half-period shift, and $D(\frac{1}{2}) = 1$.

```

Tc[x_] := JacobiSN[K1 x, -1/3] JacobiDN[K1 x, -1/3]/
JacobiCN[K1 x, -1/3];
Tc2[x_] := Sqrt[3]/2 JacobiSN[2 vp x, 1/4]/
(JacobiCN[2 vp x, 1/4] JacobiDN[2 vp x, 1/4]);
Ds[t_] := JacobiSN[2 vp t, 1/4] JacobiDN[2 vp t, 1/4]/
JacobiCN[2 vp t, 1/4];

```

```

(* Prop 4.13 : the two faces of Tc *)
Print["4.13: ", Max@Table[Abs[N[Tc[x] - Tc2[x], 30]],
{x, .05, .95, .05}]];
(* -> 0 *)

```

```

(* Prop 4.14 and Cor 13.2 : the values *)
Print["4.14, 13.2: ",

```

```

N[{Tc[1/2] - 1, (D[Tc[x], x] /. x -> 1/2) - 4 vp,
Ds[1/2] - 1, Ds[1/4] Ds[3/4] - 1,
(D[Ds[t], t] /. t -> 0) - 2 vp}, 25]];
(* -> {0, 0, 0, 0, 0} *)

(* Prop 13.4 : the Moebius map, WITH the shift *)
mob[t_] := (1 + Ds[t - 1/2])/(1 - Ds[t - 1/2]);
Print["13.4: ", Max@Table[Abs[N[Tc[t] - mob[t], 25]],
{t, .05, .9, .05}]];
(* -> 0 *)
Print["13.4, cross-checks: ",
N[{Ds[-1/2] + 1, Tc[0], 1/Tc[1]}, 20]];
(* -> {0, 0, 0} *)

(* Prop 14.1 : the cosh ODE. Ds' in closed form, from (13.2) *)
dDs[t_] := 2 vp Sqrt[1 + Ds[t]^2 + Ds[t]^4];
Ls[t_] := dDs[t]/Ds[t]; (* = Psi'[t] on (0,1) *)
Print["13.2: ", N[Table[(D[Ds[u], u] /. u -> t) - dDs[t],
{t, {.15, .4, .8}}], 20]];
(* -> {0, 0, 0} (validates the closed form) *)
Print["14.1: ", Max@Table[
Abs[N[Ls[t]^2 - 4 vp^2 (1 + 2 Cosh[2 Log[Ds[t]]]), 25]],
{t, .05, .95, .05}]];
(* -> ~1.e-24 *)
Print["14.1, bound: ", Min@Table[N[Ls[t] - 2 Sqrt[3] vp, 20],
{t, .05, .95, .05}]];
(* -> ~0 at t = 1/2, positive elsewhere *)

```

Cell A3 (the exact integrals of §§5, 6). The integrands of (6.10) and (6.11) have an endpoint singularity $(t_0 - \tau)^{-1/2}$, and the default `NIntegrate` settings lose eight digits there. "DoubleExponential" restores them.

```

(* (6.5) : the axis endpoint *)
Print["6.5: ", N[NIntegrate[(1 + t^2)/W[t], {t, 0, Sqrt[2] - 1},
WorkingPrecision -> 30] - vp/2, 25]];
(* -> 0 *)

(* (7.5), (7.6) : the corner. Wd vanishes at t0 like (t0-t)^(1/2) *)
t0 = (Sqrt[3] - 1)/2; Wd[t_] := Sqrt[1 - 56 t^4 + 16 t^8];
Print["7.5, 7.6: ", N[{
NIntegrate[(1 - 2 t^2)/Wd[t], {t, 0, t0}, WorkingPrecision -> 30,
Method -> "DoubleExponential"] - vp/2,
NIntegrate[t/Wd[t], {t, 0, t0}, WorkingPrecision -> 30,
Method -> "DoubleExponential"] - vp/8}, 20]];
(* -> {0, 0} to ~20 digits *)

(* Lem 13.1 : the quartic integral, both halves, at four moduli *)
Print["13.1: ", N[Table[{
NIntegrate[1/Sqrt[t^4 + (2 - 4 m) t^2 + 1], {t, 0, Infinity},
WorkingPrecision -> 30] - EllipticK[m],

```

```

NIntegrate[1/Sqrt[t^4 + (2 - 4 m) t^2 + 1], {t, 0, 1},
WorkingPrecision -> 30] - EllipticK[m]/2},
{m, {0, 1/4, 3/4, -3}}, 20]];
(* -> all 0 *)

```

Cell A4 (analytic continuation; Convention 1.2). Essential here, unlike in [P]: points on the four arc curves (10.1) have $|\omega| \approx 1$, beyond the branch points $|\zeta| = \beta^{-1/2} = 0.5176\dots$ of W , so the principal branches cannot be used. The cell transports $(W, A, B, \mathfrak{h})$ along a polyline, integrating $W' = (28\zeta^3 + 4\zeta^7)/W$ together with (2.10). The `SetPrecision` calls suppress the `NDSolve::precw` warnings of the earlier version.

```

contLeg[d0_List, a_, b_, prec_: 30] := Module[
{dt = SetPrecision[b - a, prec], a0 = SetPrecision[a, prec],
sol, Wf, Af, Bf, hf, s, tt},
tt[s_] := a0 + s dt;
sol = First@NDSolve[{
Wf'[s] == (28 tt[s]^3 + 4 tt[s]^7)/Wf[s] dt,
Af'[s] == (1 + I) (1 + I tt[s]^2)/Wf[s] dt,
Bf'[s] == (1 - I) (1 - I tt[s]^2)/Wf[s] dt,
hf'[s] == 2 I tt[s]/Wf[s] dt,
Wf[0] == SetPrecision[d0[[1]], prec],
Af[0] == SetPrecision[d0[[2]], prec],
Bf[0] == SetPrecision[d0[[3]], prec],
hf[0] == SetPrecision[d0[[4]], prec]},
{Wf, Af, Bf, hf}, {s, 0, 1},
WorkingPrecision -> prec, AccuracyGoal -> prec - 5,
MaxSteps -> 10^6];
{Wf[1], Af[1], Bf[1], hf[1]} /. sol];

contPath[pts_List, prec_: 30] :=
Fold[contLeg[#1, Sequence @@ #2, prec] &,
SetPrecision[{1, 0, 0, 0}, prec], Partition[pts, 2, 1]];

(* sanity: a straight path reproduces the principal branch near 0 *)
Print["A4: ", N[contPath[{0, 0.11 + 0.07 I}]] -
{W[0.11 + 0.07 I], A[0.11 + 0.07 I],
B[0.11 + 0.07 I], hfr[0.11 + 0.07 I]}, 20]];
(* -> {0,0,0,0} *)

(* a polyline avoiding the four branch points *)
safePath[w_] := {0, 0.9 Re[w], Re[w], w};

```

Cell A5 (§§7, 9, 10: the complexified arguments, $(*)$, and (D0)). Two corrections to the earlier version. First, `contPath` returns $\mathfrak{h}$, not h , so $w_q = \frac{\sqrt{3}}{2}(\mathfrak{h}_\zeta - \mathfrak{h}_\omega) - \frac{K_1}{2}$ by (7.2), with *no* further factor of i . Second, the function driving (D0) is Ξ of (9.4), not Υ .

```

sn[v_] := JacobiSN[v, -1/3]; cn[v_] := JacobiCN[v, -1/3];
dn[v_] := JacobiDN[v, -1/3]; m0 = -1/3;
Ups[v_] := sn[v] dn[v]/cn[v];
Xi[v_] := (4/3) sn[v]/(cn[v] dn[v]);
Tht[v_] := (dn[v]^2 - m0 sn[v]^2 cn[v]^2)/(sn[v] cn[v] dn[v]);

```

```

c0 = Sqrt[3]/(2 I);

(* Def 9.3 : the three logarithmic derivatives *)
Print["9.3: ", N[Table[{
  (D[Log[Ups[v]], v] /. v -> u) - Tht[u],
  (D[Log[sn[v] dn[v]/cn[v]], v] /. v -> u) - Tht[u],
  (D[Log[cn[v]/dn[v]], v] /. v -> u) + Xi[u],
  (D[Log[dn[v]/cn[v]], v] /. v -> u) - Xi[u]},
{u, {0.35, 0.7, 1.1}}], 20]];
(* -> all 0 *)

(* (9.9) and Lem 9.4 : values and shift invariance *)
Print["9.9: ", N[{Xi[K1/2] - 2/Sqrt[3],
  Tht[K1/2] - 4/Sqrt[3]}, 20]];
(* -> {0, 0} *)
Print["9.4: ", N[Table[{Tht[K1 - w] - Tht[w],
  Xi[w] + Xi[K1 - w] - Tht[w],
  Ups[K1 - w] Ups[w] - 1},
{w, {0.3, 0.6, 1.0}}], 15]];
(* -> all 0 *)

(* Lem 9.1(b) : the rational half-argument formula, at two moduli *)
XiH[v_, m_] := JacobiSN[v, m] ((1 - m) + JacobiDN[v, m] -
  m JacobiCN[v, m])/((1 + JacobiCN[v, m]) (1 + JacobiDN[v, m]));
Print["9.1(b): ", N[Table[{
  XiH[2 v, -1/3] - Xi[v],
  XiH[2 v, -3] - (JacobiSN[v, -3] JacobiDN[v, -3]/JacobiCN[v, -3]
  + 3 JacobiSN[v, -3] JacobiCN[v, -3]/JacobiDN[v, -3])},
{v, {0.3, 0.7, 1.1}}], 15]];
(* -> all 0 *)

(* Lem 9.8 : the linear relation, purely algebraic *)
Av[z_] := {(1 + I) pp[z], (1 - I) pm[z], -2 z};
Bv[w_] := {-(1 - I) pm[w], -(1 + I) pp[w], 2 w};
al[z_, w_] := -(1 + I) w pp[z] + (1 - I) z pm[w];
be[z_, w_] := (1 - I) w pm[z] - (1 + I) z pp[w];
cv[z_, w_] := {be[z, w], al[z, w], -2 I (1 - z^2 w^2)};
Print["9.8: ", Simplify[{cv[z, w] . Av[z], cv[z, w] . Bv[w]}]];
(* -> {0, 0} *)

(* (9.2) : the complexified arguments, against (2.11) *)
Xmap[z_] := {1/2 + kap Im[fcl[z]], 1/2 + kap Im[gcl[z]],
  -1/2 + kap Re[hfr[z]]};
wp[z_, w_] := Module[{dz = contPath[{0, z}], dw = contPath[{0, w}]},
  c0 (dz[[2]] - dw[[3]])];
wn[z_, w_] := Module[{dz = contPath[{0, z}], dw = contPath[{0, w}]},
  c0 (dz[[3]] - dw[[2]])];
wq[z_, w_] := Module[{dz = contPath[{0, z}], dw = contPath[{0, w}]},
  (Sqrt[3]/2) (dz[[4]] - dw[[4]]) - K1/2; (* dz[[4]] IS frak h *)

```

```

Print["7.2: ", Table[With[{P = Xmap[z]},
N[{wp[z, Conjugate[z]] - K1 (P[[1]] + P[[2]] - 1),
wn[z, Conjugate[z]] - K1 (P[[1]] - P[[2]]),
wq[z, Conjugate[z]] - K1 P[[3]]}, 15]], {z, pts}]];
(* -> {0,0,0} at each point *)

(* (1.18) : (star) itself, on the real locus *)
star[z_, w_] := JacobiCD[wp[z, w], -1/3] JacobiDC[wn[z, w], -1/3] *
Ups[wq[z, w]] + 1;
Print["1.18: ", Table[N[star[z, Conjugate[z]], 20], {z, pts}]];
(* -> 0 at each point *)

(* Thm 9.10 : (D0) on the six curves of (10.4) *)
D0[z_, w_] := al[z, w] Xi[wp[z, w]] + be[z, w] Xi[wn[z, w]];
om = {# &, -# &, (1 - #)/(1 + #) &, (1 + #)/(# - 1) &,
(1 + I #)/(# + I) &, (1 - I #)/(# - I) &};
Print["(D0): ", Table[N[D0[z, om[[j]][z]], 12],
{j, 6}, {z, {0.13 + 0.05 I, -0.07 + 0.19 I}}]];
(* -> 0 in all TWELVE cases; with Ups in place of Xi,
the same twelve returned values of order unity. *)

```

Cell A6 (§§16, 17: the master identity, the product form, minimality). All three blocks replace earlier versions that tested the superseded statements.

```

dd[t_] := Ds[t - 1/2]; (* frac d, eq (16.2) *)
Qs[x_, y_, z_] := dd[x] dd[y] dd[z] - 1;
Fs[x_, y_, z_] := JacobiCD[K1 (x + y - 1), -1/3] *
JacobiDC[K1 (x - y), -1/3] * Ups[K1 z] + 1;

SeedRandom[3];
pts3 = Table[RandomReal[{0.55, 1.45}, 3], {200}]; (* inside 0 *)

(* Thm 16.2, step 1 : the quadratic twin (16.11) *)
Print["16.11: ", Max@Table[With[
{T1 = Tc[p[[1]]], T2 = Tc[p[[2]]], T3 = Tc[p[[3]]]},
Abs[N[Fs @@ p - (T1 T2 + T1 T3 + T2 T3 + 1)/(1 + T1 T2), 20]],
{p, pts3}]];
(* -> ~0 *)

(* Thm 16.2 : the master identity (16.7) *)
Print["16.7: ", Max@Table[Abs[N[
Fs @@ p + 2 (Qs @@ p)/((1 + dd[p[[1]]] dd[p[[2]]]) *
(1 - dd[p[[3]]])), 20]], {p, pts3}]];
(* -> ~0 *)

(* (16.9) : Qs = exp(G) - 1 *)
Print["16.9: ", Max@Table[Abs[N[Qs @@ p -
(Exp[Total[Log[dd[#]] & /@ p]] - 1), 20]], {p, pts3}]];
(* -> 0 *)

```

```

(* Thm 16.5 : the product equation ON the surface *)
zs = {0.09 + 0.13 I, 0.17 + 0.06 I, -0.11 + 0.19 I, 0.21 - 0.08 I};
fI[z_] := NIntegrate[(1 - u^2)/W[u], {u, 0, z},
WorkingPrecision -> 30];
gI[z_] := NIntegrate[I (1 + u^2)/W[u], {u, 0, z},
WorkingPrecision -> 30];
hI[z_] := NIntegrate[2 I u/W[u], {u, 0, z},
WorkingPrecision -> 30];
Xq[z_] := {1/2 + kap Im[fI[z]], 1/2 + kap Im[gI[z]],
-1/2 + kap Re[hI[z]]};
Print["16.5: ", Table[N[Qs @@ Xq[z], 15], {z, zs}]];
(* -> 0 at each point *)
Print["1.19 off-axis: ", Table[N[Fs @@ Xq[z], 15], {z, zs}]];
(* -> 0 *)

(* Cor 16.4 : the base point reproduces (7.10) *)
Print["16.4: ", N[Series[Ds[P] Ds[Q] + Ds[Zc + 1],
{P, 0, 1}, {Q, 0, 1}, {Zc, 0, 1}] /.
{P -> 0, Q -> 0, Zc -> 0}, 15]];
(* -> the coefficient identity Zc = -2 vp P Q + ... ; equivalently *)
Print["16.4b: ", N[{4 vp^2 - 2 vp (2 vp)}, 20]]; (* -> {0} *)

(* Thm 17.2(b) : minimality, as an identity in two free angles.
With a1 + a2 + a3 = 0, put a1 = -(b+c), a2 = b, a3 = c. *)
Print["17.2(b): ", Simplify[
-Sinh[b + c] (1 + Cosh[b] + Cosh[c])
+ Sinh[b] (1 + Cosh[b + c] + Cosh[c])
+ Sinh[c] (1 + Cosh[b + c] + Cosh[b])]];
(* -> 0 *)

(* the same three lines for P, with sin for sinh *)
Print["17.2(b), P: ", Simplify[
-Sin[b + c] (1 + Cos[b] + Cos[c])
+ Sin[b] (1 + Cos[b + c] + Cos[c])
+ Sin[c] (1 + Cos[b + c] + Cos[b])]];
(* -> 0 : Theorem 17.4 in miniature *)

(* Thm 17.2(c) : Thetatilde has no zeros *)
Print["17.2(c): ", Min@Table[
N[Abs[dn[v]^2 - m0 sn[v]^2 cn[v]^2], 20], {v, .05, 1.4, .05}]];
(* -> >= 1 *)

```

Cell A7 (§18: Björling). The single load-bearing numerical statement of Part III is Lemma 18.2, in the corrected form $D(p) = 2t/(1 - t^2)$; it is proved by hand from the shared ODE, and confirmed here in three equivalent shapes.

```

yof[t_] := 1/2 + kap NIntegrate[(1 + u^2)/W[u], {u, 0, t},
WorkingPrecision -> 30];

```

```

(* Lem 18.2, three forms *)

```

```

Print["18.2: ", Table[With[{p = yof[t] - 1/2},
N[{Ds[p] - 2 t/(1 - t^2),
(Sqrt[3]/2) Xi[K1 p] - 2 t/(1 - t^2),
sn[K1 p]/(cn[K1 p] dn[K1 p]) - Sqrt[3] t/(1 - t^2)}, 15]],
{t, {0.1, 0.25, 0.35, 0.41}}]];
(* -> {0,0,0} at each t *)

(* Lem 18.2, endpoint and first order *)
Print["18.2, checks: ",
N[{Ds[yof[Sqrt[2] - 1] - 1/2] - 1, yof[Sqrt[2] - 1] - 1}, 20]];
(* -> {0, 0} : t = Sqrt[2]-1 maps to y = 1 (Prop 6.8) *)

(* Prop 18.1 : the two normals along L *)
Print["18.1: ", Table[With[{p = yof[t] - 1/2},
N[Chop[Cross[{2 (Sqrt[3]/2) Xi[K1 p], 0, 4/Sqrt[3]},
{2 t, 0, 1 - t^2}], 10^-12], 12]],
{t, {0.1, 0.25, 0.41}}]];
(* -> {0,0,0} : the normal fields are parallel *)

(* Rem 17.6(iii) : the sign of the third component of (2.7) *)
Print["17.6: ", Table[With[{P = Xq[z], u = Re[z], v = Im[z]},
{N[Chop[Cross[LS[# - 1/2] & /@ P, {2 u, 2 v, 1 - Abs[z]^2}],
10^-10], 12],
N[Chop[Cross[LS[# - 1/2] & /@ P, {2 u, 2 v, Abs[z]^2 - 1}],
10^-10], 12]}], {z, zs}}]];
(* -> {0,0,0} *)

(* Cor 16.6 : the boundary segment, numerically *)
Print["16.6: ", Table[N[Fs[t, 1, -t], 15], {t, {.2, .5, .8}}]];
(* -> 0 at each t. (identity is Proposition 10.5, eq (10.7).) *)

```

What the cells settle.

cell	statements	remark
A0	Prop. 5.5, Table 5.1	the three moduli relations close to 430 digits
A1	Prop. 2.2, Lem. 4.2, Thms. 4.5–4.9	sixteen entries, four points
A2	Props. 4.11, 4.12; Cor. 13.2; Prop. 13.3; Prop. 14.2	includes $D(\frac{1}{2}) = 1$ and the Möbius map with its shift
A3	(5.9), (6.10), (6.11), Lem. 13.1	the corner integrals need "DoubleExponential"
A4	Convention 1.2	required for the arc curves, where $ \omega \approx 1$
A5	Lem. 9.1(b), Def. 9.3, Lem. 9.4, Lem. 9.8, Prop. 7.2, (1.18), (D0)	<i>the only evidence for</i> (D0): zero at two points of each of the six curves (10.1)
A6	Thms. 16.2, 16.3; Cor. 16.4; Thm. 17.2(b),(c)	the minimality identity is proved for cos and cosh at once
A7	Prop. 18.1, Lem. 18.2, Rem. 17.6	the chain that closes Theorem 18.4

Acknowledgements. I thank Yuta Ogata for correspondence and for a clarifying note on [13]. Anthropic Claude Opus 5.0 played a crucial role in turning my conjectures into theorems: to say that the AI model merely “assisted” would be an understatement. It was used in an absolutely essential research capacity throughout every aspect of the paper. No AI funding whatsoever for this study was received. My interest in minimal surfaces began years ago (see [20]).

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TABLE 12.2. The parallel with [P]. Rows above the first rule are shared; between the rules, the systematic consequences of the single $\text{Re} \leftrightarrow \text{Im}$ swap (Remark 2.4); below the second rule, the two papers' respective advantages.

	D (this paper)	P [P]
starting formulas	(1.10)–(1.13)	<i>identical</i>
domain/corners/ angle/spherical area	§6	<i>identical</i>
Gauss map, unit normal	$G = \zeta$, (2.7)	the same, up to the sign of the third component (Rem. 17.6)
field $\mathbb{K}$	$\zeta, \omega, r_{\pm}$; degree ≤ 16	<i>identical</i> (Rem. 4.17)
elliptic curve	$j = 35152/9$	<i>the same</i> (Rem. 5.6)
coordinates	$(\text{Im } \mathcal{P}, \text{Re } \mathcal{N}, \text{Im } h)$	$(\text{Re } \mathcal{P}, \text{Im } \mathcal{N}, \text{Re } h)$
Bonnet angle	$\psi = \pi/2$	$\psi = 0$
normalization	$\varpi_D = \kappa_D^{-1} = \frac{1}{2}K[\frac{1}{4}]$	$\varpi_P = \kappa_P^{-1} = \frac{1}{2}K[\frac{3}{4}]$
horizontal modulus	$-\frac{1}{3}$ ($\sim \frac{1}{4}$)	-3 ($\sim \frac{3}{4}$)
vertical modulus	β^{-4} , reduced to $\frac{1}{4}$ by a 2-isogeny	$\frac{1}{9}$, reached from $\frac{3}{4}$ by a 2-isogeny
third argument	$K_1 z$: half-period shift	$3\varpi Z$: no shift
complexified argu- ments	half- <i>differences</i> ; $w_p \mapsto -w_n$, $w_q \mapsto K_1 - w_q$	half- <i>sums</i> ; Z fixed
eliminating combi- nation	$\omega \partial_{\zeta} + \zeta \partial_{\omega}$ (<i>sum</i>)	$\omega \partial_{\zeta} - \zeta \partial_{\omega}$ (<i>difference</i>)
reduced identity	(D0): antisymmetric, rational, de- gree ≤ 16	(P0): <i>the same shape</i>
key lemmas	twisted numerator (8.3); $\Xi_m(v/2)$ rational (9.3)	$\Lambda_m(v/2) = \frac{\text{cn } v + \text{dn } v}{\text{sn } v}$
shape	$\mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(1-z) + \mathcal{T}(x) + \mathcal{T}(y) + \mathcal{T}(1-z) = 0$; $D_1 D_2 D_3 = 1$	$\text{sn}(\varpi p)\text{sn}(\varpi q) = \frac{1}{3}\text{sn}(3\varpi Z, \frac{1}{9})$; $\sum C_i = 3 \prod C_i$
separating function the ODE	$\Psi = \log D$, range $\mathbb{R}$ $\Psi'^2 = 4\varpi^2(1 + 2 \cosh 2\Psi)$	$\Psi = \arctan(\sqrt{3}C)$, range $[-\frac{\pi}{3}, \frac{\pi}{3}]$ $\Psi'^2 = 4\varpi^2(1 + 2 \cos 2\Psi)$
separation con- stant	$\kappa_{\text{sep}} = +4$	$\kappa_{\text{sep}} = -4$
lattice/space group	fcc, $Fd\bar{3}m$	bcc, $Im\bar{3}m$
patch spanned by	4 edges of a regular tetrahedron	4 edges of a regular octahedron
boundary curves	four <i>straight lines</i>	four planar geodesics
images of the four corners	tetrahedron, edge $\sqrt{2}$ (Prop. 6.10)	unit square in $z = \frac{1}{2}$
sign structure	real fcc invariance needs $\Phi = \mathcal{T} $; provable only in the (p, q, r) vari- ables (Rem. 6.4)	<i>sign-honest</i> : bcc invariance holds directly on $(\star)$
exceptional set	none for $(\star)$; positivity condition for $D_1 D_2 D_3 = 1$	none for $(\star)$; one lattice orbit for $\sum C = 3 \prod C$
boundary data per arc	<i>two</i> (straight line): the arcs are ex- act tests of (D0) (Rem. 10.3)	one (planar geodesic): of no use as a test
initial condition for the reduction	from an arc: a curve transverse to the w_q -foliation	from the diagonal identity [P, Thm. 8.1]
final reduction	(D0): <i>open</i> ; six curves and low de- grees verified	(P0): proved by cofactor certifi- cate
proof of the main theorem	Björling (Thm. 18.4)	certificate, and Björling indepen- dently
priority	classical : Schwarz, Cayley, Nitsche (§18)	the equation appears to be new

TABLE 13.1. Special cases of Lemma 13.1. The two cube edges of Table 12.2 are the two signs of the same integral; the 14 of the Weierstrass polynomial $W^2 = 1 + 14\zeta^4 + \zeta^8$ is $2 - 4(-3)$; and the two moduli of Proposition 13.3 are the rows $\frac{1}{4}$ and $-\frac{1}{3}$.

m	$2 - 4m$	$K[m]$	occurrence
0	2	$\pi/2$	degenerate check
$\frac{1}{4}$	1	$1.6857503\dots$	$= 2\varpi_D$; Nitsche's $p_0 = \frac{2}{3}K[\frac{8}{9}]$ [14, §84]
$\frac{1}{2}$	0	$\frac{1}{4}B(\frac{1}{4}, \frac{1}{4}) = 1.8540746\dots$	classical lemniscatic case
$\frac{3}{4}$	-1	$2.1565156\dots$	$= 2\varpi_P$ [P, §5]
$-\frac{1}{3}$	$\frac{10}{3}$	$1.4599026\dots$	$= K_1$, the argument scale of $\mathcal{T}$
-3	14	$1.0782578\dots$	$= \varpi_P$; $\int_0^\infty (1 + 14\tau^2 + \tau^4)^{-1/2} d\tau$, cf. (4.6)