Proposed Problem Collection

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Hello there!

As you might be aware, I've always loved writing problems to share with the math community. I've actually always written problems for my peers and siblings to do, but it was in middle school where I started sharing them online through mock contests. Now, I am the test manager of the NEMO (found at `https://nemomath.github.io/`) and a problem writer for the Online Math Open (found at `http://internetolympiad.org/pages/11-omo`). What used to be a hobby has almost become a job, and I still love doing it!

To reminisce, to give back to the community, and to marvel (and cringe) at my past problems, I've compiled a list of published problems for math competitions I've written for. I'll spare you the agony of looking at my sixth grade problems, and I've instead chosen a potpourri of my favorite problems for you to enjoy. The problems are sorted by subject and in increasing order of difficulty. There are currently **120** problems here; I expect this number to keep growing!

Unfortunately, I have not compiled an answer key for these. I've instead attached a few links at the end of the document where you can find the source (and the solutions!) of many of the problems here. Feel free to email be (at seanjli [at] gmail [dot] com) if you have any further questions.

Enjoy!

Sincerely,
Sean Li

Contents

1 Algebra

1. (Th3's Mini Math Bee R1/1) In a certain one hundred story building, each flight of stairs is fifteen steps. Between adjacent floors are three flights of stairs. These are the only flights of stairs in the entire building. How many steps are there in total in the building?[1]

2. (Mock AMC8 2017) The first day that Jaime buys his favorite chocolate cake, he eats one-fourth of the cake. Each day after that, Jaime eats 25% less cake than the day before. At the end of the third day, what fraction of the cake remains?

3. (All Algebra Assessment 2017) The *washing* of a number is the process of adding one to it, and then dividing the result by five. For example, 19 is $(19 + 1) \div 5 = 4$ after washing once, and the number 19 after washing it twice is $((19 + 1) \div 5) + 1) \div 5 = 1$. The positive integer k, after washing n times, is 2.016. Find the least possible value of $k + n$.

4. (Mock ARML 2019 R1/1) Compute the value of[2]

$$\frac{\frac{\frac{\frac{1}{1+1}}{1+2+1}}{1+2+2+1}}{\frac{1+2+3+2+1}{\frac{1+2+2+1}{\frac{1+2+1}{\frac{1+1}{1}}}}}.$$

5. (CMC 10B 2019/3 & 12B 2019/3) Euler and Fermat estimate the value of the following sum:

$$3.141 + 2.718 + 1.080 + 1.741.$$

Fermat estimates the sum by rounding each addend to the nearest whole number, then adding the resulting estimates. On the other hand, Euler estimates the sum by rounding each addend to the nearest tenth. How many times closer is Euler to the exact answer than Fermat?

6. (NEMO 2019 I2) A daycare has babies that are one year, two years, and four years old. If the two-year-old babies are excluded, the average age of the remaining babies is 2.0; if the four-year-old babies are excluded, the average age of the remaining babies is 1.9; and if the one-year-old babies are excluded, the average age of the remaining babies is A. Compute A.

7. (CMC 10B 2019/4) Aluminum, Barium, and Cadmium are eating at Cheezy's Mac n' Cheese. They each order one bowl of macaroni and cheese, which is typically comprised of 70% macaroni and 30% cheese. However, Aluminum orders a bowl with 50% more cheese (not changing the amount of macaroni). What percent of the group's order is macaroni?

8. (CMC 10B 2019/6 & 12B 2019/4) What is the sum of the real values of x which satisfy

$$\frac{x^4 - 3x^2 + 2}{x^2 - 3x + 2} = 0?$$

[1] People did not mess this up, somehow.

[2] I still think this problem is super funny.

9. (CMC 10B 2018/3 & 12B 2018/2) What is the value of $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot6}+\frac{1}{4\cdot6\cdot12}$?

10. (NEMO 2019 I4) Real numbers x, y, z satisfy the inequalities
$$-8 < x < 5, \quad -2 < y < 3, \quad -5 < z < 6.$$
There exist real numbers m and n such that $m < x \cdot y \cdot z < n$ for all choices of x, y, z. Find the minimum possible value of $n - m$.

11. (CMC 10B 2018/2 & 12B 2018/1) How many integers x satisfy $(2^x - x^2)(3^x - x^3)(5^x - x^5) = 0$?[3]

12. (NEMO 2018 I2) Anna, Bobby, Carol, and David are siblings. Their ages are all different positive integers and are greater than 5, while the sum of their ages is 55. Anna is the youngest, Bobby is the second youngest, Carol is the second oldest, and David is the oldest. What is the sum of all of Bobby's possible ages?[4]

13. (CMC 10B 2018/10) George is eating from a bag of potato chips. The caloric total of a potato chip is entirely composed of integral (in grams) amounts of fats, proteins, and carbohydrates, which have 9, 4, and 4 calories per gram, respectively. After doing some research, George finds that a single potato chip has 30 calories. In total, the bag has 120 grams of fat. How many total calories does the bag have?[5]

14. (Th3's Mini Math Bee R3/1) The mean, median, and unique mode of a set of 2017 positive integers are 2017, 2017, and 2017, respectively. What is the largest possible value of the largest element of the set?[6]

15. (CMC 10B 2019/12 & 12B 2019/9) Which of the following is closest to the area of the region bounded by $x^{1000} + y^{1000} = 1$ on the Cartesian plane?[7]

16. (CMC 10A 2018/8) Let $f_1(x) = |x - 2018|$ and define $f_{n+1}(x) = f_1(f_n(x))$. Compute
$$f_1(1) - f_2(2) + f_3(3) - f_4(4) + \cdots - f_{2018}(2018).$$

17. (e-Th3 Mock ARML 2018 I7) Let Y, E, A, R be four digits, not necessarily distinct, with Y and E not equal to zero. Find the minimum possible value of $|\underline{Y}\ \underline{E}\ \underline{A}\ \underline{R} - (Y^2 + E^0 + A^1 + R^8)|$.

18. (e-Th3 Mock ARML 2018 I9) Let $P_0(x) = x^2$ and $P_n(x) = P_{n-1}(x + n) + x^2$ for all positive integers n. Find the remainder when the coefficient of x in $P_{2018}(x)$ is divided by 1000.

19. (PP's Mock AMC10/12 2016) For a real number a, consider the system of equations
$$\begin{aligned} x^2 - y^2 &= 0, \\ x^2 + y^2 &= -a^2 + 2ax + 1. \end{aligned}$$
Let A be the largest value of a such that the above system of equations has exactly two real solutions (x, y). Find A.

[3]I laugh every time I see this problem.
[4]This is the oldest problem on this collection. I wrote this problem in fifth grade! (!!!)
[5]Inspired by health class.
[6]I thought this was very MATHCOUNTS-y, but people apparently struggled with this opener. Sorry!
[7]Har har.

20. (Mock ARML 2017) The *mediant* of two reduced common fractions $\frac{w}{x}$ and $\frac{y}{z}$ is $\frac{w+y}{x+z}$. It is known that the mediant of a and $\frac{1}{20}$ is r times the mediant of a and $\frac{1}{17}$, where a and r are reduced fractions less than 1. With these conditions, the mediant of a and r is always greater than m. What is the largest possible value of m?

21. (All Algebra Asessment 2017) A faulty calculator has one malfunctioning button, namely the $\sqrt{\ }$ sign. When given a number n, pressing the $\sqrt{\ }$ sign will decrease the current value displayed on the calculator by 10%, 20%, 30%, 40%, or 50% randomly. If one were to input 1 into the calculator and press the $\sqrt{\ }$ button for a positive number of times, the final result displayed on the calculator will be a real number q. Find the sum of all possible values of q.

22. (NEMO 2019 T8) What is the sixth smallest positive real x such that $x \cdot \lfloor x \rfloor \cdot \{x\} = 6$? (Here, $\lfloor x \rfloor$ and $\{x\}$ denote the integer and fractional parts of x, respectively.)

23. (CMC 10B 2018/16 & 12B 2018/9) The exact value of

$$\sqrt{2^{2^1} + \sqrt{2^{2^2} + \sqrt{2^{2^3} + \sqrt{2^{2^4} + \dots}}}}$$

is $a + \sqrt{b}$ for positive integers a and b. Compute $100a + b$.[8]

24. (All Algebra Assessment 2017) The four unique positive integer solutions (a, b) to the equation $25^2 + 128^2 = a^2 + b^2$ are $(25, 128), (128, 25), (m, n)$, and (n, m) for positive integers $m, n \neq 25, 128$. Find mn.[9]

25. (PP's Mock AMC10/12 2016) What is the square root of the value of

$$(\sqrt{85} + \sqrt{205} + \sqrt{218})(-\sqrt{85} + \sqrt{205} + \sqrt{218})(\sqrt{85} - \sqrt{205} + \sqrt{218})(\sqrt{85} + \sqrt{205} - \sqrt{218})?$$

26. (Th3's Mini Math Bee R1/3) Caroline correctly evaluates the expression $|(x - y)(y - z)(z - x)|$ by replacing each of the variables x, y, z with a positive integer. How many possible answers between 1 and 100 (inclusive) could she have gotten?

27. (e-Th3 Mock ARML 2018 I2) Consider the graph of $f(x) = x\lfloor x^2 \rfloor$ on the Cartesian plane. Find the area of the region above the x-axis and below $f(x)$ between $x = 1$ and $x = 10$.[10]

28. (Mock ARML 2017) What is the least $x > 1$ that satisfies[11]

$$\frac{\lfloor x \rfloor \cdot \lfloor \sqrt{x} \rfloor}{\lfloor \sqrt[3]{x} \rfloor \cdot \lfloor \sqrt[4]{x} \rfloor} = 300?$$

29. (e-Th3 Mock ARML 2018 I4) Find the number of pairs of integer (a, b) with $a, b > 1$ such that $\log_b(4096 \log_a(4096))$ is a positive integer.

30. (NEMO 2019 I19) Find all reals $x > 1$ such that $\log_2\left(\log_2 x^8\right) \cdot \log_x\left(x^4 \log_x 2\right) = 12$.

[8]The original answer extraction was the sum of the digits of $a \cdot b$, as this was on a multiple choice test.

[9]I think this problem is pretty nice. I actually came up with this by applying four-square identity to Sophie-Germain, but something more straightforward works!

[10]This is a very hard opening question (unbeknownst to my past self).

[11]I am super proud of this question.

31. (Th3's Mini Math Bee R3/2) Given positive numbers x and y satisfy $\sqrt{x}+\sqrt{y}=5$, compute the minimum possible value of $\sqrt{x+9}+\sqrt{y+16}$.[12]

32. (CIME 2018I/9) Let P be the portion of the graph of
$$y=\frac{6x+1}{32x+8}-\frac{2x-1}{32x-8}$$
located in the first quadrant (not including the x and y axes). Let the shortest possible distance between the origin and a point on P be d. Find $\lfloor 1000d \rfloor$.[13]

33. (Th3's Mini Math Bee R3/3) Compute the remainder when the sum
$$\binom{3}{3}^2+\binom{4}{3}^2+\binom{5}{3}^2+\cdots+\binom{1000}{3}^2$$
is divided by 1000.[14]

34. (Th3's Mini Math Bee R4/3) Define $f(x)=x^3+20x-17$ and let r_1, r_2, r_3 be the roots of $f(x)$. Compute $f(r_1+1)\cdot f(r_2+1)\cdot f(r_3+1)$.[15]

35. (WOOT PAIME 2018/10) Find the number of triples (a,b,c) of positive integers such that $a,b,c\le 40$ and[16]
$$\frac{ab+bc+ca}{a+b+c}=\sqrt[3]{abc}.$$

36. (All Algebra Assessment 2017) Let $t=2017^{2017}$. The value of
$$\frac{1}{\pi}\sum_{n=2}^{t}\cos^{-1}\left(\frac{1+n\sqrt{n^2-4}}{n^2-1}\right),$$
where all calculations are done in radians, is a real number A. Find the integer closest to $1000A$.

Note: The notation $\cos^{-1}x$ implies that the principal value be taken; in other words, for all x, $\cos^{-1}x\in[0,\pi]$.[17]

37. (CIME 2018II/14) Positive rational numbers $x<y<z$ sum to 1 and satisfy the equation
$$(x^2+y^2+z^2-1)^3+8xyz=0.$$
Given that $\sqrt{z}$ is also rational, it can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. If $m+n<1000$, find the maximum value of $m+n$.

38. (Mock PUMaC 2017) Let M be the value of
$$\frac{\displaystyle\sum_{a=0}^{2018}2^a\binom{2018+a}{2018-a}}{\displaystyle\sum_{b=0}^{2017}2^b\binom{2017+b}{2017-b}}.$$
Compute $\lfloor M^6 \rfloor$.

[12] Another one of my favorite problems. Also, writing this helped me win the MOP 2019 Plank Countdown!

[13] Unfortunately, it seems like most people calculus bashed this.

[14] See "Newton interpolation." When I was younger, I thought this was a very nice problem. Oh, how times have changed.

[15] I also thought this problem was very nice when I was younger. Now, I think it's okay, but not stellar.

[16] This and Problem 27 are basically the same problem, actually. Also, when I proposed this problem, I unfortunately submitted a wrong answer at first.

[17] I still ike this problem a lot, even though it is a bit ugly. Also, this problem ended up causing the most trouble to contestants; sorry!

2 Combinatorics

1. (Th3's Mini Math Bee R2/1) A whole number is said to be *th3-friendly* if it has at least one 3 in its standard representation. For example, 123, 3033, and 3017 are all th3-friendly numbers. How many th3-friendly numbers are there less than 1000?[18]

2. (NEMO 2019 T1) How many ways can Danielle select two pets of different color from five brown dogs, seven grey kittens, and eight yellow parakeets? Two animals of the same color are still considered distinguishable.

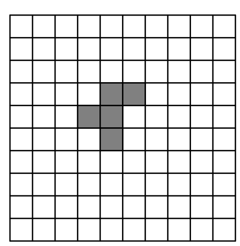
3. (NEMO 2019 I7) Carol has a fair die with faces labeled 1 to 6. She rolls the die once, and if she rolls a 1, 2, or 3, she rolls the die a second time. What is the expected value of her last roll?

4. (PP's Mock AMC10/12 2016) Ten different people are attending a party. Three of them have black shirts, four of them have red shirts, two people are wearing blue shirts, and only one person wears a white shirt. Each person then shakes hands with people of different shirt colors, and two people with the same shirt color do not shake hands. How many distinct handshakes take place?

5. (PP's Mock AMC10/12 2016) Ally is trying to create a music playlist of nine distinct songs by shuffling nine of her favorite songs. Five of them are jazz songs, three of them are hip hop, and one of them is classical. However, she wants to arrange the songs in an order such that no two songs of the same category are next to each other. How many ways can this be done?

6. (YEA CHMMC 2019 TST) Sam pulls two marbles at random from a bin of three red, four white, and five blue marbles. Given that one of his marbles is blue, what is the probability that both of his marbles are blue?[19]

7. (Mock AMC8 2017) Fabio's favorite pentomino (tile with five squares) is the F-pentomino, shaded below (the shape can be rotated or reflected). How many ways can Fabio shade five squares of a 10×10 grid such that the selected squares form an F-pentomino?



8. (Mock AMC8 2017) How many positive integers less than 1000 have a digit sum of 12? Some numbers to include are 39 and 525.

9. (Mock AMC8 2017) An urn has ten marbles, five of which are red, three of which are white, and two of which are blue. Sam selects four marbles such that there is at least one marble of each color in his set. How many ways can Sam select his set of four marbles?

10. (Mock Mandelbrot 2018) Henry and Daniel each randomly write down $1, 2$, or 3 every second for three seconds. Find the probability that the sum of Henry's numbers is equal to the sum of Daniel's numbers.

[18] A pretty unoriginal opening, but I thought it was cute anyway.

[19] Protip: conditional probability is always hard. This was #2 on the test, and had a 50% success rate.

11. (NEMO 2019 T3) A bin of one hundred marbles contains two golden marbles. Shen, Li, and Park take turns, in that order, removing marbles from the bin. If a player draws a golden marble, the game ends and the player wins. What is the probability that Shen wins?

12. (CMC 12B 2019/13) Doble writes two 1s on a whiteboard. Every second, Doble computes the sum of each pair of adjacent numbers, writing the result in between them. For example, after one second the whiteboard reads 1 2 1, and after two seconds the whiteboard reads 1 3 2 3 1. What is the sum of the numbers on the whiteboard after ten seconds?[20]

13. (Mock AMC8 2017) Al, Alex, Alexa, Alexander, and Alexandra all start with \$100. Whenever one person gives money to someone else, they give 10% of their money to that person. This encounter is called an *exchange*. Among the five people, five exchanges occur, where each person gives money to someone else at least once. After this process, what is the maximum amount of money that Al can end up with?[21]

14. (Mock ARML 2019 I2) Consider the set of rectangles formed by the unit squares of an 8×8 checkerboard. How many of these rectangles contain exactly four black squares?[22]

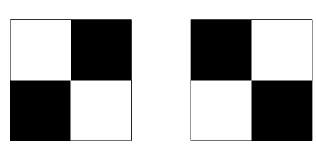
15. (CIME 2019II/6) A frog is initially at the point $(0, 0)$. Every second, the frog jumps from its current position (x, y) to either $(x + 1, y)$, $(x, y + 1)$, or $(x + 1, y + 1)$, each with equal probability. The probability that the frog eventually reaches the point $(3, 3)$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.[23]

16. (PP's Mock AMC 10 2016) A number is *elegant* if, for all $0 \le d \le 9$, the digit d does not appear more than d times. For example, the number 3 cannot appear more than 3 times in an elegant number (and the digit 0 cannot appear at all). How many positive 4-digit elegant numbers are there?[24]

17. (NEMO 2018 T5) Five real numbers are selected independently and at random from the interval $(0, 1)$. Let m be the minimum of the five selected numbers. What is the probability that $\frac{1}{4} < m < \frac{1}{2}$?[25]

18. (CMC 12B 2019/17) A man is standing at $(0, 0)$ in a desert and needs water. There are 4 oases nearby, shaped as squares with side length s, centered at $(1, 1), (-1, 1), (-1, -1)$, and $(1, -1)$, whose sides are parallel to the coordinate axes. The man doesn't know where the oases are, so he picks a direction at random and walks forever. The probability that he eventually encounters an oasis is $1/3$. What is the value of s?

19. (PP's Mock AMC 10 2016) Consider a pair of 2×2 grids. Each cell of each grid is colored either black or white, with equal probability. Any two colorings are considered non-distinct if they only differ by a rotation. For example, the two colorings shown are non-distinct. If the two grids of four squares are colored at random, what is the probability that the two resulting colorings are distinct?



[20] The original question asks to round the answer to the nearest ten thousand.
[21] Written while listening to "Bodak Yellow" by Cardi B. I wanted to call the exchanges "money moves" at first.
[22] This is a very easy problem to mess up! The team actually argued over whether to put this on the test due to its trickiness.
[23] Compare with AIME 2019I/5. Funnily enough, I did notice a similarity during the test, but I did not notice that the problems are *exactly the same* (up to a factor of 3) until after the test.
[24] This is actually pretty tricky. Choose your casework wisely!
[25] I think this is really nice!

20. (Mock MATHCOUNTS 2017) Two red triangles, two blue triangles, two red circles, two blue circles, two red squares, and two blue squares are arranged in a line such that every red shape is next to at least one other red shape and every blue shape is next to at least one other blue shape. How many such arrangements are possible?

21. (Mock ARML 2017) Consider a randomly chosen two-digit integer $\underline{a}\ \underline{b}$, with each two-digit number having an equal chance of being chosen. Let $c = a + b$ and $d = a - b$. Given that both c and d are both integers from 0 to 9, inclusive, what is the expected value of $\underline{c}\ \underline{d}$?

22. (CMC 10B 2019/24) A math class consists of five pairs of best friends who need to form five groups of two for an upcoming project. However, to promote synergy in the classroom, the teacher forbids any student to work with his best friend. In how many ways can the students pair up?

23. (Mock Mandelbrot 2018) The set of points $S = \{P(x^2, y^2), x, y \in \mathbb{Z}\}$ is plotted on the coordinate plane. How many squares with sides parallel to the coordinate axes have their four vertices in S and area at most 2018?

24. (NEMO 2019 I15) The county of NEMO-landia has five towns, with no roads built between any two of them. How many ways can the NEMO-landian board build five roads between five different pairs of towns such that it is possible to get from any town to any other town using the roads?

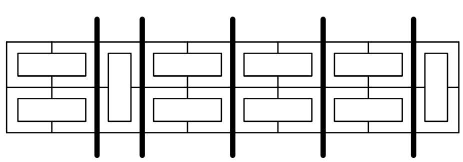
25. (YEA CHMMC 2019 TST) Daniel writes the 30 letters of the string

$$\underbrace{\texttt{CHMCHM...CHM}}_{\text{10 times}}$$

on a whiteboard. How many ways can David erase 25 of the letters such that the remaining five letters spell `CHMMC`, in that order? (The order that the letters are erased does not matter.)

26. (CIME 2019II/12) Max is playing a video game with 99 levels, labeled $1, 2, \ldots, 99$. Whenever Max completes a level, he begins the next one immediately. However, for all $1 \le n \le 99$, Max fails the n^{th} level with probability $(n+1)^{-2}$. Whenever he fails a level, he quits for the day and attempts the level again the next day. If Max first attempts the first level on Day 1 and completes the 99^{th} level on Day K, then the expected value of K can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.[26]

27. (Th3's Mini Math Bee R2/3) The interior of a 2 by 10 grid is tiled by 2 by 1 dominoes (with sides parallel to the sides of the grid). Consider a specific tiling of the grid. A line passing through the interior of the grid is *clean-cut* if the line does not pass through the interior of any dominoes. A tiling's *clean-cut number* is the number of distinct clean-cut lines of the tiling. For example, the tiling below has a clean-cut number of 5 (the clean-cut lines are bolded). What is the average clean-cut number over all possible tilings of the grid?[27]



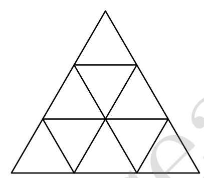
[26] Written with Eric Shen and Kyle Lee, two other members of the CMC Team. The common consensus was that this problem was too easy for its placement.

[27] Yes, horizontal lines are clean-cut. An incident regarding this happened on the contest thread, leading me to accept both answers.

28. (Mock ARML 2017) For any positive integer n, let $\#(n)$ denote the number of pairs of consecutive equal digits in the decimal representation of n. For example, $\#(330) = 1$, $\#(33003) = 2$, and $\#(3333) = 3$. Compute
$$\#(\#(1) + \#(2) + \#(3) + \cdots + \#(10^{2017})).$$

29. (CIME 2018I/9) A set of integers is *close* if is a nonempty set consisting of consecutive numbers. For instance, $\{1, 2, 3, 4\}$, $\{5, 6\}$, and $\{7\}$ are all close, but $\{2, 4, 5\}$ is not. Angela, Bill, and Charles each independently and randomly choose a close subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The expected number of elements in the intersection of the three chosen sets is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

30. (NEMO 2019 T10) Daniel paints each of the nine triangles in the diagram either crimson, scarlet, or maroon. Given that any pair of triangles sharing a side are painted different colors, how many ways can Daniel paint the diagram? Two diagrams that differ by a rotation or reflection are considered distinct.[28]



31. (e-Th3 Mock ARML 2018 I10) A six-digit positive integer is in $\mathcal{F}$ if
    - each digit is either 1, 2, 3, or 6, and
    - every pair of adjacent digits is relatively prime.

    Let $p(n)$ be defined over the positive integers as the product of the digits of n. Compute the total product of $p(k)$ as k ranges over the elements of $\mathcal{F}$.[29]

32. (CIME 2019II/9) A 19×19 checkerboard has black corners. Let $\mathcal{S}$ be the set of rectangles formed by the squares of the checkerboard. A rectangle's *darkness* is the number of black squares within its interior. The average darkness of a rectangle in $\mathcal{S}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.[30]

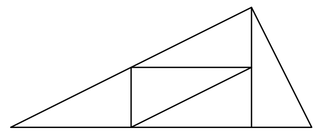
[28] I had to ask a friend for names of shades of red. Why red? I dunno.

[29] This problem ended up being easier than I thought. Hint for the nice (but hard) solution: biject to a shaded grid, where the top row represents 2 and the bottom row represents 3.

[30] The original wording unfortunately contains a dangling modifier. Oops!

3 Geometry

1. (Th3's Mini Math Bee R1/2) The following triangle can be split into five congruent right triangles. Given that one of these right triangles has perimeter 36, compute the perimeter of the larger triangle.[31]



2. (Th3's Mini Math Bee R4/1) Similar triangles ABC and DEF satisfy $AB = FD$ and $BC^2 = EF = CA$. Compute the largest possible integer perimeter of ABC.

3. (NEMO 2019 I9) Let $ABCD$ be a square with center O, and let P be a randomly chosen point in the square's interior. What is the probability that triangles AOP, BOP, COP, and DOP are all obtuse?

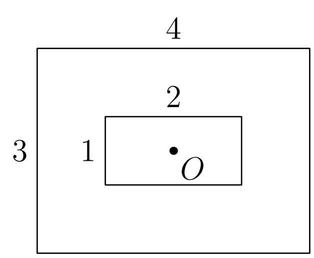
4. (NEMO 2019 T4) Points $A(5,1)$, $B(1,7)$, $C(3,10)$, and $D(7,10)$ are on the coordinate plane. Circle γ is tangent to $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$. What is the area of γ?

5. (Mock MATHCOUNTS 2017) What is the area of the triangle with side lengths $\sqrt{5}, \sqrt{12}, \sqrt{13}$?

6. (CMC 10B 2019/17) Rectangle $ABCD$ is a negligibly thin piece of paper with $AB = 10$ and $BC = 13$ sitting atop the surface of a table. Let M be the midpoint of $\overline{AB}$. Creases are made through $\overline{CM}$ and $\overline{DM}$, forming two triangular flaps. These two flaps are folded upwards so that $\overline{AM}$ and $\overline{BM}$ coincide above the surface of the table. What is the height of the resulting structure?

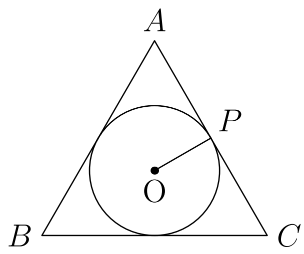
7. (Mock MATHCOUNTS 2017) Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$ and $AB : BC : CD : DA = 1 : 1 : 1+\sqrt{3} : 2$. The angle bisector of angle D meets the perimeter of $ABCD$ at X. What is the ratio $\frac{XB}{XC}$?

8. (Mock Mandelbrot 2018) The diagram given depicts a 1×2 rectangle and a 3×4 rectangle centered at a common point O. Points F and G are selected on the boundary of the inner and outer rectangle, respectively. What is the maximum possible area of $\triangle FOG$?[32]



9. (Mock AMC8 2017) Consider equilateral triangle ABC. We draw circle O such that it is tangent to sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ and let P be the point of tangency between circle O and side $\overline{CA}$. There exists a point D on side $\overline{BC}$ such that $\overline{AD}$ bisects $\overline{OP}$. Given that $OP = 7$, what is CD?

[31] The most common incorrect answer to this problem was 78, which assumes that all triangles are 9-12-15. Why so many people put this, I'll never really know.
[32] You can eyeball this!

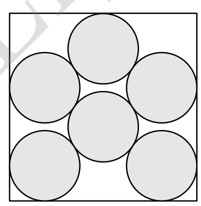


10. (PP's Mock AMC10/12 2016) Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Let $W_1X_1Y_1Z_1$ be a square such that W_1 lies on side AB, X_1 lies on side AC, and Y_1 and Z_1 lie on side BC. Then, for each integer $k \geq 2$, square $W_kX_kY_kZ_k$ has W_k on side AB, X_k on side BC, and Y_k and Z_k on side $W_{k-1}X_{k-1}$. Compute $[W_1X_1Y_1Z_1] + [W_2X_2Y_2Z_2] + [W_3X_3Y_3Z_3] + \cdots$.

11. (e-Th3 Mock ARML 2018 I5) Consider rectangle $ABCD$ with $AB = 20$ and $BC = 18$. Points X and Y are selected on sides BC and AD such that $BX/CX = 1/4$ and $AY/DY = 2/3$, respectively. Let lines AX and BY intersect at point P, and let line CP intersect $\overline{AB}$ at Z. Compute AZ/BZ.

12. (Mock MATHCOUNTS 2017) If a point X is randomly selected inside equilateral triangle ABC, what is the probability that exactly two of $\angle AXB, \angle BXC, \angle CXA$ are obtuse?

13. (CIME 2019I/2) Six circles of radius 1 are packed within a square of side length ℓ as shown. Adjacent circles are tangent to each other, while the five outer circles are tangent to the sides of the square. If $\ell = \dfrac{a}{\sqrt{b}} + c$ for positive integers a, b, c where b is not divisible by the square of any prime, compute $a + b + c$.



14. (CIME 2018II/2) Garfield and Odie are situated at $(0, 0)$ and $(25, 0)$, respectively. Suddenly, Garfield and Odie dash in the direction of the point $(9, 12)$ at speeds of 7 and 10 units per minute, respectively. During this chase, the minimum distance between Garfield and Odie can be written as $\frac{m}{\sqrt{n}}$ for relatively prime positive integers m and n. Find $m + n$.

15. (PP's Mock AMC10/12 2017) Let k be a real number. Define P_i to be the point $(i, \log_k i)$ for positive integers i on the Cartesian plane. Given that the area of the polygon $P_1P_2P_3 \dots P_{2017}$ is 2017, the value of k^{2017} can be expressed as $\sqrt{\frac{a}{b}}$, where a and b are relatively prime positive integers. What are the last two digits of $a + b$?[33]

16. (CMC 10A 2018/18) Consider quadrilateral $ABCD$ with $AB = 3$, $BC = 4$, $CD = 5$, and $DA = 6$. Circle Γ passes through A, B, C and intersects DA again at X. Given that $CX = 4$, compute the area of quadrilateral $ABCD$.[34]

[33] So I initially intended to propose this to WOOT PAIME. However, I accidentally closed the text document where I was painstakingly typing up the solution, so I ragequit. Oops.

[34] This might actually require Law of Cosines to solve. I'd like to see a solution that doesn't use it!

17. (NEMO 2019 I16) Externally tangent circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have radii 6 and 8, respectively. A line ℓ intersects $\mathcal{C}_1$ at A and B, and $\mathcal{C}_2$ at C and D. Given that $AB = BC = CD$, find AD.

18. (Mock MATHCOUNTS 2017, adapted) Farmer John is painting some region of the Cartesian plane every minute. Specifically, at the beginning of the n-th minute, Farmer John paints the square with vertices $(n, n), (n, 2n), (2n, 2n), (2n, n)$. After 20 minutes, what is the area of region that is painted?[35]

19. (OMO Fall 2019/9) Convex equiangular hexagon $ABCDEF$ has $AB = CD = EF = 1$ and $BC = DE = FA = 4$. Congruent and pairwise externally tangent circles γ_1, γ_2, and γ_3 are drawn such that γ_1 is tangent to side $\overline{AB}$ and side $\overline{BC}$, γ_2 is tangent to side $\overline{CD}$ and side $\overline{DE}$, and γ_3 is tangent to side $\overline{EF}$ and side $\overline{FA}$. Then the area of γ_1 can be expressed as $\frac{m\pi}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

20. (Mock MATHCOUNTS 2017) Regular[36] hexagon $SPRINT$ is drawn. Let O be the midpoint of $\overline{SP}$. Points B and X are on $\overline{NT}$ and $\overline{RI}$, respectively, such that $\triangle BOX$ is an isosceles right triangle. What fraction of $SPRINT$ is covered by BOX?[37]

21. (Mock MATHCOUNTS 2017) A 3×4 rectangle is rotated about its diagonal to form a 3D solid. What is the volume of the figure, in cubic units?[38]

22. (CMC 10B 2018/22 & 12B 2018/20) Consider triangle $\triangle ABC$ with $AB = 2$, $BC = 5$, and $\angle B = 60°$. Select point P_1 on $\overline{AC}$ such that $\angle BP_1C = 150°$. Points $P_2, P_3, P_4, \ldots$ are sequentially constucted such that for all positive integers k,
    - line BC meets the circumcircle of $\triangle ABP_{3k-2}$ again at point P_{3k-1},
    - line AB meets the circumcircle of $\triangle ACP_{3k-1}$ again at point P_{3k}, and
    - line AC meets the circumcircle of $\triangle BCP_{3k}$ again at point P_{3k+1}.

    What is the distance between points A and P_{2018}?

23. (CIME 2018I/4) Triangle $\triangle ABC$ has $AB = 3$, $BC = 4$, and $AC = 5$. Let M and N be the midpoints of AC and BC, respectively. If line AN intersects the circumcircle of triangle $\triangle BMC$ at points X and Y, then $XY^2 = \frac{m}{n}$ for some relatively prime positive integers m, n. Find $m + n$.

24. (CIME 2018II/8) Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. The internal angle bisector of $\angle ABC$ intersects side CA at X. The circumcircles of triangles AXB and BXC intersect sides BC and AB at M and N, respectively. The value of MN^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

25. (NEMO 2019 T9) Triangle ABC has $AB = 6$, $BC = 11$, and $CA = 7$. Let M be the midpoint of $\overline{BC}$. Points E and O are on $\overline{AC}$ and $\overline{AB}$, respectively, and point N lies on line AM. Given that quadrilateral $NEMO$ is a rectangle, find its area.

26. (Th3's Mini Math Bee R4/2) Triangle XYZ has $XY = 3, YZ = 4$ and $ZX = 5$. Let M be the midpoint of ZX and let ℓ be the line passing through points M and Y. Triangle XYZ is then reflected about ℓ to form triangle $X'YZ'$. Compute the area of the union of XYZ and $X'YZ'$.

[35] The original numbers for this problem were *nasty*.

[36] Apparently I did not include this on the original test. Yikes.

[37] I thought this problem was very good when I was younger. I now think it's somewhat okay, but the rotation solution is still nice.

[38] "Express your answer to the nearest hundredth." And, as you'll see, for good reason.

27. (e-Th3 Mock ARML 2018 I8) Regular hexagon $ABCDEF$ has side length 1. Equilateral triangle XYZ is drawn such that A is on $\overline{XY}$, C is on $\overline{YZ}$, and E is on $\overline{ZX}$. If 70% of the interior of $ABCDEF$ is also within the interior of XYZ, find XY.[39]

28. (Mock Mandelbrot 2018) Let $\triangle ABC$ be an equilateral triangle with side length 1, and let $\mathcal{T}$ be the locus of all points P such that exactly one triangle in the set $\{\triangle APB, \triangle BPC, \triangle CPA\}$ is obtuse. The points in $\mathcal{T}$ form multiple regions of finite area. Find the total area of these regions.[40]

[39] One of my favorite problems to date. The numbers work out so well!

[40] This is extremely tricky; try to get it right first try!

4 Number Theory

1. (All Algebra Assessment 2017) Find the sum of the distinct prime factors of $2^{11} + (2016 - 17)$.
2. (NEMO 2019 T2) Find the unique three-digit positive integer which
   - has a tens digit of 9, and
   - has three distinct prime factors, one of which is also a three-digit positive integer.
3. (PP's Mock AMC10/12 2017) Let $d(n)$ denote the number of positive divisors of the positive integer n. Consider the equation $d(d(d(x))) = 3$. What is the smallest possible value of x?
4. (Mock MATHCOUNTS 2017) How many divisors does $2017 \cdot (4031^2 + 3) - 4$ have?
5. (PP's Mock AMC10/12 2017) Given a positive integer n with prime factorization $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ for pairwise-distinct primes $p_1, p_2, \cdots, p_k$ and positive integers $r_1, r_2, \cdots, r_k$, let $\wedge(n)$ be the value of $r_1 + r_2 + \cdots + r_k$. For example, $\wedge(96) = \wedge(2^5 \cdot 3^1) = 5 + 1 = 6$ and $\wedge(210) = \wedge(2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1) = 1 + 1 + 1 + 1 = 4$. What is the sum of the digits of
$$\wedge\left(\wedge\left(2016^{2016^{2016}}\right)\right)?$$
6. (CIME 2018I/1) A positive integer n is a *stepstool number* if n has one less positive divisor than $n+1$. For example, 3 is a stepstool number, as 3 has 2 divisors and 4 has $2 + 1 = 3$ divisors. Find the sum of all stepstool numbers less than 300.
7. (e-Th3 Mock ARML 2018 I3) Karina writes out the positive integers less than 2018 whose (not necessarily distinct) digits are all in the set $\{2, 0, 1, 8\}$. How many of her integers are divisible by 6?
8. (Mock MATHCOUNTS 2017) A positive integer's number of divisors is equal to a quarter of its value. What is the least possible value of this number?
9. (NEMO 2019 I14) Find all primes $p \geq 5$ such that p divides $(p-3)^{p-3} - (p-4)^{p-4}$.
10. (Th3's Mini Math Bee R2/2) What is the largest positive integer n such that $n!$ divides $\dfrac{(5!)!}{(4!)!}$?[41]
11. (Mock ARML 2017) A lock's dial is offset by one degree. Two different operations can be applied to the dial: rotating the dial 41 degrees in any direction, or rotating the dial 43 degrees in any direction. The lock will open once the dial is not offset, but the lock will not open in the middle of an operation. What is the minimum number of operations needed to open the lock?
12. (PP's Mock AMC10/12 2016) The polynomial $x^3 + ax^2 + bx + c$ has three roots $\alpha \leq \beta \leq \gamma$, all of which are positive integers. Given that $4a + 2b + c = -2048$, what is the sum of all possible values of γ?[42]
13. (OMO Fall 2019/12) Let $F(n)$ denote the smallest positive integer greater than n whose sum of digits is equal to the sum of the digits of n. For example, $F(2019) = 2028$. Compute $F(1) + F(2) + \cdots + F(1000)$.
14. (OMO Fall 2019/13) Compute the number of subsets S with at least two elements of $\{2^2, 3^3, \ldots, 216^{216}\}$ such that the product of the elements of S has exactly 216 positive divisors.

[41] One of my favorite problems to date. It "tastes" like olympiad math.
[42] This problem is a mess.

15. (Mock MATHCOUNTS 2017) Given a list of numbers, a *factor pass* is a process in which each element in the list is replaced by a list of the original element's positive divisors. For example, after factor passing the number 4, one gets $1, 2, 4$. After two factor passes, 4 becomes $1,\ 1, 2,\ 1, 2, 4$. How many elements are in the list which results after factor passing 201^7 four times?[43]

16. (YEA CHMMC 2019 TST) William chooses two numbers a and b, independently and at random, from the set $\{0, 1, \dots, 2018\}$. The probability that $a^{12} - b^{12}$ is divisible by 2019 is $\frac{n}{2019^2}$. Find n.

17. (Mock ARML 2017) For $n = 3^a 67^b$ for nonnegative integers a, b, define $f(n)$ to be equal to $a + b$. Compute
$$\sum_{m,n \mid 201^7} f\left(\gcd\left(\gcd(m,n), \frac{m}{\gcd(m,n)}\right) \cdot \gcd\left(\gcd(m,n), \frac{n}{\gcd(m,n)}\right)\right).$$
Note that m and n are not necessarily distinct.[44]

[43] Compare with CIME 2019I/15. And this was on a Mock MATHCOUNTS!

[44] Written with Justin Lee. Super hard mode: replace 201^7 with 20^{17}, and $(3, 67)$ with $(2, 5)$.

5 Miscellaneous

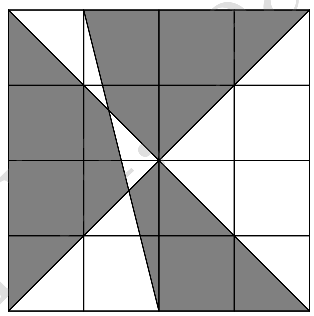
5.1 Olympiad

1. (CJMO 2019/2) Prove that if a, b, c are real numbers, and the polynomial $P(x) = x^3 + ax^2 + bx + c$ has only real roots, then
$$(b-1)^2 \leq \left(\frac{a^2-2b}{3}+1\right)^3,$$
and determine when equality holds.[45]

2. (ELMO SL 2019 A2) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all surjective functions $g : \mathbb{Z} \to \mathbb{Z}$, $f + g$ is also surjective. (A function g is *surjective* over $\mathbb{Z}$ if for all integers y, there exists an integer x such that $g(x) = y$.)[46]

5.2 Other

R1-1. (Mock ARML 2017)[47] What fraction of the square is shaded?



R1-2. (Mock ARML 2017) Let $T = TNYWR$. Let k be the nearest integer to $50 \cdot T$. An *ARML string* of length n consists of the word ARML repeated until the string has length n. For example, an ARML string of length 9 would be ARMLARMLA. Let M be the number of possible permutations of the letters of an ARML string of length k. Compute the remainder when M is divided by $k + 1$.

R1-3. (Mock ARML 2017) Let $T = TNYWR$. The monic cubic with roots T, x, and y takes the form $p^3 + ap^2 + 20p + 17$ for some real number a. Let $f(p)$ be the monic quartic with roots T, x, y, and a. Compute $f(0)$.

[45] Written with Justin Lee. I think he actually wrote most (all?) of it, but I was credited for writing the solution or something.
[46] Written while falling asleep. This problem from $\mathbb{R} \to \mathbb{R}$ drove me crazy for a while.
[47] The three problems are in a relay; $TNYWR$ means "the number you will receive," i.e. the answer from the previous problem.

6 Problem Sources

- All Algebra Assessment: https://artofproblemsolving.com/community/c594864h1359258.
- Christmas Math Competitions (CMC): https://artofproblemsolving.com/community/c798404_christmas_mathematics_competitions_year_2.
- e-Th3's Mock ARML: https://artofproblemsolving.com/community/c594864h1641711.
- Mock MATHCOUNTS Nationals[48]: https://artofproblemsolving.com/community/c5h1375861.
- "Name Eliminated" Math Open[49]: https://nemomath.github.io/.
- Online Math Open: http://internetolympiad.org/pages/11-omo.
- PP's Mock AMCs: https://artofproblemsolving.com/community/c147536_personpsychopaths_mock_contests.
- Th3Numb3rThr33's[50] Mini Math Bee: https://artofproblemsolving.com/community/c5h1478276.

[48] Recommendation: do NOT take this under contest conditions. The problems are way too hard.
[49] The acronym changes every year.
[50] Yup, that's me.