Dimension reduction for semidefinite programming

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Semidefinite programs (SDPs)

$$\begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \mathbb{S}^n_+ \end{array}$$

Formulated over vector space $\mathbb{S}^n$ of $n \times n$ symmetric matrices.

- variable $X \in \mathbb{S}^n$
- $\mathcal{A} \subseteq \mathbb{S}^n$ an affine subspace, $C \in \mathbb{S}^n$ cost matrix
- $\mathbb{S}^n_+$ cone of psd matrices

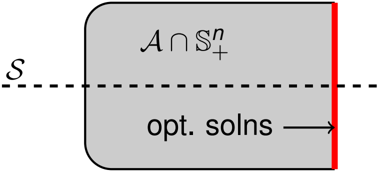
Efficiently solvable in theory; in practice, solving some instances impossible unless special structure is exploited.

Dimension reduction

Reformulate problem over subspace $\mathcal{S} \subseteq \mathbb{S}^n$ intersecting set of optimal solutions

$$\begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \mathbb{S}^n_+ \end{array} \qquad \begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \mathbb{S}^n_+ \cap \mathcal{S} \end{array}$$

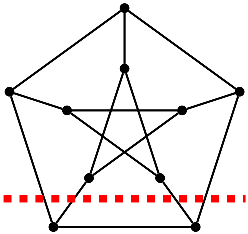
(Reformulation)



where $\mathbb{S}^n_+ \cap \mathcal{S}$ equals product $\mathcal{K}_i \times \cdots \times \mathcal{K}_m$ of 'simple' cones.

Reduction methods: *symmetry reduction* and *facial reduction*

Symmetry reduction (MAXCUT relaxation example)



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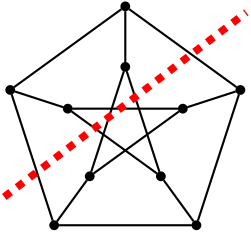

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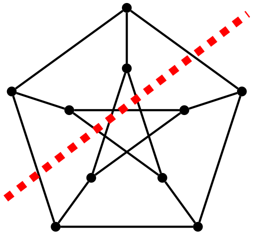

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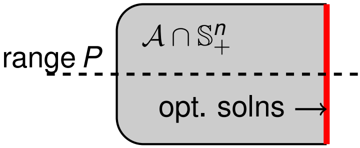
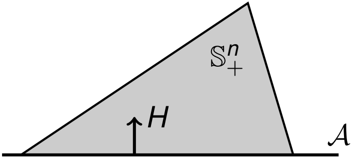


Idea: find special projection map P

- $P(X)$ optimal when X optimal.
- P explicitly constructed from automorphism group of graph.
- Range 'block-diagonal'—a direct-sum of matrix algebras.

(e.g., Schrijver '79; Gatermann-P. '03)

Facial reduction



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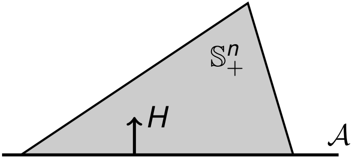

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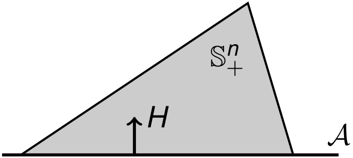

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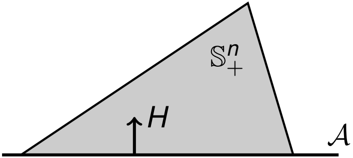

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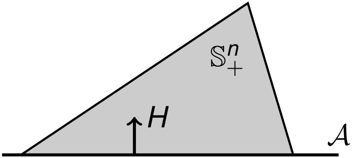

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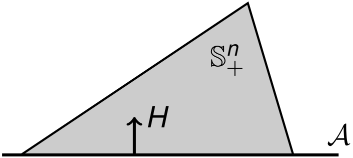

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Borwein-Wolkowicz ’81; Pataki ’00; Permenter-P. ’14

Application specific approaches

Facial reduction:

- MAXCUT (Anjos, Wolkowicz)
- QAP (Zhao,Wolkowicz)
- Sums-of-squares optimization (Permenter-P., Waki-Muramatsu)
- Matrix completion (Krislock,Wolkowicz)
- ...

Symmetry reduction:

- MAXCUT (earlier example),
- QAP (de Klerk, Sotirov);
- Markov chains (Boyd *et al.*);
- codes (Schrijver; Laurent)
- ...

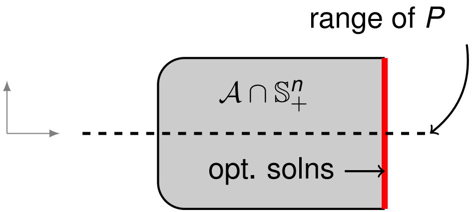
Our approach

This talk: a reduction method subsuming symmetry reduction

- Notion of ‘optimal’ reductions.
- A general purpose algorithm with optimality guarantees
- Jordan algebra interpretation; hence, easy extension to symmetric cone optimization (e.g., LP, SOCP).
- Combinatorial refinements for computational efficiency

How does symmetry reduction work?

Given SDP $\min_{X \in \mathcal{A} \cap \mathbb{S}^n_+} \operatorname{Tr} CX$, method finds special orthogonal projection $P : \mathbb{S}^n \to \mathbb{S}^n$



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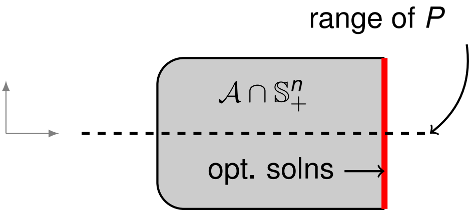

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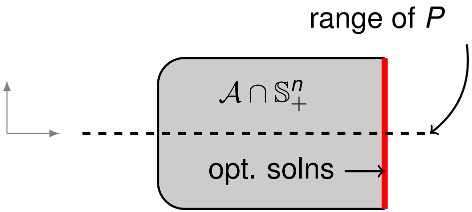

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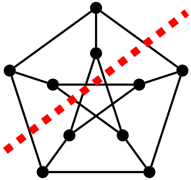
- Hence, if X feasible then $P(X)$ feasible with equal cost:

Example: a MAXCUT SDP relaxation



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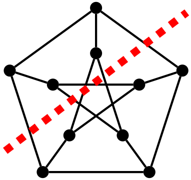

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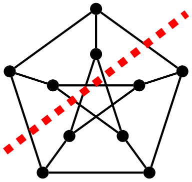

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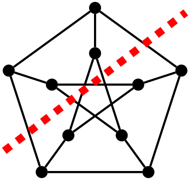

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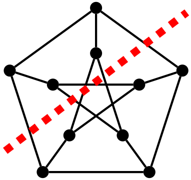

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Hence, range of P contains solutions: when X feasible, $P(X)$ feasible with equal cost.

Our approach: optimize over projections

Given SDP $\min_{X \in \mathcal{A} \cap \mathbb{S}^n_+} \langle C, X \rangle$, find map P that solves

$$
\begin{array}{ll}
\text{minimize} & \operatorname{rank} P \\
\text{subject to} & P(C) = C, P(I) = I \\
& P(\mathcal{A}) \subseteq \mathcal{A} \\
& P(\mathbb{S}^n_+) \subseteq \mathbb{S}^n_+ \\
& P : \mathbb{S}^n \to \mathbb{S}^n \text{ an orthogonal projection.}
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Main properties:

- Can be solved in polynomial time.
- Range of P structured: a *Jordan subalgebra* of $\mathbb{S}^n$.
- $\mathbb{S}^n_+ \cap \operatorname{range} P$ equals a product of symmetric cones.

Unital and positive projections and the squaring map

Theorem (Størmer)

Let $P : \mathbb{S}^n \to \mathbb{S}^n$ be an orthogonal projection satisfying $P(I) = I$. The following are equivalent.

1. $P(\mathbb{S}^n_+) \subseteq \mathbb{S}^n_+$, *i.e., P is positive.*
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Proof ($1 \Rightarrow 2$):

- $P(I) = I$ and P positive implies *Kadison's inequality*

$$P(X^2) - P(X)P(X) \succeq 0.$$

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- For $X \in \operatorname{range} P$
$$\begin{aligned}\langle I, P(X^2) - X^2\rangle &= \langle P(I), P(X^2) - X^2\rangle \\ &= \langle I, P^2(X^2) - P(X^2)\rangle \\ &= \langle I, P(X^2) - P(X^2)\rangle.\end{aligned}$$

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Hence, trace of psd matrix $P(X^2) - X^2$ is zero.

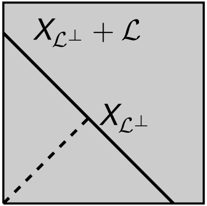
Invariant affine subspaces of projections

Theorem

For an orth. proj. map $P : \mathbb{S}^n \to \mathbb{S}^n$ and affine set $\mathcal{A} := X_{\mathcal{L}^\perp} + \mathcal{L}$ the following are equivalent.

1. $P(\mathcal{A}) \subseteq \mathcal{A}$
2. *The range of P contains $X_{\mathcal{L}^\perp}$ and is invariant under $P_{\mathcal{L}}$.*

- $X_{\mathcal{L}^\perp}$ the min.-Frobenius-norm pt. of $\mathcal{A}$
- $\mathcal{L}$ a linear subspace
- $P_{\mathcal{L}}$ the orthogonal projection map onto $\mathcal{L}$.



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The optimal subspace of $\min_{X \in \mathcal{A} \cap \mathbb{S}^n_+} \langle C, X \rangle$

Theorem (Permenter-P.)

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$$\begin{array}{ll} \textit{minimize} & \dim \mathcal{S} \\ \textit{subject to} & \mathcal{S} \ni I, X_{\mathcal{L}^\perp}, C \\ & \mathcal{S} \supseteq P_{\mathcal{L}}(\mathcal{S}) \\ & \mathcal{S} \supseteq \{X^2 : X \in \mathcal{S}\}, \end{array}$$

where affine set $\mathcal{A} = X_{\mathcal{L}^\perp} + \mathcal{L}$

Subspace optimization and solution algorithm

$$\begin{array}{ll} \text{minimize} & \dim \mathcal{S} \\ \text{subject to} & \mathcal{S} \ni C, X_{\mathcal{L}^\perp}, I \\ & \mathcal{S} \supseteq P_{\mathcal{L}}(\mathcal{S}) \\ & \mathcal{S} \supseteq \{X^2 : X \in \mathcal{S}\} \end{array}$$

$\mathcal{S} \leftarrow \operatorname{span}\{C, X_{\mathcal{L}^\perp}, I\}$
repeat
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until converged.

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Properties of algorithm:

- Optimal subspace contains each iterate (induction)

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- At termination, subspace feasible; hence, optimal.

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- At termination, subspace feasible; hence, optimal.

Properties of minimization problem:

- Feasible set closed under intersection (lattice)

Subspace optimization and solution algorithm

$$\begin{array}{ll} \text{minimize} & \dim \mathcal{S} \\ \text{subject to} & \mathcal{S} \ni C, X_{\mathcal{L}^\perp}, I \\ & \mathcal{S} \supseteq P_{\mathcal{L}}(\mathcal{S}) \\ & \mathcal{S} \supseteq \{X^2 : X \in \mathcal{S}\} \end{array}$$

$\mathcal{S} \leftarrow \mathrm{span}\{C, X_{\mathcal{L}^\perp}, I\}$
repeat
  $\mathcal{S} \leftarrow \mathcal{S} + P_{\mathcal{L}}(\mathcal{S})$
  $\mathcal{S} \leftarrow \mathcal{S} + \mathrm{span}\{X^2 : X \in \mathcal{S}\}$
until converged.

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Properties of minimization problem:

- Feasible set closed under intersection (lattice)
- A unique solution.

Combinatorial descriptions

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Can tradeoff *dimension* with *sparsity of a basis*?

Yes! Three kinds of sparse bases for $\mathcal{S}$:

- *Partition* subspaces: defined by a partition of $[n] \times [n]$.
- *Coordinate* subspaces: defined by a sparsity pattern
- *Combinatorial* subspaces: orthogonal basis of 0/1 matrices

E.g.,

$$\begin{bmatrix} a & a & b \\ a & a & b \\ b & b & c \end{bmatrix} \quad \text{vs.} \quad \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & d \end{bmatrix} \quad \text{vs.} \quad \begin{bmatrix} a & 0 & b \\ 0 & a & c \\ b & c & b \end{bmatrix}$$

Finding optimal structured subspaces

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& \mathcal{S} \text{ is a partition subspace}
\end{array}
$$

$\mathcal{P} \leftarrow \mathrm{Part}\{C, X_{\mathcal{L}^\perp}, I\}$
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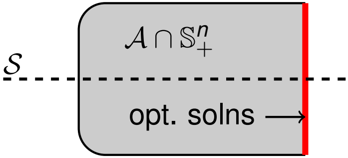
Great! But there's more...

Decomposition via Jordan algebras

Given SDP $\min_{X \in \mathcal{A} \cap \mathbb{S}^n_+} \langle C, X \rangle$, we've found a subspace invariant under $X \mapsto X^2$ containing optimal solutions:



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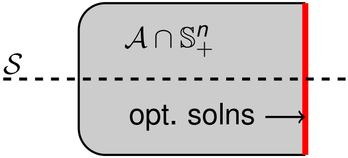

$$\mathcal{S} \supseteq \{X^2 : X \in \mathcal{S}\}$$

- Subspaces invariant under $X \mapsto X^2$ have decomposition

$$\mathcal{S} = Q \begin{pmatrix} \mathcal{S}_1 & 0 & \dots & 0 \\ 0 & \mathcal{S}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathcal{S}_m \end{pmatrix} Q^T,$$

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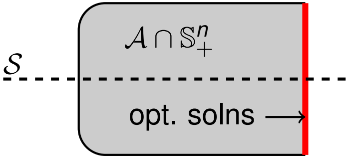

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$\mathcal{S}_i$ are *simple Jordan algebras*

- Number of distinct eigenvalues of generic element equals *rank* of $\mathcal{S}_i$—a complexity measure.

Minimizing dimension optimizes decomposition

$$\begin{array}{ll} \text{minimize} & \dim \mathcal{S} \\ \text{subject to} & \mathcal{S} \ni X_{\mathcal{L}^\perp}, C, I \\ & \mathcal{S} \supseteq P_{\mathcal{L}}(\mathcal{S}) \\ & \mathcal{S} \supseteq \{X^2 : X \in \mathcal{S}\}, \end{array}$$

All feasible subspaces have decomp. $\mathcal{S} = \oplus_{i=1}^{d_\mathcal{S}} \mathcal{S}_i$. In what sense does solution $\mathcal{S}^*$ optimize the ranks of each $\mathcal{S}_i$?

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- $\mathcal{S}^*$ minimizes $\sum_i \operatorname{rank} \mathcal{S}_i$ and $\max_i \operatorname{rank} \mathcal{S}_i$

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- $\mathcal{S}^*$ minimizes $\sum_i \operatorname{rank} \mathcal{S}_i$ and $\max_i \operatorname{rank} \mathcal{S}_i$
- *Majorization* inequalities hold, i.e., for each $m \geq 1$

$$\sum_{i=1}^{m} \operatorname{rank} \mathcal{S}_i^* \leq \sum_{i=1}^{m} \operatorname{rank} \mathcal{S}_i$$

(ranks sorted in decreasing order)

Majorization example

Subspaces (parametrized by u_i and v_i) and their rank vectors

$$\begin{pmatrix} u_1 & u_2 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_4 & 0 & 0 \\ 0 & 0 & 0 & u_5 & u_6 \\ 0 & 0 & 0 & u_6 & u_7 \end{pmatrix} \qquad \begin{pmatrix} v_1 & v_2 & 0 & 0 & 0 \\ v_2 & v_3 & 0 & 0 & 0 \\ 0 & 0 & v_4 & v_5 & v_6 \\ 0 & 0 & v_5 & v_7 & v_8 \\ 0 & 0 & v_6 & v_8 & v_9 \end{pmatrix}$$

$$r_u = (2, 1, 2) \qquad r_v = (2, 3)$$

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$$r_u = (2, 1, 2) \qquad\qquad r_v = (2, 3)$$

Vector $r'_u = (2, 2, 1)$ majorized by $r'_v = (3, 2, 0)$:

$$2 \leq 3, \qquad 2 + 2 \leq 3 + 2, \qquad 2 + 2 + 1 \leq 3 + 2 + 0$$

Jordan algebras

- Jordan algebras are commutative algebras satisfying *Jordan identity*
$$(X \circ Y) \circ X^2 = X \circ (Y \circ X^2)$$
- The vector space $\mathbb{S}^n$ a Jordan algebra if equipped with product
$$X \circ Y := \frac{1}{2}(XY + YX)$$
- The *subalgebras* of $\mathbb{S}^n$ precisely the sets closed under squaring map $X \mapsto X^2$ since
$$XY + YX = (X + Y)^2 - X^2 - Y^2.$$
- Structure theorem of Jordan-von Neumann-Wigner describes subalgebras of $\mathbb{S}^n$....

Decomposition of $\mathcal{S} \cap \mathbb{S}^n_+$

If $\mathcal{S} \subset \mathbb{S}^n$ a Jordan subalgebra, it equals direct-sum $\oplus_{i=1}^m \mathcal{S}_i$, where each $\mathcal{S}_i$ is isomorphic to one of the following:

- Algebra of Hermitian matrices with real, complex or quaternion entries
- A spin-factor algebra

Implies *cone-of-squares* $\mathcal{S} \cap \mathbb{S}^n_+$ isomorphic to product of

- PSD cones with real/complex/quaternion entries
- Lorentz cones

Yields reformulation of original SDP over this product

$$\begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \mathbb{S}^n_+ \end{array} \qquad \begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \underbrace{T(\mathcal{K}_1 \times \cdots \times \mathcal{K}_m)}_{\mathcal{S} \cap \mathbb{S}^n_+} \end{array}$$

Computational results

Comparison with reduction method of de Klerk ’10 survey (generating *-algebras from data):

instance	$\mathcal{S}^*$	$\mathcal{S}_{data}$
hamming_7_5_6	5	8256
hamming_8_3_4	5	32896
hamming_9_5_6	6	131328
hamming_9_8	6	131328
hamming_10_2	7	524800

- Table list dimension of our subspace $\mathcal{S}^* \subseteq \mathbb{S}^n$ and subspace $\mathcal{S}_{data} \subseteq \mathbb{S}^n$ found by generating *-algebra.
- Decomposing $\mathcal{S}^*$ yields a linear program.

Results: SOSOPT (Seiler ’13) Demo scripts

Script Name	n (before)	n (after)
`sosoptdemo2`	13, 3	3, 2 $\times$ 3, 1 $\times$ 7
`sosoptdemo4`	35	5 $\times$ 5, 1 $\times$ 10
`gsosoptdemo1`	9, 5	6, 3 $\times$ 2, 2
`IOGainDemo_3`	15, 8	10, 5 $\times$ 2, 3
`Chesi(1|2)_IterationWithVlin`	9, 5	6, 3 $\times$ 2, 2
`Chesi3_GlobalStability`	14, 5	8, 6, 3, 2
`Chesi(3|4)_IterationWithVlin`	9, 5	6, 3 $\times$ 2, 2
`Chesi(5|6)_Bootstrap`	19, 9	13, 6 $\times$ 2, 3
`Chesi(5|6)_IterationWithVlin`	19, 9	13, 6 $\times$ 2, 3
`Coutinho3_IterationWithVlin`	9, 5	6, 3 $\times$ 2, 2
`HachichoTibken_Bootstrap`	19, 9	12, 7, 6, 3
`HachichoTibken_IterationWithVlin`	19, 9	12, 7, 6, 3
`Hahn_IterationWithVlin`	9, 5	6, 3, 3, 2
`KuChen_IterationWithVlin`	19, 9	13, 6 $\times$ 2, 3
`Parrilo1_GlobalStabilityWithVec`	3, 2	2, 1 $\times$ 3
`Parrilo2_GlobalStabilityWithMat`	3, 2	2, 1 $\times$ 3
`VDP_IterationWithVball`	5, 4	3 $\times$ 2, 2, 1
`VDP_IterationWithVlin`	9, 5	6, 3 $\times$ 2, 2
`VDP_LinearizedLyap`	9, 5	6, 3 $\times$ 2, 2
`VannelliVidyasagar2_Bootstrap`	19, 9	13, 6 $\times$ 2, 3
`VannelliVidyasagar2_IterationWithVlin`	19, 9	13, 6 $\times$ 2, 3
`VincentGrantham_IterationWithVlin`	9, 5	6, 3 $\times$ 2, 2
`WTBenchmark_IterationWithVlin`	19, 9	13, 6 $\times$ 2, 3

Conclusions

New reduction method for SDP.

- Generalizes symmetry reduction and *-algebra-methods
- Fully algorithmic, don't need to compute automorphisms!
- Yields optimal 'block-diagonalization' (majorization)
- Can exploit combinatorial description of subspace
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Thanks for your attention!