

# Near-Potential Games: Geometry and Dynamics

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Potential games are a special class of games for which many adaptive user dynamics converge to a Nash equilibrium. In this article, we study properties of near-potential games, that is, games that are close in terms of payoffs to potential games, and show that such games admit similar limiting dynamics.

We first focus on finite games in strategic form. We introduce a distance notion in the space of games and study the geometry of potential games and sets of games that are equivalent, with respect to various equivalence relations, to potential games. We discuss how, given an arbitrary game, one can find a nearby game in these sets. We then study dynamics in near-potential games by focusing on continuous-time perturbed best response dynamics. We characterize the limiting behavior of this dynamics in terms of the upper contour sets of the potential function of a close potential game and approximate equilibria of the game. Exploiting structural properties of approximate equilibrium sets, we strengthen our result and show that for games that are sufficiently close to a potential game, the sequence of mixed strategies generated by this dynamics converges to a small neighborhood of equilibria whose size is a function of the distance from the set of potential games.

In the second part of the article, we study continuous games and show that our approach for characterizing the limiting sets in near-potential games extends to continuous games. In particular, we consider continuous-time best response dynamics and a variant of it (where players update their strategies only if there is at least  $\epsilon$  utility improvement opportunity) in near-potential games where the strategy sets are compact and convex subsets of a Euclidean space. We show that these update rules converge to a neighborhood of equilibria (or the maximizer of the potential function), provided that the potential function of the nearby potential game satisfies some structural properties. Our results generalize the known convergence results for potential games to near-potential games.

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## 1. INTRODUCTION

Potential games play an important role in game-theoretic analysis because of their appealing structural and dynamic properties. One property which is particularly relevant in the justification and implementation of equilibria is that many reasonable adaptive user dynamics converge to a Nash equilibrium in potential games (see Monderer and

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Shapley [1996a, 1996b], Fudenberg and Levine [1998], and Young [2004]). This motivates the question whether such convergence behavior extends to larger classes of games. A natural starting point is to consider games which are “close” to potential games. In this article, we formalize this notion and study *near-potential* games.

In the first part of the article, we focus on finite games in strategic form. We start by defining a distance notion on the space of games, and present a convex optimization formulation for finding the closest potential game to a given game in terms of it. We also consider best response (in pure and mixed strategies) and von Neumann-Morgenstern equivalence relations, and focus on sets of games that are equivalent (with respect to these equivalence relations) to potential games. Two games that are equivalent with respect to these equivalence relations generate the same trajectories of strategy profiles under update rules such as best response dynamics and fictitious play. Therefore, sets of games that are equivalent to potential games share the same limiting dynamic behavior as potential games. Identifying such sets enables us to extend the favorable dynamic properties of potential games to a larger set of games. Sets of games that are von Neumann-Morgenstern equivalent to potential games coincide with the set of weighted potential games, a generalization of potential games. We study the geometry of sets of games that are equivalent to potential games and the set of ordinal potential games (which is another generalization of potential games [Monderer and Shapley 1996b]). We show that these sets are nonconvex subsets of the space of games, and hence finding the closest game that is equivalent to a potential game, or the closest ordinal potential game, requires solving a nonconvex optimization problem.

Our next set of results uses this distance notion to provide a quantitative characterization of the limiting set of dynamics in finite games. We first focus on continuous-time perturbed best response dynamics, an update rule that is extensively studied in the literature, and can be used to characterize the convergence properties of stochastic fictitious play (see Hofbauer and Sandholm [2002] and Fudenberg and Levine [1998]). The trajectories of perturbed best response dynamics are characterized by differential equations which involve each player updating its strategy according to a (single-valued) smoothed best response function. For potential games, the limiting behavior of the trajectories generated by this update rule can be analyzed using a Lyapunov function that consists of two terms: the potential function and a term related to the smoothing in the best responses. Previous work has established convergence of the trajectories to the equilibrium points of the differential equations [Hofbauer and Sandholm 2002]. We first show that these equilibrium points are contained in a set of approximate equilibria, which is a subset of a neighborhood of (mixed) Nash equilibria of the underlying game, when the smoothing factor is small (Theorem 4.2, and Corollary 4.3).

We then focus on near-potential games and show that in these games, the potential function of a nearby potential game increases along the trajectories of continuous-time perturbed best response dynamics outside some approximate equilibrium set. This result is then used to establish convergence to a set of strategy profiles characterized by the approximate equilibrium set of the game, and the upper contour sets of the potential function of a close potential game (Theorem 4.4).

Exploiting the properties of the approximate equilibrium sets we strengthen our result and show that if the original game is sufficiently close to a potential game (and the smoothing factor is small) then trajectories converge to a small neighborhood of equilibria whose size approaches zero as the distance from potential games (and the smoothing factor) goes to zero (Theorem 4.5). Our analysis relies on the following observation: In games with finitely many equilibria, for sufficiently small  $\epsilon$ , the  $\epsilon$ -equilibria are contained in small disjoint neighborhoods of the equilibria. Using this observation we first establish that after some time instant the trajectories of the continuous-time perturbed best response dynamics visit only one such neighborhood. Then using the

fact that the potential function increases outside a small approximate equilibrium set, and the Lipschitz continuity of the mixed extension of the potential function, we quantify how far the trajectories can get from this set.

In the second part of the article, we focus on continuous games. The strategy sets of continuous games are nonempty compact metric spaces, and utility functions are continuous mappings from set of strategy profiles to real numbers. We also impose convexity of strategy sets on the games we consider. The distance notion we introduced for finite games naturally extends to continuous games. Using this distance notion, we define continuous near-potential games. By developing a framework similar to the one for finite games, we characterize the limiting behavior of dynamics in continuous near-potential games.

We analyze two update rules for such games: (i) continuous-time best response dynamics, (ii) best response dynamics with  $\epsilon$ -stopping. The first update rule is mathematically analogous to the continuous-time perturbed best response dynamics we considered for finite games. When defining continuous-time best response dynamics, unlike perturbed best response dynamics, we do not require a perturbation of payoffs (or smoothing of best responses). We relax this perturbation assumption, since the best response maps are single valued for continuous games under appropriate concavity assumptions on payoffs. The second update rule also has an  $\epsilon$ -stopping condition: players update their strategies only if they have a utility<sup>1</sup> improvement opportunity of at least  $\epsilon > 0$ . This  $\epsilon$ -stopping condition captures scenarios where agents (perhaps due to unmodeled costs) do not update their strategies unless there is a significant utility improvement possibility.<sup>2</sup> The trajectories of both of these update rules are generated by differential equations which are similar to the ones that described continuous-time perturbed best response dynamics. Intuitively, both update rules involve (continuous-time) strategy updates in the direction of best responses.

For the first update rule, we follow a similar approach to Theorem 4.4, and show that in near-potential games, strategy updates outside an  $\epsilon$ -equilibrium lead to an increase in the potential. This result allows us to characterize the limiting set of dynamics in terms of the upper contour sets of the potential function of a close potential game (Theorem 5.2). We also show that if the potential function satisfies additional concavity conditions, then the trajectories of dynamics converge to a neighborhood of the maximizer of the potential function, whose size we explicitly characterize (Corollary 5.3).

The differential equations that describe the evolution of trajectories for best response dynamics with  $\epsilon$ -stopping have discontinuous right-hand sides. To deal with this issue, we adopt a solution concept that involves differential inclusions (at the points of discontinuity), and allow for multiple trajectories corresponding to a single initial condition. We establish that when the  $\epsilon$  parameter is larger than the distance of the original game from a potential game (and the smoothing parameter), all trajectories of this update rule converge to an  $\epsilon$ -equilibrium set (Theorem 5.7). Moreover, for small  $\epsilon$ , this set is contained in a neighborhood of the equilibria of the game.

Other than the papers cited earlier, our article is related to a long line of literature studying convergence properties of various user dynamics in potential games: see Monderer and Shapley [1996b], Young [2004], Sandholm [2010], and Hofbauer and Sandholm [2002] for better/best response dynamics, Monderer and Shapley [1996a], Shamma and Arslan [2004], Marden et al. [2009], and Hofbauer and Sandholm [2002],

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<sup>1</sup>In this article, we use the terms utility and payoff interchangeably.

<sup>2</sup>It is documented in experimental economics that decision makers disproportionately stick with the status quo in settings that involve repeated decision making (see Samuelson and Zeckhauser [1988]). The update rule we study can be viewed as a model of status quo inertia.

and for fictitious play, and Blume [1993, 1997], Alós-Ferrer and Netzer [2010], and Marden and Shamma [2008] for logit response dynamics. It is most closely related to the recent papers Candogan et al. [2010b, 2011a]. In Candogan et al. [2010b, 2011a], we developed a framework for obtaining the closest potential game to a given game (using a distance notion slightly different than the one we employ in this article). In Candogan et al. [2011b], we studied convergence behavior of discrete-time update processes in near-potential games. We showed that the trajectories of discrete-time better/best response dynamics converge to approximate equilibrium sets while the empirical frequencies of fictitious play converge to a neighborhood of (mixed) Nash equilibrium. Moreover, the sizes of these sets diminish when the distance of the original game from the set of potential games goes to zero. This article provides a more in-depth study of the geometry of the set of potential games and sets of games that are equivalent to potential games, and investigates the limiting behavior of continuous-time update rules in near-potential games. Also it extends this framework for analyzing finite games to continuous games, where the strategy sets of players contain uncountably many elements, and to update rules with  $\epsilon$ -stopping, where players update their strategies only when they have significant utility opportunity.

The rest of the article is organized as follows: We present the game-theoretic preliminaries for our work in Section 2. In Section 3, we show how to find a close potential game to a given game, and discuss the geometry of the sets of games that are equivalent to potential games. In Section 4, we analyze continuous-time perturbed best response dynamics in near-potential games. We focus on an analogous update rule for continuous games, continuous-time best response dynamics, and a variant of it, and characterize their limiting behavior in Section 5. We close in Section 6 with concluding remarks and future work.

## 2. PRELIMINARIES

In this section, we present the game-theoretic background that is relevant to our work. Additionally, we introduce some features of potential games and structural properties of mixed equilibria that are used in the rest of the article.

### 2.1. Finite Strategic Form Games

Our main focus in the first part of the article is on finite strategic form games. In this section, we present the game-theoretic background on finite games. The relevant background on continuous games can be found in Section 5, where we extend our results to continuous games.

A (noncooperative) finite game in strategic form consists of:

- a finite set of players, denoted by  $\mathcal{M} = \{1, \dots, M\}$ ;
- strategy spaces: a finite set of strategies (or actions)  $E^m$ , for every  $m \in \mathcal{M}$ ; and
- utility functions:  $u^m : \prod_{k \in \mathcal{M}} E^k \rightarrow \mathbb{R}$ , for every  $m \in \mathcal{M}$ .

We denote a (strategic form) game by the tuple  $(\mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}})$ , the number of players in this game by  $|\mathcal{M}| = M$ , and the joint strategy space of this game by  $E = \prod_{m \in \mathcal{M}} E^m$ . We refer to a collection of strategies of all players as a *strategy profile* and denote it by  $\mathbf{p} = (p^1, \dots, p^M) \in E$ . The collection of strategies of all players but the  $m$ th one is denoted by  $\mathbf{p}^{-m}$ .

The basic solution concept in a noncooperative game is that of a (pure) Nash Equilibrium (NE). A pure Nash equilibrium is a strategy profile from which no player can unilaterally deviate and improve its payoff. Formally,  $\mathbf{p}$  is a Nash equilibrium if

$$u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m}) \leq 0,$$

for every  $q^m \in E^m$  and  $m \in \mathcal{M}$ .

To address strategy profiles that are approximately a Nash equilibrium, we use the concept of  $\epsilon$ -equilibrium. A strategy profile  $\mathbf{p} \triangleq (p^1, \dots, p^M)$  is an  $\epsilon$ -equilibrium ( $\epsilon \geq 0$ ) if

$$u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m}) \leq \epsilon$$

for every  $q^m \in E^m$  and  $m \in \mathcal{M}$ . Note that a Nash equilibrium is an  $\epsilon$ -equilibrium with  $\epsilon = 0$ .

## 2.2. Potential Games

We next describe a particular class of games that is central in this article, the class of potential games [Monderer and Shapley 1996b].

*Definition 2.1.* Consider a noncooperative game  $\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$ . If there exists a function  $\phi : E \rightarrow \mathbb{R}$  such that for every  $m \in \mathcal{M}$ ,  $p^m, q^m \in E^m$ ,  $\mathbf{p}^{-m} \in E^{-m}$ , then:

- (1)  $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m})$ , then  $\mathcal{G}$  is an *exact potential game*;
- (2)  $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = w_m(u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}))$ , for some strictly positive weight  $w_m > 0$ , then  $\mathcal{G}$  is a *weighted potential game*; and
- (3)  $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) > 0 \Leftrightarrow u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}) > 0$ , then  $\mathcal{G}$  is an *ordinal potential game*.

The function  $\phi$  is referred to as a *potential function* of the game.

This definition suggests that potential games are games in which the utility changes due to unilateral deviations for each player coincide with the corresponding change in the value of a global potential function  $\phi$ . Note that every exact potential game is a weighted potential game with  $w_m = 1$  for all  $m \in \mathcal{M}$ . From the definitions it also follows that every weighted potential game is an ordinal potential game. In other words, ordinal potential games generalize weighted potential games, and weighted potential games generalize exact potential games.

Definition 2.1 ensures that in exact, weighted, and ordinal potential games unilateral deviations from a strategy profile that maximizes the potential function (weakly) decrease the utility of the deviating player. Thus, this strategy profile corresponds to a Nash equilibrium, and it follows that every ordinal potential game has a pure Nash equilibrium.

In this article, our main focus is on exact potential games. The only exception is Section 3 where we discuss the geometries of sets of games that are equivalent to potential games and weighted and ordinal potential games. For this reason, whenever there is no confusion, we refer to exact potential games as *potential games*.

We conclude this section by providing necessary and sufficient conditions for a game to be an exact or ordinal potential game. Before we formally state these conditions, we first provide some definitions which will be used in Section 3.

*Definition 2.2 (Path – Closed Path – Improvement Path).* A *path* is a collection of strategy profiles  $\gamma = (\mathbf{p}_0, \dots, \mathbf{p}_N)$  such that  $\mathbf{p}_i$  and  $\mathbf{p}_{i+1}$  differ in the strategy of exactly one player. A path is a *closed path* (or a *cycle*) if  $\mathbf{p}_0 = \mathbf{p}_N$ . A path is an *improvement path* if  $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$  where  $m_i$  is the player who modifies its strategy when the strategy profile is updated from  $\mathbf{p}_{i-1}$  to  $\mathbf{p}_i$ .

The transition from strategy profile  $\mathbf{p}_{i-1}$  to  $\mathbf{p}_i$  is referred to as step  $i$  of the path. We refer to a closed improvement path, such that the inequality  $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$  is strict for at least a single step of the path, as a *weak improvement cycle*. We say that

a closed path is *simple* if no strategy profile other than the first and the last strategy profiles is repeated along the path. For any path  $\gamma = (\mathbf{p}_0, \dots, \mathbf{p}_N)$ , let  $I(\gamma)$  represent the “utility improvement” along the path, that is;

$$I(\gamma) = \sum_{i=1}^N u^{m_i}(\mathbf{p}_i) - u^{m_i}(\mathbf{p}_{i-1}),$$

where  $m_i$  is the index of the player that modifies its strategy in the  $i$ th step of the path.

The following proposition provides an alternative characterization of exact and ordinal potential games. This characterization will be used when studying the geometry of sets of different classes of potential games (cf. Theorem 3.4).

**PROPOSITION 2.3.** [MONDERER AND SHAPLEY 1996B; VOORNEVELD AND NORDE 1997]. (i) A finite game  $\mathcal{G}$  is an exact potential game if and only if  $I(\gamma) = 0$  for all simple closed paths  $\gamma$ .

(ii) A finite game  $\mathcal{G}$  is an ordinal potential game if and only if it does not include weak improvement cycles.

### 2.3. Mixed Strategies and $\epsilon$ -Equilibria

Our study of perturbed best response dynamics relies on the notion of mixed strategies and structural properties of mixed equilibrium sets in games. In this section we provide the relevant definitions and properties of mixed equilibria.

We start by introducing the concept of mixed strategies in games. For each player  $m \in \mathcal{M}$ , we denote by  $\Delta E^m$  the set of probability distributions on  $E^m$ . For  $x^m \in \Delta E^m$ ,  $x^m(p^m)$  denotes the probability player  $m$  assigns to strategy  $p^m \in E^m$ . We refer to the distribution  $x^m \in \Delta E^m$  as a *mixed strategy of player  $m \in \mathcal{M}$*  and to the collection  $\mathbf{x} = \{x^m\}_{m \in \mathcal{M}} \in \prod_m \Delta E^m$  as a *mixed strategy profile*. The mixed strategy profile of all players but the  $m$ th one is denoted by  $\mathbf{x}^{-m}$ . We use  $\|\cdot\|$  to denote the standard 2-norm on  $\prod_m \Delta E^m$ , that is, for  $\mathbf{x} \in \prod_m \Delta E^m$ , we have  $\|\mathbf{x}\| = \sqrt{\sum_{m \in \mathcal{M}} \sum_{p^m \in E^m} (x^m(p^m))^2}$ .

By slight (but standard) abuse of notation, we use the same notation for the mixed extension of utility function  $u^m$  of player  $m \in \mathcal{M}$ , that is;

$$u^m(\mathbf{x}) = \sum_{\mathbf{p} \in E} u^m(\mathbf{p}) \prod_{k \in \mathcal{M}} x^k(p^k), \quad (1)$$

for all  $\mathbf{x} \in \prod_m \Delta E^m$ . In addition, if player  $m$  uses some pure strategy  $q^m$  and other players use the mixed strategy profile  $\mathbf{x}^{-m}$ , the payoff of player  $m$  is denoted by

$$u^m(q^m, \mathbf{x}^{-m}) = \sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(q^m, \mathbf{p}^{-m}) \prod_{k \in \mathcal{M}, k \neq m} x^k(p^k).$$

Similarly, we denote the mixed extension of the potential function by  $\phi(\mathbf{x})$ , and we use the notation  $\phi(q^m, \mathbf{x}^{-m})$  to denote the potential when player  $m$  uses some pure strategy  $q^m$  and other players use the mixed strategy profile  $\mathbf{x}^{-m}$ .

A mixed strategy profile  $\mathbf{x} = \{x^m\}_{m \in \mathcal{M}} \in \prod_m \Delta E^m$  is a *mixed  $\epsilon$ -equilibrium* if for all  $m \in \mathcal{M}$  and  $p^m \in E^m$ ,

$$u^m(p^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \leq \epsilon. \quad (2)$$

Note that if the inequality holds for  $\epsilon = 0$ , then  $\mathbf{x}$  is referred to as a *mixed Nash equilibrium* of the game. We use the notation  $\mathcal{X}_\epsilon$  to denote the set of mixed  $\epsilon$ -equilibria.

We conclude this section with two technical lemmas which summarize some continuity properties of the mixed equilibrium mapping and the mixed extensions of potential

and utility functions. These results appeared earlier in Candogan et al. [2011b], and we refer the interested reader to this paper for the proofs.

Before we state these lemmas we first provide the relevant definitions (see Berge [1963] and Fudenberg and Tirole [1991]).

*Definition 2.4 (Upper Semicontinuous Function).* A function  $g : X \rightarrow Y \subset \mathbb{R}$  is upper semicontinuous at  $x_*$ , if, for each  $\epsilon > 0$  there exists a neighborhood  $U$  of  $x_*$ , such that  $g(x) < g(x_*) + \epsilon$  for all  $x \in U$ . We say  $g$  is upper semicontinuous, if it is upper semicontinuous at every point in its domain.

Alternatively,  $g$  is upper semicontinuous if  $\limsup_{x_n \rightarrow x_*} g(x_n) \leq g(x_*)$  for every  $x_*$  in its domain.

*Definition 2.5 (Upper Semicontinuous Multivalued Function).* A multivalued function  $g : X \rightrightarrows Y$  is upper semicontinuous at  $x_*$ , if for any open neighborhood  $V$  of  $g(x_*)$  there exists a neighborhood  $U$  of  $x_*$  such that  $g(x) \subset V$  for all  $x \in U$ . We say  $g$  is upper semicontinuous, if it is upper semicontinuous at every point in its domain and  $g(x)$  is a compact set for each  $x \in X$ .

Alternatively, when  $Y$  is compact,  $g$  is upper semicontinuous if its graph is closed, that is, the set  $\{(x, y) | x \in X, y \in g(x)\}$  is closed.

The first lemma establishes the Lipschitz continuity of the mixed extensions of the payoff functions and the potential function (with respect to the 2-norm defined earlier). The second lemma proves upper semicontinuity of the approximate equilibrium mapping.<sup>3</sup>

LEMMA 2.6. [CANDOGAN ET AL. 2011B]. *Let  $v : \prod_{m \in \mathcal{M}} E^m \rightarrow \mathbb{R}$  be a mapping from pure strategy profiles to real numbers. Its mixed extension is Lipschitz continuous with a Lipschitz constant of  $M \sum_{p \in E} |v(\mathbf{p})|$ .*

LEMMA 2.7. [CANDOGAN ET AL. 2011B]. *Let  $g : \mathbb{R} \rightrightarrows \prod_{m \in \mathcal{M}} \Delta E^m$  be a multivalued function such that  $g(\alpha) = \mathcal{X}_\alpha$ . This multivalued function is upper semicontinuous.*

Upper semicontinuity of the approximate equilibrium mapping implies that for any given neighborhood of the  $\epsilon$ -equilibrium set, there exists an  $\epsilon' > \epsilon$  such that  $\epsilon'$ -equilibrium set is contained in this neighborhood. In particular, this implies that every neighborhood of equilibria of the game contains an  $\epsilon'$ -equilibrium set for some  $\epsilon' > 0$ . Hence, if there are finitely many equilibria, the disjoint neighborhoods of these equilibria contain the  $\epsilon'$ -equilibrium set for a sufficiently small  $\epsilon' > 0$ . In Section 4, we use this observation to establish convergence of continuous-time perturbed best response dynamics to small neighborhoods of equilibria of near-potential games.

### 3. EQUIVALENCE CLASSES AND GEOMETRY OF POTENTIAL GAMES

In this section, we focus on finite games and introduce a distance notion in the space of games. We show how to find the closest potential game to a given game with respect to this notion (Section 3.1). This notion is invariant under constant additions to payoffs of players, and the invariance can be used to define an equivalence relation in the space of games. We then introduce other equivalence relations for games, such as best response equivalence in pure and mixed strategies. These equivalence relations allow for identifying sets of games that have similar dynamical properties to potential games, and thereby extending our analysis to a larger set of nearby games

<sup>3</sup>Here we fix the game, and discuss upper semicontinuity with respect to the  $\epsilon$  parameter characterizing the  $\epsilon$ -equilibrium set. We note that this is different than the common results in the literature which discuss upper semicontinuity of the equilibrium set with respect to changes in the utility functions of the underlying game (see Fudenberg and Tirole [1991]).

(Section 3.2). We also show that while the set of potential games is a subspace, the sets of equivalent games that are considered here are nonconvex subsets of the space of games. Hence, it is possible to find the closest potential game to a given game by solving a convex optimization problem, but finding the closest game that is equivalent to a potential game requires solving a nonconvex optimization problem.

### 3.1. Maximum Pairwise Difference and Near-Potential Games

We start this section by formally defining a notion of distance between games.

*Definition 3.1 (Maximum Pairwise Difference).* Let  $\mathcal{G}$  and  $\hat{\mathcal{G}}$  be two games with set of players  $\mathcal{M}$ , set of strategy profiles  $E$ , and collections of utility functions  $\{u^m\}_{m \in \mathcal{M}}$  and  $\{\hat{u}^m\}_{m \in \mathcal{M}}$  respectively. The Maximum Pairwise Difference (MPD) between these games is defined as

$$d(\mathcal{G}, \hat{\mathcal{G}}) \triangleq \max_{\mathbf{p} \in E, m \in \mathcal{M}, q^m \in E^m} |(u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m})) - (\hat{u}^m(q^m, \mathbf{p}^{-m}) - \hat{u}^m(p^m, \mathbf{p}^{-m}))|.$$

Note that the (pairwise) utility difference  $u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m})$  quantifies how much player  $m$  can improve its utility by unilaterally deviating from strategy profile  $(p^m, \mathbf{p}^{-m})$  to strategy profile  $(q^m, \mathbf{p}^{-m})$ . Thus, the MPD captures how different two games are in terms of the utility improvements due to unilateral deviations. We refer to pairs of games with small MPD as *close games*, and games that have a small MPD to a potential game as *near-potential games*.<sup>4</sup>

The MPD measures the closeness of games in terms of the difference of utility changes rather than the difference of their utility functions, that is, using terms of the form

$$|(u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m})) - (\hat{u}^m(q^m, \mathbf{p}^{-m}) - \hat{u}^m(p^m, \mathbf{p}^{-m}))|$$

instead of terms of the form  $|u^m(p^m, \mathbf{p}^{-m}) - \hat{u}^m(p^m, \mathbf{p}^{-m})|$ . This is because, as discussed in the existing literature, the difference in utility changes provides a better characterization of the strategic similarities (equilibrium and dynamic properties) between two games than the difference in utility functions. For instance, constant-sum games share similar strategic properties with zero-sum games [Fudenberg and Tirole 1991], and these games are equivalent in terms of utility changes due to unilateral deviations but not in terms of their utility functions. More generally consider two games with payoffs  $u^m(\mathbf{p})$  and  $u^m(\mathbf{p}) + v^m(\mathbf{p}^{-m})$  for every player  $m$  and strategy profile  $\mathbf{p}$ , where  $v^m : \prod_{k \neq m} E^k \rightarrow \mathbb{R}$  is an arbitrary function. It can be seen from the definition of Nash equilibrium that despite a potentially nonzero difference of their utility functions, these two games share the same equilibrium set. Intuitively, since the  $v^m$  term does not change the utility improvement due to unilateral deviations, it does not affect any of the strategic considerations in the game. While these two games have nonzero utility difference, the MPD between them is equal to zero.<sup>5</sup> Hence MPD identifies a strategic equivalence between these games.<sup>6</sup> We defer the discussion of other relevant notions of strategic equivalence to Section 3.2.

<sup>4</sup>Note that while defining MPD, we take the maximum over  $\mathbf{p} \in E$  and  $q^m \in E^m$ , as opposed to  $\mathbf{x} \in \Delta E$  and  $y^m \in \Delta E^{-m}$ . We want to emphasize that these are equivalent, as the maximum pairwise difference over mixed strategies is upper bounded by that over pure strategies.

<sup>5</sup>Since two games that are not identical may have MPD equal to zero it follows that MPD is not a norm in the space of games. However, it can be easily shown that it is a seminorm.

<sup>6</sup>If two games have zero MPD, then the equilibrium sets of these games are identical. However, payoffs at equilibria may differ, and hence they may be different in terms of their efficiency (such as Pareto efficiency) properties (see Candogan et al. [2011a]).

Using the distance notion defined before and Definition 2.1, we next formulate the problem of finding the closest potential game to a given game. Assume that a game with utility functions  $\{u^m\}_m$  is given. The closest potential game (in terms of MPD) to this game, with payoff functions  $\{\hat{u}^m\}_m$ , and potential function  $\phi$  can be obtained by solving the following optimization problem.

$$(P) \quad \min_{\phi, \{\hat{u}^m\}_m} \max_{\mathbf{p} \in E, m \in \mathcal{M}, q^m \in E^m} \left| (u^m(q^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})) - (\hat{u}^m(q^m, \mathbf{p}^{-m}) - \hat{u}^m(\mathbf{p}^m, \mathbf{p}^{-m})) \right|$$

subject to  $\phi(\bar{q}^m, \bar{\mathbf{p}}^{-m}) - \phi(\bar{\mathbf{p}}^m, \bar{\mathbf{p}}^{-m}) = \hat{u}^m(\bar{q}^m, \bar{\mathbf{p}}^{-m}) - \hat{u}^m(\bar{\mathbf{p}}^m, \bar{\mathbf{p}}^{-m})$ ,  
for all  $m \in \mathcal{M}$ ,  $\bar{\mathbf{p}} \in E$ ,  $\bar{q}^m \in E^m$ .

Note that the difference  $(u^m(q^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})) - (\hat{u}^m(q^m, \mathbf{p}^{-m}) - \hat{u}^m(\mathbf{p}^m, \mathbf{p}^{-m}))$  is linear in  $\{\hat{u}^m\}_m$ . Thus, the objective function is the maximum of such linear functions, and hence is convex in  $\{\hat{u}^m\}_m$ . The constraints of this optimization problem guarantee that the game with payoff functions  $\{\hat{u}^m\}_m$  is a potential game with potential function  $\phi$ . Note that these constraints are linear. Hence, the closest potential game (in terms of MPD) to a given game can be found by solving convex optimization problem (P). Additionally, the preceding problem can be reformulated as a linear program, hence the closest potential game can be found in polynomial time in input size (i.e., the number of strategy profiles). For general games, the input is exponential in the number of players, hence the runtime will not be polynomial in the number of players.<sup>7</sup>

Efficient computation of closest potential games provides a valuable tool for analysis of dynamics in arbitrary games. It allows for systematically approximating a given game with a potential game, which can be used to characterize the limiting behavior of dynamics in the original game.

### 3.2. Equivalence Classes and Geometry

In this section, we define different equivalence relations for games and focus on the sets of games that are equivalent to potential games. The games that belong to these sets have similar dynamic behavior to potential games under different update rules. This enables our analysis of dynamics in the subsequent sections to apply more broadly to games that will be nearby such sets.

We start by introducing an equivalence relation that involves MPD. The MPD between two games can be zero even when these games are not identical. For instance, as discussed in the previous section the MPD between games with payoff functions  $u^m(\mathbf{p})$  and  $u^m(\mathbf{p}) + v^m(\mathbf{p}^{-m})$  equal to zero. We refer to games that have MPD equal to zero as *MPD-equivalent games*. It was established in Candogan et al. [2011a] that two games with payoffs  $\{u^m\}$  and  $\{\hat{u}^m\}$  have identical (pairwise) utility differences (hence zero MPD) if and only if there exist functions  $v^m : \prod_{k \neq m} E^k \rightarrow \mathbb{R}$  such that for all players  $m$  and strategy profiles  $\mathbf{p}$ , we have  $u^m(\mathbf{p}) = \hat{u}^m(\mathbf{p}) + v^m(\mathbf{p}^{-m})$ . This result provides a necessary and sufficient condition for MPD-equivalence between games.

We first consider games that are MPD-equivalent to potential games. It can be seen that any game that is MPD-equivalent to an (exact) potential game is also an (exact) potential game with the same potential function. Therefore, the set of games that are MPD-equivalent to potential games is the set of potential games itself.

<sup>7</sup>For games that admit a compact representation, such as graphical games, the complexity of the problem also decreases (see Candogan et al. [2012]).

We next introduce three additional equivalence relations that enable us to extend the set of potential games to larger sets with similar dynamic properties.

*Definition 3.2.* Let  $\mathcal{G}$  and  $\hat{\mathcal{G}}$  be two games with set of players  $\mathcal{M}$ , set of strategy profiles  $E$ , and collections of utility functions  $\{u^m\}_{m \in \mathcal{M}}$  and  $\{\hat{u}^m\}_{m \in \mathcal{M}}$  respectively. These games are:

- *pure strategy best response equivalent* if for any player  $m$  and pure strategy profile  $\mathbf{p}^{-m}$  we have  $\arg \max_{q^m \in E^m} u^m(q^m, \mathbf{p}^{-m}) = \arg \max_{q^m \in E^m} \hat{u}^m(q^m, \mathbf{p}^{-m})$ ,
- *mixed strategy best response equivalent* if for any player  $m$  and mixed strategy profile  $\mathbf{x}^{-m}$  we have  $\arg \max_{q^m \in E^m} u^m(q^m, \mathbf{x}^{-m}) = \arg \max_{q^m \in E^m} \hat{u}^m(q^m, \mathbf{x}^{-m})$ , and
- *VNM-equivalent* if for every player  $m$ , there exists a nonnegative constant  $w^m > 0$ , and a function  $v^m : \prod_{k \neq m} E^k \rightarrow \mathbb{R}$  such that for any strategy profile  $\mathbf{p}$ , we have  $u^m(\mathbf{p}) = w^m \hat{u}^m(\mathbf{p}) + v^m(\mathbf{p}^{-m})$ .

It can be easily seen from the definition that these are valid equivalence relations. The definition also suggests that VNM-equivalence reduces to MPD-equivalence when the weights  $\{w^m\}$  are chosen to be equal to 1 for all players, that is, two games that are MPD-equivalent are VNM-equivalent. Moreover, two games which are VNM-equivalent have identical best responses for every player  $m$  and every opponent mixed strategy profile  $\mathbf{x}^{-m}$ . This shows that games that are VNM-equivalent are mixed strategy best response equivalent. Clearly two games that are mixed strategy best response equivalent are pure strategy best response equivalent. Hence MPD, VNM, mixed and pure strategy best response equivalences define progressively more inclusive equivalence relations between games. An important consequence for our purposes is that these equivalence relations define a sequence of nested sets (by considering sets of games that are equivalent to potential games) containing the set of potential games. The paper Morris and Ui [2004] studies other relevant equivalence relations and it follows from their analysis that VNM-equivalence is strictly included in the mixed strategy best response equivalence, that is, there are games that are mixed strategy best response equivalent, but not VNM-equivalent.<sup>8</sup>

It was shown in Morris and Ui [2004] that the set of games that are VNM-equivalent to potential games is the set of weighted potential games. To see this note from Definitions 2.1 and 3.2 that a game which is VNM-equivalent (with weights  $\{w^m\}$ ) to an exact potential game with potential function  $\phi$  is a weighted potential game with the same weights and potential function.

<sup>8</sup>The definition of best response equivalence in Morris and Ui [2004] is closely related to the equivalence relations defined here. In this article, for each player  $m$ , the authors first define the function  $\Lambda_m$  such that

$$\Lambda_m(q_m, X | u^m) = \left\{ \lambda_m \in \Delta E^{-m} \mid \sum_{\mathbf{p}^{-m} \in E^{-m}} \lambda_m(\mathbf{p}^{-m}) (u^m(q^m, \mathbf{p}^{-m}) - u^m(p^m, \mathbf{p}^{-m})) \geq 0 \text{ for all } p^m \in X \right\},$$

where  $\Delta E^{-m}$  denotes the set of probability distributions over strategy profiles that belong to  $\prod_{k \neq m} E^k$ . That is for a given strategy  $q^m \in E^m$ ,  $\Lambda_m(q_m, X | u^m)$  denotes the set of distributions over  $\prod_{k \neq m} E^k$  such that when the strategy profile of its opponents is drawn according to a distribution from this set, player  $m$  prefers strategy  $q^m$  to all other strategies in  $X$ . In Morris and Ui [2004], authors define two games to be best response equivalent if  $\Lambda_m(q_m, E^m | u^m) = \Lambda_m(q_m, E^m | \hat{u}^m)$  for all players  $m$  and strategies  $q^m \in E^m$ . Note that this equivalence relation is contained in mixed strategy best response equivalence considered here, as it allows for distributions over  $\prod_{k \neq m} E^k$  that are not necessarily mixed strategy profiles. In Morris and Ui [2004], it is established that there is nontrivial gap between best response equivalence and VNM-equivalence, that is, there are games that are best response equivalent, but not VNM-equivalent. Since their best response equivalence is included in our mixed strategy best response equivalence, the same holds for our mixed strategy best response equivalence.

Pure strategy best response equivalence is relevant in the context of best response dynamics (in pure strategies) since it leads to similar trajectories under this dynamics. Similarly, mixed strategy best response equivalence applies when considering dynamics that involve each player choosing a best response against its opponents' mixed strategies; an example of such dynamics is fictitious play (see Monderer and Shapley [1996a], and also Section 4). Two games with payoff functions  $\{u^m\}$  and  $\{w^m u^m\}$ , where  $w^m > 0$  are per-player weights, are VNM-equivalent (and hence mixed and pure strategy best response equivalent). Therefore, VNM-equivalence, while relevant for best-response-type dynamics (in pure and mixed strategies), does not apply when considering dynamics that involve absolute values of payoff gains due to unilateral deviations (such as logit response; see Blume [1993, 1997]). For such dynamics, it is natural to focus on classes of games which are MPD-equivalent.

We conclude this section by showing that the sets of games that are pure/mixed strategy best response or VNM-equivalent to potential games are nonconvex. This implies that for a given game the problem of finding the closest game in these equivalence classes requires solving a nonconvex optimization problem. In contrast, as explained in the previous subsection, the closest exact potential game to a given game can be found by solving a convex optimization problem.

We denote the space of all games with set of players  $\mathcal{M}$  and set of strategy profiles  $E$  by  $\mathcal{G}_{\mathcal{M},E}$ . Before we present our result on the sets of potential games, we introduce the notion of convexity for sets of games.

*Definition 3.3.* Let  $B \subset \mathcal{G}_{\mathcal{M},E}$ . The set  $B$  is said to be convex if and only if for any two game instances  $\mathcal{G}_1, \mathcal{G}_2 \in B$  with collections of utilities  $u = \{u^m\}_{m \in \mathcal{M}}$ ,  $v = \{v^m\}_{m \in \mathcal{M}}$  respectively

$$\langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{\alpha u^m + (1 - \alpha)v^m\}_{m \in \mathcal{M}} \rangle \in B,$$

for all  $\alpha \in [0, 1]$ .

Shortly we show that the set of exact potential games is a subspace of the space of games (i.e., a subset characterized in terms of linear equalities), whereas the sets of games that are equivalent to potential games are nonconvex.

**THEOREM 3.4.** (i) *The sets of exact potential games are a subspace of  $\mathcal{G}_{\mathcal{M},E}$ .*

(ii) *The sets of games that are pure/mixed strategy best response or VNM-equivalent to potential games are nonconvex subsets of  $\mathcal{G}_{\mathcal{M},E}$ .*

**PROOF.** (i) Definition 2.1 implies that the set of exact potential games is the subset of space of games where the utility functions satisfy the condition

$$u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}) = \phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}),$$

for some function  $\phi$ , and all strategy profiles  $\mathbf{p}$ , and players  $m$ . Note that for each  $\mathbf{p}$  and  $m$  this is a linear equality constraint on the utility functions  $\{u^m\}$  and potential function  $\phi$ . The set of all utility functions and potential functions that correspond to a potential game is the intersection of the sets defined by these linear equality constraints, and hence is a subspace. Projecting this set onto the collection of utility functions, or to  $\mathcal{G}_{\mathcal{M},E}$ , it follows that the set of exact potential games is also a subspace of the space of games.

(ii) We prove the claim by showing that the convex combination of two games that are VNM-equivalent to potential games is not pure strategy best response equivalent to a potential game. This implies that the sets of games that are pure/mixed strategy best response or VNM-equivalent to potential games are nonconvex, since games that are VNM-equivalent to potential games are also mixed strategy best response equivalent,

Table I. Payoffs and Potential Function in  $\mathcal{G}_1$ 

$(u^1, u^2)$	A	B	$\phi$	A	B
A	0,0	0,4	A	0	12
B	2,0	8,6	B	2	20

Table II. Payoffs and Potential Function in  $\mathcal{G}_2$ 

$(u^1, u^2)$	A	B	$\phi$	A	B
A	4,2	6,0	A	20	18
B	0,8	0,0	B	8	0

Table III. Payoffs in  $\mathcal{G}_3$ 

$(u^1, u^2)$	A	B
A	2,1	3,2
B	1,4	4,3

and games that are mixed strategy best response equivalent to potential games are pure strategy best response equivalent.

In Table I we present the payoffs and the potential function in a two-player game,  $\mathcal{G}_1$ , where each player has two strategies. Given strategies of both players the first table shows payoffs of players (the first number denotes the payoff of the first player), the second table shows the corresponding potential function. In both tables the first column stands for actions of the first player and the top row stands for actions of the second player. Note that this game is a weighted potential game with weights  $w_1 = 1$ ,  $w_2 = 3$ , hence it is VNM-equivalent to a potential game.

Similarly, another game  $\mathcal{G}_2$  is defined in Table II. Note that this game is also a weighted potential game (and VNM-equivalent to a potential game) with weights  $w_1 = 3$ ,  $w_2 = 1$ .

In Table III, we consider a game  $\mathcal{G}_3$ , in which the payoffs are averages (hence convex combinations) of payoffs of  $\mathcal{G}_1$  and  $\mathcal{G}_2$ .

Note that this game has a best response cycle.

$$(A, A) - (A, B) - (B, B) - (B, A) - (A, A)$$

On the other hand, games that are pure strategy best response equivalent to potential games cannot have best response cycles [Voorneveld and Norde 1997].

The previous example shows that the sets of two-player games that are pure/mixed strategy best response or VNM-equivalent to potential games is nonconvex. For general  $n$ -player games, the claim immediately follows by constructing two  $n$ -player weighted potential games, and embedding  $\mathcal{G}_1$  and  $\mathcal{G}_2$  in these games.  $\square$

The example provided in the proof of part (ii) of Theorem 3.4 also shows that the sets of weighted and ordinal potential games are also nonconvex. To see this note that  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are weighted potential games and hence ordinal potential games, but their convex combination,  $\mathcal{G}_3$ , cannot be pure or mixed strategy best response equivalent to an exact potential game, due to presence of an improvement cycle (see Proposition 2.3).

Theorem 3.4 suggests that the closest exact potential game to a given game can be systematically obtained by solving a convex optimization problem, whereas for finding the closest games in the equivalence classes of potential games there is no such framework. This suggests that in order to analyze dynamics in a given game the closest (in terms of MPD) potential game can be obtained (by solving a convex optimization problem), and the closeness of these games can be exploited. On the other hand, a similar approach which involves using a game that is equivalent to a potential game needs to

employ a heuristic for finding such a game close to the original game, as finding the closest such game requires solving a nonconvex optimization problem.

In the rest of the article, we shift our attention to analysis of dynamics in near-potential games. In these sections, we do not discuss how a close potential game to a given game is obtained, but we just assume that a close exact potential game with potential function  $\phi$  is known and the MPD between this game and the original game is  $\delta$ . We provide characterization results on limiting dynamics for a given game in terms of  $\phi$  and  $\delta$ .

#### 4. PERTURBED BEST RESPONSE DYNAMICS

In this section, we analyze convergence of continuous-time perturbed best response dynamics in near-potential games. This update rule assumes that each agent continuously updates its strategy, by using a “noisy” best response to its opponents’ strategies. Perturbed best response dynamics is widely studied in the literature, and it is closely related to stochastic fictitious play (an update rule similar to discrete-time fictitious play, where the payoffs of agents are subject to stochastic shocks) through stochastic approximation theory [Benaim 1996; Benaim and Hirsch 1999; Hofbauer and Sandholm 2002]: “limit points” of stochastic fictitious play are the “recurrent states” of perturbed best response dynamics. Hence, limiting behavior of stochastic fictitious play can be characterized in terms of the limiting behavior of continuous-time perturbed best response dynamics, which in general admits a more tractable analysis than fictitious play. Using this approach, convergence properties of different versions of (stochastic) fictitious play have been established [Benaim and Hirsch 1999; Hofbauer and Sandholm 2002; Fudenberg and Levine 1998; Fudenberg and Takahashi 2008].

It is known that for potential games the trajectories of perturbed best response dynamics converge [Hofbauer and Sandholm 2002; Shamma and Arslan 2004], and the convergence proof follows from a Lyapunov-function argument. Extending these results, we first establish that in potential games, provided that the smoothing factor is small, the limiting point of perturbed best response dynamics is contained in a small neighborhood of equilibria of the game. Then we show that similar convergence results can be obtained for near-potential games. In particular, we establish that provided that the game is sufficiently close to a potential game, the trajectories of perturbed best response dynamics converge to a neighborhood of equilibria, whose size is characterized in terms of the distance of the original game from a potential game.

Before we define perturbed best response dynamics, we introduce the notion of a smoothed best response. The smoothed (mixed strategy) best response of player  $m$  is a function  $\tilde{\beta}^m : \prod_{k \neq m} \Delta E^k \rightarrow \Delta E^m$  such that

$$\tilde{\beta}^m(\mathbf{x}^{-m}) = \operatorname{argmax}_{y^m \in \Delta E^m} u^m(y^m, \mathbf{x}^{-m}) + H^m(y^m). \quad (3)$$

Here  $H^m : \Delta E^m \rightarrow \mathbb{R}_+$  is a strictly concave function such that  $\max_{y^m \in \Delta E^m} H^m(y^m) = \tau > 0$  for all  $y^m \in \Delta E^m$ , and  $\|\nabla H^m(y^m)\|$  approaches to infinity as  $y^m$  approaches to the boundary of  $\Delta E^m$ . Since  $H^m$  is strictly concave and  $u^m(y^m, \mathbf{x}^{-m})$  is linear in  $y^m$ , it follows that the smoothed best response is single valued.<sup>9</sup> We refer to  $H^m$  as the smoothing function associated with the update rule, and  $\tau$  as the smoothing parameter.

<sup>9</sup>Linearity in  $y^m$  follows since the mixed extension of the payoff function is linear in the probability distribution chosen by the agent, that is,  $u^m(y^m, \mathbf{x}^{-m}) = \sum_{q^m \in E^m} y^m(q^m) u^m(q^m, \mathbf{x}^{-m})$ .

A smoothing function that is commonly used in the literature is the entropy function [Hofbauer and Sandholm 2002; Fudenberg and Levine 1998], which is given by

$$H^m(x^m) = -c \sum_{q^m \in E^m} x^m(q^m) \log(x^m(q^m)), \quad x^m \in \Delta E^m, \quad (4)$$

where  $c > 0$  is a fixed constant. Note that for this function, if the value of  $c$  is small, then strategy  $\tilde{\beta}^m$  is an approximate best response of player  $m$ .

We are now ready to provide a formal definition of continuous-time perturbed best response dynamics.

*Definition 4.1 (Perturbed Best Response Dynamics).* Perturbed best response dynamics is the update rule where the mixed strategy of each player  $m \in \mathcal{M}$  evolves according to the differential equation

$$\dot{x}^m = \tilde{\beta}^m(\mathbf{x}^{-m}) - x^m. \quad (5)$$

Note that presence of the smoothing factor guarantees that the smoothed best response is single valued and continuous, and therefore the differential equation in (5) is well defined. We say that continuous-time perturbed best response dynamics converges to a set  $S$ , if starting from any mixed strategy profile, the trajectory defined by the preceding differential equation converges to  $S$ , that is,  $\inf_{\mathbf{x} \in S} \|\mathbf{x}_t - \mathbf{x}\| \rightarrow 0$  as  $t \rightarrow \infty$ .

If the original game is a potential game with potential function  $\phi$ ,  $V(\mathbf{x}) = \phi(\mathbf{x}) + \sum_m H^m(x^m)$  is a strict Lyapunov function for the continuous-time perturbed best response dynamics (see Hofbauer and Sandholm [2002]). Thus, it follows that in potential games, perturbed best response dynamics converges to the set of strategy profiles for which  $\dot{x} = 0$ , or equivalently  $\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m = 0$  for all players  $m$ . Note that due to the presence of the smoothing term, this set does not coincide with the set of equilibria of the game. Our first result shows that if the smoothing term is bounded by  $\tau$ , this set is contained in the  $\tau$ -equilibrium set of the game.

**THEOREM 4.2.** *In potential games, the trajectory  $\mathbf{x}_t$  of continuous-time perturbed best response dynamics with smoothing parameter  $\tau$  converges to  $\mathcal{X}_\tau$ , the set of  $\tau$ -equilibria of the game.*

**PROOF.** For each  $\mathbf{x} \notin \mathcal{X}_\tau$ , there exists a player  $m$  and strategy  $y^m \in \Delta E^m$  such that  $u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) > \tau$ . Since  $0 \leq H^m(z^m) \leq \tau$  for all  $z^m \in \Delta E^m$  it follows that  $u^m(y^m, \mathbf{x}^{-m}) + H^m(y^m) - u^m(x^m, \mathbf{x}^{-m}) - H^m(x^m) > 0$ , and thus  $x^m \neq \tilde{\beta}^m(\mathbf{x}^{-m})$ . Since perturbed best response dynamics converges to the set of strategy profiles for which  $\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m = 0$  for all players  $m$ , it follows that no  $\mathbf{x} \notin \mathcal{X}_\tau$  belongs to the limiting set. Hence, the limiting points of perturbed best response dynamics are contained in  $\mathcal{X}_\tau$ .  $\square$

The smoothing term present in the definition of continuous-time perturbed best response dynamics ensures that the best responses are single valued, and the corresponding differential equation is well defined. In the context of stochastic fictitious play, the smoothing parameter quantifies the size of the stochastic shocks applied to payoffs of players. Using the structure of the mixed equilibrium sets of games and the previous result, we next show that in potential games provided that the smoothing parameter is small, continuous-time perturbed best response dynamics converges to a small neighborhood of the equilibria of the game.

**COROLLARY 4.3.** *Consider a potential game  $\mathcal{G}$ , and let a constant  $r > 0$  be given. There exists a sufficiently small  $\tau > 0$  such that the trajectory  $\mathbf{x}_t$  of continuous-time perturbed best response dynamics with smoothing parameter  $\tau$  converges to  $r$ -neighborhood of the equilibria of the game.*

PROOF. Lemma 2.7 (ii) implies that for small enough  $\tau > 0$ ,  $\mathcal{X}_\tau$  is contained in a  $r$ -neighborhood of the equilibria. Using this observation, the result immediately follows from Theorem 4.2.  $\square$

We next focus on near-potential games and investigate the convergence behavior of continuous-time perturbed best response dynamics in such games. Our first result establishes that in near-potential games, starting from a strategy profile that is not an  $\epsilon$ -equilibrium (where  $\epsilon > M(\delta + \tau)$ ), the potential of a nearby potential game increases with rate at least  $\epsilon - M(\delta + \tau)$ . Using this result we also characterize the limiting behavior of continuous-time perturbed best response dynamics in near-potential games in terms of the approximate equilibrium set of the game and the upper contour sets of the potential function of a close potential game.

**THEOREM 4.4.** *Consider a game  $\mathcal{G}$  and let  $\hat{\mathcal{G}}$  be a close potential game such that  $d(\mathcal{G}, \hat{\mathcal{G}}) \leq \delta$ . Denote the potential function of  $\hat{\mathcal{G}}$  by  $\phi$ , and the smoothing parameter by  $\tau > 0$ .*

*Let  $\mathbf{x}_t$  be the trajectory of continuous-time perturbed best response dynamics in  $\mathcal{G}$ . Then:*

(i)  $\dot{\phi}(\mathbf{x}_t) > \epsilon - M(\delta + \tau)$  if  $\mathbf{x}_t \notin \mathcal{X}_\epsilon$ .

(ii) *The trajectory  $\mathbf{x}_t$  converges to the set of mixed strategy profiles which have potential larger than the minimum potential in the  $M(\delta + \tau)$ -equilibrium set of the game, that is,  $\mathbf{x}_t$  converges to  $\{\mathbf{x} | \phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_{M(\delta+\tau)}} \phi(\mathbf{y})\}$  as  $t \rightarrow \infty$ .*

PROOF. (i) When continuous-time perturbed best response dynamics is used, players always modify their mixed strategies in the direction of their best responses. From the definition of mixed extension of  $\phi$  it follows that

$$\dot{\phi}(\mathbf{x}_t) = \sum_m \nabla_m \phi^T(\mathbf{x}_t) (\tilde{\beta}^m(\mathbf{x}_t^{-m}) - x_t^m) = \sum_m (\phi(\tilde{\beta}^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - \phi(x_t^m, \mathbf{x}_t^{-m})), \quad (6)$$

where  $\nabla_m$  denotes the collection of partial derivatives with respect to the strategies of player  $m$ , or equivalently the entries of the  $x^m$  vector.

Observe that if  $\mathbf{x}_t \notin \mathcal{X}_\epsilon$ , then there exists a player  $m$  and strategy  $y^m$  such that

$$u^m(y^m, \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) > \epsilon. \quad (7)$$

By definition of  $\tilde{\beta}^m$  it follows that

$$u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) + H^m(\tilde{\beta}^m(\mathbf{x}^{-m})) \geq u^m(y^m, \mathbf{x}^{-m}) + H^m(y^m). \quad (8)$$

Since  $0 \leq H^m(x^m) \leq \tau$ , we obtain

$$u^m(\tilde{\beta}^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) \geq u^m(y^m, \mathbf{x}^{-m}) - \tau. \quad (9)$$

Eqs. (8) and (9) together with the definition of MPD and the aforesaid inequality imply that

$$\begin{aligned} \phi(\tilde{\beta}^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - \phi(x_t^m, \mathbf{x}_t^{-m}) &\geq u^m(\tilde{\beta}^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) - \delta \\ &\geq u^m(y^m, \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) - \delta - \tau \\ &> \epsilon - \delta - \tau. \end{aligned} \quad (10)$$

Using the same equations, for players  $k \neq m$ , it follows that

$$\begin{aligned} \phi(\tilde{\beta}^k(\mathbf{x}_t^{-k}), \mathbf{x}_t^{-k}) - \phi(x_t^k, \mathbf{x}_t^{-k}) &\geq u^k(\tilde{\beta}^k(\mathbf{x}_t^{-k}), \mathbf{x}_t^{-k}) - u^k(x_t^k, \mathbf{x}_t^{-k}) - \delta \\ &\geq u^k(y^k, \mathbf{x}_t^{-k}) - u^k(x_t^k, \mathbf{x}_t^{-k}) - \delta - \tau \\ &\geq -\delta - \tau, \end{aligned} \quad (11)$$

where  $y^k$  denotes the best response of player  $k$  to  $\mathbf{x}_t^{-k}$ , that is,  $u^k(y^k, \mathbf{x}_t^{-k}) = \max_{z^k} u^k(z^k, \mathbf{x}_t^{-k})$ . Thus, if  $\mathbf{x}_t \notin \mathcal{X}_\epsilon$ , then summing the preceding inequalities ((10) and (11)) over all players, and using Eq. (6) we obtain

$$\dot{\phi}(\mathbf{x}_t) > \epsilon - M(\delta + \tau). \quad (12)$$

(ii) Let  $\epsilon > M(\delta + \tau)$  be given. The first part of the theorem implies that outside  $\mathcal{X}_\epsilon$ , the potential increases with a rate of at least  $\epsilon - M(\delta + \tau) > 0$ . Since the mixed extension of the potential function is a bounded function, it follows that starting from any strategy profile, the set  $\mathcal{X}_\epsilon$  is reached in finite time. It is immediate that  $\phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_\epsilon} \phi(\mathbf{y})$  for any  $\mathbf{x} \in \mathcal{X}_\epsilon$ . Since  $\phi(\mathbf{x}) > \epsilon - M(\delta + \tau) > 0$  for  $\mathbf{x} \notin \mathcal{X}_\epsilon$ , and  $\phi$  is bounded it follows that when the trajectory leaves  $\mathcal{X}_\epsilon$ ,  $\phi(\mathbf{x})$  cannot decrease below  $\min_{\mathbf{y} \in \mathcal{X}_\epsilon} \phi(\mathbf{y})$ . Thus, we conclude that after  $\mathcal{X}_\epsilon$  is reached for the first time, the trajectory of dynamics satisfies  $\phi(\mathbf{x}_t) \geq \max_{\mathbf{y} \in \mathcal{X}_\epsilon} \phi(\mathbf{y})$ .

Therefore it follows that the trajectory converges to the set of mixed strategies  $\{\mathbf{x} | \phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_\epsilon} \phi(\mathbf{y})\}$ . Since this is true for any  $\epsilon > M(\delta + \tau)$ , we obtain convergence to  $\{\mathbf{x} | \phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_{M(\delta+\tau)}} \phi(\mathbf{y})\}$ .  $\square$

This theorem characterizes the limiting behavior of continuous-time perturbed best response dynamics in near-potential games in terms of the approximate equilibrium set of the game, and the upper contour sets of the potential function of a close potential game. As the deviation from a potential game increases, the set in which dynamics will be contained gradually becomes larger. Thus, this characterization is more accurate for games that are closer to potential games.

For exact potential games, as we established in Corollary 4.3, the trajectories of continuous-time perturbed best response dynamics converge to a small neighborhood of the equilibria of the game, provided that the smoothing parameter  $\tau$  is small. However, even for potential games (i.e., when  $\delta = 0$ ) the previous theorem provides a characterization of the limiting behavior only in terms of the upper contour sets of the potential function, and hence cannot recover the convergence result for potential games. Additionally, in some cases, there may exist (mixed) equilibria where the potential takes its lowest value and the preceding characterization becomes vacuous.<sup>10</sup> Even when the potential function is not minimized by a mixed equilibrium, there may be mixed equilibria which have potential value lower than many of the (mixed) strategy profiles, for which the aforesaid result provides a weak characterization.

We next strengthen the previous convergence result by exploiting the properties of mixed approximate equilibrium sets in near-potential games. The feature of mixed equilibrium sets which plays a key role in our analysis was stated in Lemma 2.7. By considering the upper semicontinuity of the approximate equilibrium map  $g(\alpha)$  at  $\alpha = 0$ , this lemma implies that for small  $\epsilon$ , the  $\epsilon$ -equilibrium set is contained in a small neighborhood of equilibria.

It was established in part (i) of Theorem 4.4 that under continuous-time perturbed best response dynamics the potential function of a nearby potential game (with MPD  $\delta$  to the original game), evaluated at the current mixed strategy profile  $\mathbf{x}$ , increases when  $\mathbf{x}$  is outside the  $M(\delta + \tau)$ -equilibrium set of the original game. As discussed earlier, if  $\delta$  and  $\tau$  are sufficiently small, then the  $M(\delta + \tau)$ -equilibria of the game will be contained in a small neighborhood of the equilibria. Thus, for sufficiently small  $\delta$

<sup>10</sup>When an equilibrium is a global minimizer of the potential function, unilateral deviations from this strategy profile can neither increase nor decrease the potential value. Consequently, in such games many strategy profiles share the same potential value. Note that generically this cannot be the case, and hence an equilibrium cannot be the global minimizer of the potential.

and  $\tau$ , it is possible to establish that the potential of a close potential game increases outside a small neighborhood of the equilibria of the game. In Theorem 4.5, we use this observation to show that for sufficiently small  $\delta$  and  $\tau$  the trajectories of perturbed best response dynamics converge to a neighborhood of an equilibrium. We state the theorem under the assumption that the original game has finitely many equilibria. This assumption generically holds, that is, for any game a (nondegenerate) random perturbation of payoffs will lead to such a game with probability one (see Fudenberg and Tirole [1991]). When stating our result, we make use of the Lipschitz continuity of the mixed extension of the potential function, as established in Lemma 2.6.

**THEOREM 4.5.** *Consider a game  $\mathcal{G}$  and let  $\hat{\mathcal{G}}$  be a close potential game such that  $d(\mathcal{G}, \hat{\mathcal{G}}) \leq \delta$ . Denote the potential function of  $\hat{\mathcal{G}}$  by  $\phi$ , and the Lipschitz constant of the mixed extension of  $\phi$  by  $L$ . Assume that  $\mathcal{G}$  has finitely many equilibria.*

*Let  $\mathbf{x}_t$  be the trajectory of continuous-time perturbed best response dynamics in  $\mathcal{G}$ . There exists some  $\delta, \bar{\epsilon} > 0$ , (which are functions of utilities of  $\mathcal{G}$  but not  $\delta$ ) such that if  $\delta + \tau < \bar{\delta}$ , then  $\mathbf{x}_t$  converges to*

$$\left\{ \mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_k^*\| \leq \frac{2MLf(M(\delta + \tau))}{\epsilon} + f(M(\delta + \tau) + \epsilon), \text{ for some equilibrium } \mathbf{x}_k^* \right\}, \quad (13)$$

for any  $\bar{\epsilon} \geq \epsilon > 0$ , where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an upper semicontinuous function satisfying  $f(x) \rightarrow 0$  as  $x \rightarrow 0$ .

The proof of this theorem can be found in the Appendix. As explained earlier, for small  $\delta$ ,  $\tau$ , and  $\epsilon$ , the  $M(\delta + \tau) + \epsilon$ -equilibrium set of the game is contained in small neighborhoods of the equilibria of the game. If there are finitely many equilibria, these neighborhoods are disjoint and each of them contains a different component of the  $M(\delta + \tau) + \epsilon$ -equilibrium set. In the proof we show that the potential increases if the played strategy profile is outside this approximate equilibrium set. Then, we quantify the increase in the potential, when the trajectory leaves this approximate equilibrium set and returns back to it at a later time instant. Using this increase condition we show that after some time, the trajectory can visit the component of the approximate equilibrium set in the neighborhood of only a single equilibrium. This holds since the increase condition guarantees that the potential increases significantly when the trajectory leaves the neighborhood of an equilibrium and reaches to that of another equilibrium. Finally, using the increase condition one more time, we establish that if after time  $T$ , the trajectory visits the approximate equilibrium set only in the neighborhood of a single equilibrium, we can construct a neighborhood of this equilibrium, which contains the trajectory for all  $t > T$ . This neighborhood is expressed in Eq. (13).<sup>11</sup>

If the original game is a potential game (hence  $\delta = 0$ ), and the smoothing parameter  $\tau$  is arbitrarily small, the preceding theorem implies (by choosing  $\epsilon$  small as well, and using upper semicontinuity of  $f$  and the fact that  $f(0) = 0$ ) that the trajectory of continuous-time perturbed best response dynamics converges to a small neighborhood of the equilibria. Moreover, for any given  $r > 0$ , there exists a sufficiently small  $\tau$ , such that  $\epsilon$  can be chosen to guarantee  $f(M(\delta + \tau) + \epsilon) \leq r/2$  and  $2MLf(M(\delta + \tau))/\epsilon \leq r/2$ . Thus, the previous theorem recovers the convergence result of continuous-time perturbed best response dynamics for potential games, stated in Corollary 4.3. Hence,

<sup>11</sup>A similar proof technique was employed in Candogan et al. [2011b] for establishing convergence of discrete-time fictitious play to a neighborhood of equilibria in near-potential games. We want to emphasize that the proof of Theorem 4.5 exploits the properties of continuous-time dynamics, and obtains a convergence result for continuous-time perturbed best response dynamics. Moreover, the proof has important technical differences that lead to a different construction for the limiting set of continuous-time dynamics.

we conclude that the convergence result of continuous-time perturbed best response dynamics in near-potential games (Theorem 4.5) is a natural extension of the convergence result for potential games (Corollary 4.3).

## 5. CONTINUOUS GAMES AND DYNAMICS

In this section, we extend the framework for studying continuous-time dynamics in finite games, introduced in the previous section, to continuous games, and characterize the limiting behavior of similar dynamics in such games.

We first focus on continuous-time best response dynamics. This update rule is mathematically analogous to the perturbed best response dynamics discussed in Section 4, and it involves players who (continuously) modify their strategy in the direction of their best response. We show that in continuous near-potential games, the limiting behavior of this update rule can be characterized in terms of the upper contour sets of the potential function of a potential game that is close to the original game (Section 5.1). Moreover, when the potential function satisfies additional conditions (that relate to strong concavity), we show that the limiting set is a subset of a neighborhood of the maximizer of the potential function, where the size of the neighborhood is characterized in terms of the distance of the original game from the potential game.

Second, we extend our results to best response dynamics with  $\epsilon$ -stopping condition, where players update their strategies only when there is possibility of significant utility improvement (Section 5.2). This update rule is characterized with a strategy update threshold  $\epsilon$ , that is, only players who have at least  $\epsilon$  utility improvement opportunity update their strategies. Presence of the  $\epsilon$  threshold captures unmodeled decision making costs, which prevent players from updating their strategies unless they can guarantee significant payoff improvement. Due to the presence of the  $\epsilon$  threshold for strategy updates, the differential equations describing this dynamical process have discontinuous right-hand sides. Hence, in order to analyze this update rule, we introduce a solution concept that involves differential inclusions. We also present an invariance theorem which is used to characterize the limiting behavior of these differential inclusions. Using this machinery and exploiting properties of nearby potential games, we characterize the limiting behavior of continuous-time best response dynamics with  $\epsilon$  stopping in near-potential games. In particular, we show that in continuous near-potential games trajectories of this update rule converge to a set of approximate equilibria. This set is contained in a small neighborhood of equilibria, provided that the original game is sufficiently close to a potential game and  $\epsilon$  is small.

### 5.1. Continuous-Time Best Response Dynamics

We start by introducing continuous games, and the notation that will be used in the rest of this section. A continuous game consists of:

- a finite set of players, denoted by  $\mathcal{M} = \{1, \dots, M\}$ ;
- strategy spaces: a compact metric space  $E^m$ , for every  $m \in \mathcal{M}$ ; and
- continuous utility functions  $u^m : \prod_{k \in \mathcal{M}} E^k \rightarrow \mathbb{R}$ , for every  $m \in \mathcal{M}$ .

Note that the main difference between these games and finite games introduced in Section 2 are the strategy spaces. While the strategy spaces in finite games are finite sets, in continuous games we allow for more general strategy spaces. In this section, we impose additional structure on the strategy spaces of continuous games. In particular we assume that for every  $m \in \mathcal{M}$ ,  $E^m$  is a convex and compact subset of a product space of real numbers, that is,  $E^m \subset \mathbb{R}^K$ , for every  $m \in \mathcal{M}$ . We denote a strategy of player  $m$  by  $x^m \in E^m$ . Similarly, strategies of all players but  $m$  are denoted by  $\mathbf{x}^{-m}$ .

Since  $E^m \subset \mathbb{R}^K$ , for every  $m \in \mathcal{M}$ ,  $x^m$  and  $\mathbf{x}^{-m}$  can be thought of vectors in a Euclidean space.

A continuous game, which satisfies the conditions of Definition 2.1 (1), is a (continuous) exact potential game. In this section we refer to such games simply as potential games. Under differentiability conditions, additional characterizations of continuous potential games can be provided (see Monderer and Shapley [1996b]). We do not introduce such equivalent characterizations here since they are not necessary in establishing our results.

The distance measure MPD, which was introduced in Section 3, immediately generalizes to continuous games by replacing the maximum with the supremum operator. In this section, we measure the distance between continuous games in terms of this generalized version of MPD, and denote the MPD between two continuous games  $\mathcal{G}$  and  $\hat{\mathcal{G}}$  by  $d(\mathcal{G}, \hat{\mathcal{G}})$ . We refer to continuous games that are close to potential games in terms of this distance notion as near-potential games.

The definitions of Nash equilibria and  $\epsilon$ -equilibria in finite and continuous games are identical (see Section 2 for a definition). For this reason, with a slight abuse of notation, in this section we denote the set of pure  $\epsilon$  equilibria of a game by  $\mathcal{X}_\epsilon$ , that is,  $\mathcal{X}_\epsilon = \{\mathbf{x} \in E \mid u^m(x^m, \mathbf{x}^{-m}) \geq u^m(y^m, \mathbf{x}^{-m}) - \epsilon, \text{ for all } m \in M, y^m \in E^m\}$ .

In order to simplify the discussion, in the rest of the article we assume that the utility functions are continuous and for any  $m \in \mathcal{M}$ , and for all  $\mathbf{y}^{-k} \in E^{-k}$ ,  $u^m(x^m, \mathbf{y}^{-k})$  is (strictly) concave in  $x^m$ . Such games are sometimes referred to as concave games, and they always have pure Nash equilibria [Rosen 1965].

Next we focus on continuous-time best response dynamics in continuous games. Before we provide a formal definition of this update rule, we revisit the definition of best response. The best response of player  $m \in \mathcal{M}$ , is defined as follows.

$$\beta^m(\mathbf{x}^{-m}) = \arg \max_{y^m \in E^m} u^m(y^m, \mathbf{x}^{-m}) \quad \text{for all } \mathbf{y}^{-m} \in E^{-m} \quad (14)$$

Note that since  $u^m(x^m, \mathbf{y}^{-k})$  is strictly concave in  $x^m$ , best responses of players are single valued, even in the absence of a smoothing term. Using the notion of the best responses, the continuous-time best response dynamics can be defined as follows.

*Definition 5.1 (Continuous-Time Best Response Dynamics).* Continuous-time best response dynamics is the update rule, where the strategy of each player  $m \in \mathcal{M}$  evolves according to the differential equation

$$\dot{x}^m = \beta^m(\mathbf{x}^{-m}) - x^m. \quad (15)$$

Continuous-time best response dynamics and its variants have been studied in the existing literature [Candogan et al. 2010a; Fudenberg and Levine 1998; Hofbauer and Sorin 2006]. A number of papers focused on their limiting behavior and applications to evolutionary game theory [Hofbauer and Sigmund 2003; Sandholm 2010]. It is also known that when the underlying game is a potential game with a potential function that is concave in the strategy of each player, this update rule converges to a Nash equilibrium (see Candogan et al. [2010a], and also Theorem 5.2).

Observe that in continuous-time best response dynamics, agents find their best responses and adjust their strategies in the *direction* of best responses, as opposed to directly implementing the best responses.<sup>12</sup> Adopting the best responses may require

<sup>12</sup>Since agents update their strategies in the direction of their best responses, an alternative name for this update rule would be continuous-time best response direction dynamics. In this article we use the slightly shorter name continuous-time best response dynamics.

significant changes in the current strategy of the agents. Thus, in many realistic settings (such as engineering applications; see Candogan et al. [2010a] for a wireless power control application), it may be infeasible or costly to adopt best responses at every infinitesimal time period. Continuous best response dynamics, studied in this section, avoids such significant changes in the strategies of agents, by requiring arbitrarily small strategy adjustments in the direction of their best responses. In settings where agents are allowed to make bounded changes in their strategies over unit time, this update rule can be viewed as a natural limit of the discrete-time best response dynamics [Candogan et al. 2010a].<sup>13</sup>

Comparing Definitions 4.1 and 5.1, it can be seen that the trajectories of continuous-time best response dynamics and perturbed best response dynamics evolve according to similar differential equations. Continuous-time best response dynamics does not include a smoothing term in the best response, and its domain is any compact convex subset of  $\mathbb{R}^K$  (as opposed to the probability simplex, as in the case of perturbed best response dynamics in the context of finite games). In Definition 4.1, presence of a smoothing term ensured that the best response function is single valued, and the corresponding differential equations are well defined. Due to the concavity assumption on the payoffs, we do not need a smoothing term in Definition 5.1 to guarantee that the differential equations are well defined.

We next characterize the limiting behavior of this dynamics in near-potential games. We start our analysis by showing that outside an  $\epsilon$ -equilibrium set, the potential of the nearby potential game increases as a result of the strategy updates. Then, we use this observation to identify a limiting set in terms of the upper contour sets of this potential function, in an analogous fashion to Theorem 4.4. Due to the continuous nature of the problem, in Theorem 5.2 we require additional structure on the potential function of the nearby potential game (such as differentiability, and concavity in the strategy of each player). Note that these conditions are naturally satisfied in Theorem 4.4 by the properties of the mixed extension of the potential function in finite games.

**THEOREM 5.2.** *Consider a continuous game  $\mathcal{G}$  and let  $\hat{\mathcal{G}}$  be a close continuous potential game such that  $d(\mathcal{G}, \hat{\mathcal{G}}) \leq \delta$ . Assume that the potential function of  $\hat{\mathcal{G}}$ , denoted by  $\phi$ , is differentiable, and for all  $\mathbf{y}^{-m} \in E^{-m}$ , and  $m \in \mathcal{M}$ ,  $\phi(x^m, \mathbf{y}^{-m})$  is concave in  $x^m$ .*

*Let  $\mathbf{x}_t$  be the trajectory of continuous-time best response dynamics in  $\mathcal{G}$ . Then:*

- (i)  $\dot{\phi}(\mathbf{x}_t) > \epsilon - M\delta$  if  $\mathbf{x}_t \notin \mathcal{X}_\epsilon$ ,
- (ii) *the trajectory  $\mathbf{x}_t$  converges to the set of strategy profiles which have potential larger than the minimum potential in the  $M\delta$ -equilibrium set of the game, that is,  $\mathbf{x}_t$  converges to  $\{\mathbf{x} | \phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_{M\delta}} \phi(\mathbf{y})\}$  as  $t \rightarrow \infty$ .*

**PROOF.** (i) From the definition of the continuous-time best response dynamics and the concavity of  $\phi$  in each player's strategy it follows that

$$\dot{\phi}(\mathbf{x}_t) = \sum_m \nabla_m \phi^T(\mathbf{x}_t) (\beta^m(\mathbf{x}_t^{-m}) - x_t^m) \geq \sum_m (\phi(\beta^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - \phi(x_t^m, \mathbf{x}_t^{-m})),$$

<sup>13</sup>Other update rules in continuous time are also modeled by differential equations, allowing for only arbitrarily small changes in the strategies of agents at every infinitesimal time period. One example of such update rules is the gradient-type dynamics, where agents update their strategies in the *direction* of the payoff gradient [Anderson et al. 2004; Arrow and Hurwicz 1960; Friedman and Ostrov 2010]. We also note that in Friedman and Ostrov [2010], it is established that when strategy updates have a particular cost structure, arbitrarily small adjustments in the direction of the payoff gradient naturally emerge.

where  $\nabla_m$  denotes the collection of partial derivatives with respect to the strategies of player  $m$ , or equivalently the entries of the  $x^m$  vector. Using the definition of MPD this implies that

$$\begin{aligned} \dot{\phi}(\mathbf{x}_t) &\geq \sum_m \phi(\beta^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - \phi(x_t^m, \mathbf{x}_t^{-m}) \\ &\geq \sum_m u^m(\beta^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) - \delta. \end{aligned} \quad (16)$$

If  $\mathbf{x}_t$  does not belong to an  $\epsilon$  equilibrium we have  $u^m(\beta^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) > \epsilon$  for at least one player. Additionally, for all  $m$ ,  $u^m(\beta^m(\mathbf{x}_t^{-m}), \mathbf{x}_t^{-m}) - u^m(x_t^m, \mathbf{x}_t^{-m}) \geq 0$  by definition of best responses. Hence, Eq. (16) implies that  $\dot{\phi}(\mathbf{x}_t) > \epsilon - M\delta$ .

(ii) Since the strategy sets of players are compact subsets of  $\mathbb{R}^K$ , and the potential is continuous, it follows that it is bounded. Hence, the claim in the second part follows by an argument identical to that in the proof of Theorem 4.4 (ii).  $\square$

Theorem 5.2 implies convergence of continuous-time best response dynamics to a Nash equilibrium set in potential games, where  $\phi$  is concave in the strategy of each player. This follows since in such a game  $\delta$  is equal to 0, and part (i) of the preceding theorem implies that potential increases outside the equilibrium set. The potential function is bounded in games where the strategy spaces are compact, hence this ensures that the trajectories of dynamics converge to a Nash equilibrium set.

If the nearby potential game has additional structure, the previous result can also be used to provide an explicit characterization of the limiting set of dynamics in near-potential games. One useful structure, which is exploited in the following theorem, relates to the strong concavity of the potential function.<sup>14</sup>

**COROLLARY 5.3.** *Consider a continuous game  $\mathcal{G}$  and let  $\hat{\mathcal{G}}$  be a close continuous potential game such that  $d(\mathcal{G}, \hat{\mathcal{G}}) \leq \delta$ . Let  $\phi : \mathbb{R}^{KM} \rightarrow \mathbb{R}$ , be a strongly concave and twice continuously differentiable function. Assume that when its domain is restricted to  $\prod_m E^m$ ,  $\phi$  corresponds to the potential function of  $\hat{\mathcal{G}}$ , and it satisfies:*

- (a) for all  $\mathbf{x} \in \mathbb{R}^{KM}$ ,  $-\alpha_2 I \geq \nabla^2 \phi(\mathbf{x}) \geq -\alpha_1 I$ , where  $I$  is the identity matrix, that is, the Hessian matrix  $\nabla^2 \phi(\mathbf{x})$  has its eigenvalues in  $[-\alpha_1, -\alpha_2]$ ; and
- (b) for all  $\mathbf{x}^{-m} \in E^{-m}$ , and  $m \in \mathcal{M}$ ,  $\arg \max_{y^m} \phi(y^m, \mathbf{x}^{-m})$  belongs to the interior of  $E^m$ .

Let  $\mathbf{x}_t$  be the trajectory of continuous-time best response dynamics in  $\mathcal{G}$ . Then,  $\mathbf{x}_t$  converges to:

$$(i) \{ \mathbf{x} \in E \mid \phi(\mathbf{x}) \geq \max_{\mathbf{y} \in E} \phi(\mathbf{y}) - \frac{\alpha_1 M(M+1)\delta}{\alpha_2} \},$$

$$(ii) \{ \mathbf{x} \in E \mid \|\mathbf{x} - \mathbf{x}^*\| \leq \frac{4\alpha_1^2 M(M+1)\delta}{\alpha_2^2} \}, \text{ where } \mathbf{x}^* = \arg \max_{\mathbf{x} \in E} \phi(\mathbf{x}).$$

**PROOF.** (i) In order to characterize the limiting set, we first provide a lower bound for the potential  $\phi(\mathbf{x})$  when  $\mathbf{x} \in \mathcal{X}_{M\delta}$ , and then use this characterization together with Theorem 5.2.

If  $\mathbf{x} \in \mathcal{X}_{M\delta}$ , then since the MPD between the potential game and original game is  $\delta$ , it follows that  $\max_{y^m \in E^m} \phi(y^m, \mathbf{x}^{-m}) - \phi(x^m, \mathbf{x}^{-m}) \leq (M+1)\delta$ . Condition (b) and the concavity of the potential imply that the unconstrained maximization problem  $\max_{y^m \in \mathbb{R}^K} \phi(y^m, \mathbf{x}^{-m})$  also has a solution in the interior of  $E^m$ . Condition (a) implies that (see Boyd and Vandenberghe [2004] on using this condition for obtaining upper

<sup>14</sup>A differentiable function  $f$  is strongly concave if and only if  $-mI \geq \nabla^2 f(x)$  for all  $x$ , where  $I$  is the identity matrix, and  $m > 0$ .

and lower bounds on the norm of the gradient of a function), when the solution is in the interior (see Footnote 15)

$$\|\nabla_m \phi(\mathbf{x})\|^2 \leq 2\alpha_1(M+1)\delta, \quad (17)$$

where as before  $\nabla_m$  stands for the collection of partial derivatives of its argument with respect to the components of strategies of player  $m$ . This implies that

$$\|\nabla \phi(\mathbf{x})\|^2 \leq \sum_m 2\alpha_1(M+1)\delta = 2\alpha_1 M(M+1)\delta. \quad (18)$$

On the other hand, condition (a) also implies that (see Footnote 15)  $\max_{\mathbf{y} \in \mathbb{R}^{KM}} \phi(\mathbf{y}) - \phi(\mathbf{x}) \leq \frac{1}{2\alpha_2} \|\nabla \phi(\mathbf{x})\|^2$ . It follows from Condition (b) and the concavity of the potential that  $\arg \max_{\mathbf{y} \in \mathbb{R}^{KM}} \phi(\mathbf{y}) = \arg \max_{\mathbf{y} \in E} \phi(\mathbf{y})$ , and hence we conclude  $\max_{\mathbf{y} \in E} \phi(\mathbf{y}) - \phi(\mathbf{x}) \leq \frac{1}{2\alpha_2} \|\nabla \phi(\mathbf{x})\|^2$ . Together with Eq. (18), this implies that

$$\max_{\mathbf{y} \in E} \phi(\mathbf{y}) - \phi(\mathbf{x}) \leq \frac{\alpha_1 M(M+1)\delta}{\alpha_2}. \quad (19)$$

Thus, if  $\mathbf{x} \in \mathcal{X}_{M\delta}$ ,  $\phi(\mathbf{x})$  is lower bounded by  $\max_{\mathbf{y} \in E} \phi(\mathbf{y}) - \frac{\alpha_1 M(M+1)\delta}{\alpha_2}$ .

It follows from Theorem 5.2 that the trajectory of dynamics converges to  $\{\mathbf{x} \mid \phi(\mathbf{x}) \geq \min_{\mathbf{y} \in \mathcal{X}_{M\delta}} \phi(\mathbf{y})\}$ . Using Eq. (19) this implies that continuous-time best response dynamics converges to the set  $\{\mathbf{x} \in E \mid \phi(\mathbf{x}) \geq \max_{\mathbf{y} \in E} \phi(\mathbf{y}) - \frac{\alpha_1 M(M+1)\delta}{\alpha_2}\}$ .

(ii) Consider a strategy profile  $\mathbf{x} \in \{\mathbf{z} \mid \phi(\mathbf{z}) \geq \max_{\mathbf{y} \in E} \phi(\mathbf{y}) - \frac{\alpha_1 M(M+1)\delta}{\alpha_2}\}$ . Using Conditions (a) and (b), in a similar fashion to Eq. (17) we obtain

$$\|\nabla \phi(\mathbf{x})\|^2 \leq 2\alpha_1 \frac{\alpha_1 M(M+1)\delta}{\alpha_2}. \quad (20)$$

These conditions also imply that (see Footnote 15)

$$\|\mathbf{x} - \mathbf{x}^*\| \leq \frac{2}{\alpha_2} \|\nabla \phi(\mathbf{x})\|^2 \leq \frac{4\alpha_1^2 M(M+1)\delta}{\alpha_2^2}. \quad (21)$$

Thus, the desired convergence result follows.  $\square$

Condition (b) of the preceding corollary is satisfied for any potential function that decreases as its argument approaches to the boundary of the feasible region (in a similar fashion to a barrier function). Condition (a), on the other hand, is used in the literature to provide bounds on the condition numbers of upper (or lower) contour sets of strongly concave (or convex) functions (see Boyd and Vandenberghe [2004]).<sup>15</sup>

Note that this corollary implies that if the original game is a potential game, then trajectories of continuous-time best response dynamics converge to the maximizer of the potential function (which is the unique Nash equilibrium due to the strict concavity of the potential). For an arbitrary game, the limiting set of continuous-time best response dynamics can still be characterized in terms of the maximizer of the potential function, distance between the games, and features of the potential.

<sup>15</sup> Assume that a function  $f$  satisfies this condition. It is known that (see Boyd and Vandenberghe [2004]) the following inequalities hold: (i)  $f(y) - f(x) \leq \frac{1}{2\alpha_2} \|\nabla f(x)\|^2$ , (ii)  $\|x - x^*\| \leq \frac{1}{2\alpha_2} \|\nabla f(x)\|^2$ , where  $x^* = \arg \max_x f(x)$ , and (iii)  $\|\nabla f(x)\|^2 \leq 2\alpha_1(f(x^*) - f(x))$ .

## 5.2. Best Response Dynamics with $\epsilon$ -Stopping Condition

In this section, we focus on best response dynamics with  $\epsilon$ -stopping condition. We start by providing a formal definition of this update rule. The definition involves the concept of best responses, as introduced in the previous section.

*Definition 5.4 (Best Response Dynamics with  $\epsilon$ -Stopping).* Best response dynamics with  $\epsilon$ -stopping is the update rule, where the strategy of each player  $m \in \mathcal{M}$  evolves according to the differential equation

$$\dot{x}^m = \begin{cases} 0 & \text{if for all } y^m \in E^m, u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \leq \epsilon, \\ \beta^m(\mathbf{x}^{-m}) - x^m & \text{otherwise.} \end{cases} \quad (22)$$

It can be seen from this definition that, unlike the other update rules discussed in this article, in best response dynamics with  $\epsilon$ -stopping agents update their strategies only if it is possible to improve their utility by more than  $\epsilon$ . In the recent literature relevant update rules have received significant attention. For instance,  $\alpha$ -Nash dynamics, where players update their strategies to their best responses only if they can improve their payoffs by a factor of  $\alpha$ , has been considered. It was shown that this update rule leads to fast convergence to an approximate equilibrium set in potential games [Awerbuch et al. 2008; Chakraborty and Khanna 2009; Chien and Sinclair 2011; Christodoulou et al. 2006; Mirrokni and Vetta 2004]. Note that after a logarithmic transformation of the utility functions, this update rule suggests strategy updates occur only if agents can improve their payoffs by more than  $\alpha$ . Hence, the update rule we consider here can be viewed as a continuous-time analog of this update rule.

Note that the right-hand side of (22) is discontinuous, and a proper solution concept should be adopted for its analysis. We introduce two relevant solution concepts that will be discussed in this section [Alvarez et al. 2000; Clarke et al. 1998; Deimling 1992; Filippov and Arscott 1988].

*Definition 5.5 (Krasovskii Solution - Filippov Solution).* Given a differential equation  $\dot{x} = f(x)$ , with a discontinuous right-hand side: (i) a Krasovskii solution is a solution of the differential inclusion

$$\dot{x} \in \bigcap_{\theta > 0} \overline{\text{co}}f(x + \theta B), \quad (23)$$

(ii) a Filippov solution is a solution of the differential inclusion

$$\dot{x} \in \bigcap_{\theta > 0} \bigcap_{\mathcal{N}|\nu(\mathcal{N})=0} \overline{\text{co}}f(x + \theta B \setminus \mathcal{N}), \quad (24)$$

where,  $x \in \mathbb{R}^n$ ,  $\overline{\text{co}} S$  stands for the closure of the convex hull of set  $S$ ,  $\nu$  stands for the Lebesgue measure on  $\mathbb{R}^n$ ,  $B$  is the open unit ball in  $\mathbb{R}^n$ ,  $\theta > 0$  is a positive real number, and  $\theta B$  stands for a ball with radius  $\theta$ .

Intuitively, at the points of discontinuity, these definitions extend differential equations to differential inclusions, by allowing any right-hand side which is in the convex hull of the discontinuous end points of the  $f$  function. Due to the intersection over sets of measure zero ( $\nu(\mathcal{N}) = 0$ ) present in its definition, Filippov solutions disregard certain types of discontinuities<sup>16</sup>, and lead to more “robust” solutions [Ito 1979]. If  $f$  is a continuous function, both of these solutions reduce to regular differential equations.

<sup>16</sup>For instance, if one can obtain a continuous function  $f_c$ , changing the value of  $f$  function on a set of measure zero, then a Filippov solution leads to a differential equation where the right-hand side is  $f_c$ .

Note that since in these solution concepts we have differential inclusions instead of differential equations, the trajectory corresponding to a given initial condition  $x_0$  need not be unique. It can be seen from Definition 5.5 that the right-hand side of the differential inclusion for Filippov solutions is a subset of that of the Krasovskii solution, hence any trajectory obtained from a Filippov solution is also a Krasovskii solution. In the rest of the article, we restrict our attention to Krasovskii solutions, and note that convergence for Filippov solutions immediately follows from convergence for Krasovskii solutions.

We next focus on the differential inclusion corresponding to the Krasovskii solutions of best response dynamics with  $\epsilon$ -stopping, and characterize its properties. Note that the right-hand side of (22) is discontinuous at  $\mathbf{x} \in \prod_m E^m$ , only when, for some player  $m$  and  $z^m \in E^m$ ,

$$u^m(z^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) = \epsilon, \quad (25)$$

and for all other  $y^m \in E^m$ ,  $u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \leq \epsilon$ . For every player  $m$ , we define a function  $g^m : \prod_m E^m \rightarrow 2^{[0,1]}$ , such that

$$g^m(\mathbf{x}) = \begin{cases} \{0\} & \text{for all } y^m \in E^m, u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) < \epsilon \\ \{1\} & \text{for some } y^m \in E^m, u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) > \epsilon \\ [0, 1] & \text{otherwise.} \end{cases} \quad (26)$$

Note that  $g^m$  is multivalued, if and only if (25) holds, or equivalently at the points of discontinuity of the right-hand side of Eq. (22). Additionally, at these points, the definition of the Krasovskii solution implies that  $\dot{x}^m$  belongs to the convex hull of  $\beta^m(\mathbf{x}^{-m}) - x^m$  and 0. Therefore, it follows that the differential inclusion  $F^m$  corresponding to best response dynamics with  $\epsilon$ -stopping can be expressed as

$$\dot{x}^m \in F^m(\mathbf{x}) \quad \text{for all } m \in \mathcal{M}, \quad (27)$$

where  $F^m(\mathbf{x}) = \{\theta^m(\beta^m(\mathbf{x}^{-m}) - x^m) | \theta^m \in g^m(\mathbf{x})\}$ .

We say that best response dynamics with  $\epsilon$ -stopping converges to a set  $S$ , if starting from any mixed strategy profile, all the trajectory defined by the preceding differential inclusion converges to  $S$ , that is,  $\inf_{\mathbf{x} \in S} \|\mathbf{x}_t - \mathbf{x}\| \rightarrow 0$  as  $t \rightarrow \infty$ .

We next present an invariance theorem which will be used to establish convergence of best response dynamics with  $\epsilon$ -stopping in the earlier defined sense. Before we state the theorem, we introduce some additional notation. For any continuously differentiable function  $V : \Omega \rightarrow \mathbb{R}$ , we define the function  $D_F V : \Omega \rightarrow \mathbb{R}$  associated with the multivalued function  $F$ , such that for all  $x \in \Omega$

$$D_F V(x) = \sup_{v \in F(x)} \left\{ \lim_{z \rightarrow 0} \frac{1}{z} [V(x + zv) - V(x)] \right\}. \quad (28)$$

Note that  $D_F V(x)$  is well defined as long as  $x + zv \in \Omega$  for sufficiently small  $z$  and all  $v \in F(x)$ .

The following theorem generalizes La Salle's invariance theorem to differential inclusions (see Khalil [2002] and Michel et al. [2001]) and will be used in our analysis of the best response dynamics with  $\epsilon$ -stopping.

**THEOREM 5.6.** [MICHEL ET AL. 2001]. *Consider the differential inclusion,  $\dot{x} \in F(x)$ , where the trajectories of the dynamics are contained in a compact set and there exists a bounded set  $B$  such that  $F(x) \subset B$  for all  $x \in \Omega$ . Assume that:*

(i) *there exists a continuously differentiable function  $V : \Omega \rightarrow \mathbb{R}$  and a continuous function  $W : \Omega \rightarrow \mathbb{R}$  satisfying*

$$D_F V(x) \leq -W(x) \leq 0 \quad \text{for all } x \in \Omega, \text{ and} \quad (29)$$

(ii)  $V$  is bounded below.

Then, for any solution of the differential inclusion, trajectory  $x_t$  converges to  $\{x \in \Omega \mid W(x) = 0\}$  as  $t \rightarrow \infty$ .

The collection of differential inclusions introduced in Eq. (27) can alternatively be written as

$$\dot{\mathbf{x}} \in F(\mathbf{x}), \quad (30)$$

where  $F(\mathbf{x}) = \{F^m(\mathbf{x})\}_{m \in \mathcal{M}}$ . Since  $\beta^m(\mathbf{x}^{-m}), x^m \in E^m$ , and  $E^m$  is convex and compact, it follows that  $x^m + z\nu^m \in E^m$  for every  $\nu^m \in F^m(\mathbf{x})$  and sufficiently small  $z > 0$ , and all trajectories generated by the differential inclusion in Eq. (30) are contained in the compact set  $\prod_m E^m$ .

Moreover, since  $E^m$  is a compact subset of  $\mathbb{R}^K$ , it is bounded. Hence, it follows that  $F(\mathbf{x}) = \{F^m(\mathbf{x})\}$  is a subset of a bounded set. Thus, it can be seen that the differential inclusion in Eq. (30) satisfies conditions of Theorem 5.6 on properties of  $F$ , and trajectories of dynamics (Conditions other than (i) and (ii)). We next show that Conditions (i) and (ii) also hold for appropriately defined  $V$  and  $W$  functions, and use Theorem 5.6 to characterize the limiting behavior of continuous-time best response dynamics with  $\epsilon$ -stopping in near-potential games.

**THEOREM 5.7.** *Consider a continuous game  $\mathcal{G}$  and let  $\hat{\mathcal{G}}$  be a close continuous potential game such that  $d(\mathcal{G}, \hat{\mathcal{G}}) = \delta$ . Assume that the potential function of  $\hat{\mathcal{G}}$ , denoted by  $\phi$ , is continuously differentiable, and for all  $\mathbf{x}^{-m} \in E^{-m}$ , and  $m \in \mathcal{M}$ ,  $\phi(y^m, \mathbf{x}^{-m})$  is concave in  $y^m$ .*

Let  $\mathbf{x}_t$  be a trajectory of best response dynamics with  $\epsilon$ -stopping condition in  $\mathcal{G}$ . If  $\epsilon > \delta$ , then  $\mathbf{x}_t$  converges to the set of  $\epsilon$ -equilibria of  $\mathcal{G}$ .

**PROOF.** In order to prove the theorem, we define the functions  $V : \prod_m E^m \rightarrow \mathbb{R}$  and  $W : \prod_m E^m \rightarrow \mathbb{R}$  properly and use Theorem 5.6.

We define  $V$  such that  $V(\mathbf{x}) = -\phi(\mathbf{x})$ . Note that  $V$  is a bounded function that is also continuously differentiable. Additionally, for any  $\mathbf{x} \in \prod_m E^m$ ,  $m \in \mathcal{M}$  and  $\nu^m \in F^m(\mathbf{x})$  we have  $x^m + z\nu^m \in E^m$  for small enough  $z$  as explained before. Thus, the limit  $\lim_{z \rightarrow 0} \frac{V(\mathbf{x} + z\nu) - V(\mathbf{x})}{z}$  exists for all  $\mathbf{x} \in \prod_m \Delta E^m$  and  $\nu \in F(\mathbf{x})$ . Note that this quantity corresponds to the directional derivative of  $V$  in  $\nu$  direction. Observing that  $\prod_m E^m \subset \prod_m \mathbb{R}^{|E^m|}$ , and writing the directional derivative of  $V$  in terms of the inner product of the relevant direction vector and the gradient of  $V$ , it follows that

$$\lim_{z \rightarrow 0} \frac{V(\mathbf{x} + z\nu) - V(\mathbf{x})}{z} = \nabla V^T(\mathbf{x})\nu = \sum_m \nabla_m V^T(\mathbf{x})\nu^m. \quad (31)$$

Here  $\nu^m$  is the component of the  $\nu$  vector corresponding to player  $m$ 's strategies and  $\nabla_m V$  denotes the vector of partial derivatives of  $V$  with respect to the components of player  $m$ 's strategy vector. Therefore, using the characterization of  $F$  in Eqs. (30) and (27), it follows that

$$\begin{aligned} D_F V(\mathbf{x}) &= \sup_{\nu \in F(\mathbf{x})} \left\{ \sum_m \nabla_m V^T(\mathbf{x})\nu^m \right\} \\ &= \sup_{\theta^k \in g^k(\mathbf{x}), \forall k} \sum_m \theta^m \nabla_m V^T(\mathbf{x})(\beta^m(\mathbf{x}^{-m}) - x^m) \\ &\leq \sum_m \sup_{\theta^m \in g^m(\mathbf{x})} \theta^m \nabla_m V^T(\mathbf{x})(\beta^m(\mathbf{x}^{-m}) - x^m). \end{aligned} \quad (32)$$

Since  $\phi(q^m, \mathbf{p}^{-m})$  is concave in  $q^m$  for any  $\mathbf{p}^{-m} \in E^{-m}$ , it follows that  $V(q^m, \mathbf{p}^{-m})$  is convex in  $q^m$ , and hence  $\nabla_m V^T(\mathbf{x})(\beta^m(\mathbf{x}^{-m}) - x^m) \leq V(\beta^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - V(x^m, \mathbf{x}^{-m})$ . Thus, using Eq. (32) we obtain

$$\begin{aligned} D_F V(\mathbf{x}) &\leq \sum_m \sup_{\theta^m \in g^m(\mathbf{x})} \theta^m (V(\beta^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - V(x^m, \mathbf{x}^{-m})) \\ &\leq \sum_m \sup_{\theta^m \in g^m(\mathbf{x})} \theta^m (\phi(x^m, \mathbf{x}^{-m}) - \phi(\beta^m(\mathbf{x}^{-m}), \mathbf{x}^{-m})). \end{aligned} \quad (33)$$

The definition of  $g^m$  (see Eq. (26)) implies that  $\theta^m$  can be chosen different than 0 only for agents for which

$$u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \geq \epsilon, \quad (34)$$

for some  $y^m \in E^m$ . Let  $m$  be such an agent, and  $y^m$  denote such a strategy. By definition of  $\beta^m$  it follows that

$$u^m(\beta^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) \geq u^m(y^m, \mathbf{x}^{-m}). \quad (35)$$

Subtracting  $u^m(x^m, \mathbf{x}^{-m})$  from both sides, together with Eq. (34) the previous inequality implies that

$$u^m(\beta^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \geq u^m(y^m, \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) \geq \epsilon. \quad (36)$$

Therefore, using the definition of MPD the preceding inequality implies that

$$\begin{aligned} \phi(\beta^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - \phi(x^m, \mathbf{x}^{-m}) &\geq u^m(\beta^m(\mathbf{x}^{-m}), \mathbf{x}^{-m}) - u^m(x^m, \mathbf{x}^{-m}) - \delta \\ &\geq \epsilon - \delta > 0. \end{aligned} \quad (37)$$

Since  $g^k(\mathbf{x}) = \{0\}$ , for agents which have  $u^k(y^k, \mathbf{x}^{-k}) - u^k(x^k, \mathbf{x}^{-k}) < \epsilon$  for all  $y^k$ , and for all other agents the aforesaid inequality holds, Eq. (33) can be rewritten as

$$D_F V(\mathbf{x}) \leq \sum_m \sup_{\theta^m \in g^m(\mathbf{x})} -\theta^m (\epsilon - \delta). \quad (38)$$

Let  $\mathbf{x} \notin \mathcal{X}_\epsilon$ , and  $k$  denote a player who can improve its payoff by strictly more than  $\epsilon$ , unilaterally deviating from  $\mathbf{x}$ . Since  $\theta^m \geq 0$  for  $m \neq k$  and  $g^k(\mathbf{x}) = \{1\}$  it follows that

$$D_F V(\mathbf{x}) \leq -(\epsilon - \delta) < 0. \quad (39)$$

Similarly, if  $\mathbf{x} \in \mathcal{X}_\epsilon$ , then Eq. (38) implies that

$$D_F V(\mathbf{x}) \leq 0. \quad (40)$$

Let  $dist : 2^E \setminus \{\emptyset\} \times E \rightarrow \mathbb{R}$  be such that  $dist(S, \mathbf{x}) \triangleq \inf_{\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|_\infty$ , where  $\|\cdot\|_\infty$  denotes the infinity norm in the Euclidean space that contains  $\prod_m E^m$ . Since  $\prod_m E^m$  is a compact subset of a Euclidean space, it follows that  $dist$  is a function bounded by some  $b > 0$ . Let  $W : \prod_m E^m \rightarrow \mathbb{R}$  be a function such that  $W(\mathbf{x}) = dist(\mathcal{X}_\epsilon, \mathbf{x})(\epsilon - \delta)/b$ . We have  $W(\mathbf{x}) = 0$ , for  $\mathbf{x} \in \mathcal{X}_\epsilon$  and  $0 < W(\mathbf{x}) \leq (\epsilon - \delta)$  for all  $\mathbf{x} \in \prod_m E^m \setminus \mathcal{X}_\epsilon$ . Hence,  $W : \prod_m E^m \rightarrow \mathbb{R}$  is a continuous function, and as implied by Eqs. (39) and (40),  $D_F V(\mathbf{x}) \leq -W(\mathbf{x}) \leq 0$ .

Since the trajectories of the dynamics are contained in the compact set  $\prod_m E^m$  and  $F(\mathbf{x})$  belongs to a bounded set for all  $\mathbf{x} \in \prod_m E^m$  (as  $\beta^m(\mathbf{x}^{-m}), x^m \in E^m$ ), Theorem 5.6, with the functions  $V$  and  $W$  introduced before, implies that the best response dynamics with  $\epsilon$ -stopping converges to the set for which  $W(\mathbf{x}) = 0$ . On the other hand, the definition of  $W(\cdot)$  implies that this set is equivalent to  $\mathcal{X}_\epsilon$ , the set of  $\epsilon$ -equilibria of  $\mathcal{G}$ , and the claim follows.  $\square$

The previous theorem implies that if the original game is close to a potential game (hence  $\delta \approx 0$ ), then  $\epsilon$  can be chosen small to establish convergence of the best response dynamics with  $\epsilon$ -stopping to a small approximate equilibrium set of the game. Moreover, as the deviation from a potential game increases, the set in which dynamics will be contained gradually becomes larger (for  $\epsilon$  chosen arbitrarily). Using Lemma 2.7 (appropriately generalized to continuous games) for sufficiently small  $\epsilon$ , the limiting approximate equilibrium set is contained in a small neighborhood of the equilibria of the game. Thus, we conclude that in near-potential games (and hence in potential games), even if players stop updating their strategies due to limited utility improvement opportunity (using a variant of continuous-time best response dynamics) convergence to a small neighborhood of the equilibria of the game can still be established.

## 6. CONCLUSIONS

In this article, we study properties of near-potential games, and characterize the limiting behavior of continuous-time dynamics in these games. We first introduce a distance notion in the space of games, and study the geometry of sets of potential games and games that are equivalent to potential games. We also provide a framework for finding games that are close to potential games. Then we focus on continuous-time perturbed best response dynamics, and in finite near-potential games characterize its limiting behavior in terms of the approximate equilibrium sets of the game, and the upper contour sets of the potential function of a nearby potential game. The characterization is tighter when the original game is closer to a potential game. We strengthen our result by exploiting the structure of the mixed equilibrium sets, and showing that if the original game is sufficiently close to a potential game, then convergence to a small neighborhood of the equilibria can be established. We also extend our framework to study continuous-time best response dynamics and best response dynamics with  $\epsilon$ -stopping condition in continuous near-potential games. In particular, we show that in continuous near-potential games, the limiting sets of both of these dynamics can be characterized in terms of the potential of a nearby potential game and the distance of this game from the original game. Moreover, as in the case of finite games, we establish that our characterization is tighter for games that are closer to potential games. Our results extend the known convergence properties of continuous-time dynamics in potential games to near-potential games and to settings where strategy updates take place only when there is sufficient utility improvement opportunity.

Our analysis and results motivate a number of interesting research questions. An interesting direction is to consider other learning dynamics, such as projection dynamics and replicator dynamics (see Sandholm [2010]), which are known to converge in potential games and investigate whether one can extend these convergence results to near-potential games. Other future work includes focusing on other classes of games with appealing dynamic properties, such as zero-sum games and supermodular games [Fudenberg and Levine 1998; Milgrom and Roberts 1990; Young 2004], and understanding, through an analysis similar to the one in this article, whether or not one can establish similar dynamic properties for nearby games.

## APPENDIX

**PROOF OF THEOREM 4.5.** Assume that  $\mathcal{G}$  has  $l > 0$  equilibria, denoted by  $\mathbf{x}_1^*, \dots, \mathbf{x}_l^*$ . Define the minimum pairwise distance between the equilibria as  $d \triangleq \min_{i \neq j} \|\mathbf{x}_i^* - \mathbf{x}_j^*\|$ . Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a function such that

$$f(\alpha) = \max_{\mathbf{x} \in \mathcal{X}_\alpha} \min_{k \in \{1, \dots, l\}} \|\mathbf{x} - \mathbf{x}_k^*\|, \quad (41)$$

for all  $\alpha \in \mathbb{R}_+$ . Note that  $\min_{k \in \{1, \dots, l\}} \|\mathbf{x} - \mathbf{x}_k^*\|$  is continuous in  $\mathbf{x}$ , since it is minimum of finitely many continuous functions. Moreover,  $\mathcal{X}_\alpha$  is a compact set, since  $\epsilon$ -equilibria are defined by finitely many inequality constraints of the form (2). Therefore, in Eq. (41) maximum is achieved and  $f$  is well defined for all  $\alpha \geq 0$ . From the definition of  $f$ , it follows that the union of closed balls of radius  $f(\alpha)$ , centered at equilibria, contain  $\alpha$ -equilibrium set of the game. Thus, intuitively,  $f(\alpha)$  captures the size of a closed neighborhood of equilibria, which contains  $\alpha$ -equilibria of the underlying game.

Let  $a > 0$  be such that  $f(a) < d/4$ , that is, every  $a$ -equilibrium is at most  $d/4$  distant from an equilibrium of a game. Lemma 2.7 implies (using upper semicontinuity at  $\alpha = 0$ ) that such  $a$  exists. Since  $d$  is defined as the minimum pairwise distance between the equilibria, it follows that  $a$ -equilibria of the game are contained in disjoint  $f(a) < d/4$  neighborhoods around equilibria of the game, that is, if  $\mathbf{x} \in \mathcal{X}_a$ , then  $\|\mathbf{x} - \mathbf{x}_k^*\| \leq f(a)$  for exactly one equilibrium  $\mathbf{x}_k^*$ . Moreover, for  $a_1 \leq a$ , since  $\mathcal{X}_{a_1} \subset \mathcal{X}_a$ , it follows that  $a_1$ -equilibria of the game are also contained in disjoint neighborhoods of equilibria.

We prove the theorem in 4 steps summarized shortly. First two steps explore the properties of function  $f$ , and define  $\bar{\delta}$  and  $\bar{\epsilon}$  present in the theorem statement. Last two steps are the main steps of the proof, where we establish convergence of perturbed best response dynamics to a neighborhood of equilibria.

- *Step 1:* We first show that  $f$  is (i) weakly increasing, (ii) upper semicontinuous, and it satisfies (iii)  $f(0) = 0$ , (iv)  $f(x) \rightarrow 0$  as  $x \rightarrow 0$ .
- *Step 2:* We show that there exists some  $\bar{\delta} > 0$ ,  $\bar{\epsilon} > 0$  such that the following inequalities hold. We have

$$M\bar{\delta} + \bar{\epsilon} < a, \quad (42)$$

and

$$f(M\bar{\delta} + \bar{\epsilon}) < \frac{d(a - M\bar{\delta})}{16ML}. \quad (43)$$

We will prove the statement of the theorem assuming that  $0 \leq \delta + \tau < \bar{\delta}$  and establish convergence to the set in Eq. (13), for any  $\bar{\epsilon} \geq \epsilon > 0$ . As can be seen from the definition of  $a$  and  $f$  (see Eq. (41)), the first inequality guarantees that  $\bar{\epsilon} + M\bar{\delta}$ -equilibrium set is contained in disjoint neighborhoods of equilibria, and the second one guarantees that these neighborhoods are small.

- *Step 3:* In this step we prove that after some time perturbed best response dynamics can visit the  $\bar{\epsilon} + M(\delta + \tau)$ -equilibrium set contained in the neighborhood of only one equilibrium.
- *Step 4:* In this step, using the fact that the neighborhood of only one equilibrium is visited after some time, we show that the trajectory converges to the set given in the theorem statement.

Next we prove each of these steps.

*Step 1.* By definition  $\mathcal{X}_{\alpha_1} \subset \mathcal{X}_\alpha$  for any  $\alpha_1 \leq \alpha$ . Since the feasible set of the maximization problem in Eq. (41) is given by  $\mathcal{X}_\alpha$ , this implies that  $f(\alpha_1) \leq f(\alpha)$ , that is,  $f$  is a weakly increasing function of its argument. Note that the feasible set of the maximization problem in Eq. (41) can be given by the multivalued function  $g(\alpha) = \mathcal{X}_\alpha$ , which is upper semicontinuous in  $\alpha$  as shown in Lemma 2.7. Since as a function of  $\mathbf{x}$ ,  $\min_{k \in \{1, \dots, l\}} \|\mathbf{x} - \mathbf{x}_k^*\|$  is continuous it follows from Berge's maximum theorem (see Berge [1963]) that for  $\alpha \geq 0$ ,  $f(\alpha)$  is an upper semicontinuous function.

The set  $\mathcal{X}_0$  corresponds to the set of equilibria of the game, hence  $\mathcal{X}_0 = \{\mathbf{x}_1^*, \dots, \mathbf{x}_l^*\}$ . Thus, the definition of  $f$  implies that  $f(0) = 0$ . Moreover, upper semicontinuity of  $f$  implies that for any  $\epsilon > 0$ , there exists some neighborhood  $V$  of 0, such that  $f(x) \leq \epsilon$

for all  $x \in V$ . Since  $f(x) \geq 0$  by definition, this implies that  $\lim_{x \rightarrow 0} f(x)$  exists and equals to 0.

*Step 2.* Let  $\bar{\delta}, \bar{\epsilon} > 0$  be small enough such that  $M\bar{\delta} + \bar{\epsilon} < a/2$ . Since  $\lim_{x \rightarrow 0} f(x) = 0$ , it follows that for sufficiently small  $\bar{\delta}, \bar{\epsilon}$  we obtain  $f(M\bar{\delta} + \bar{\epsilon}) < \frac{ad}{32ML} < \frac{(a-M\bar{\delta})d}{16ML}$ .

*Step 3.* Assume that the trajectory of the dynamics leaves the component of the  $\bar{\epsilon} + M(\delta + \tau)$ -equilibrium set contained in the neighborhood of equilibrium  $\mathbf{x}_k^*$ , and enters the component of this approximate equilibrium set in the neighborhood of equilibrium  $\mathbf{x}_l^*$ . Since  $\bar{\epsilon} + M(\delta + \tau) \leq \bar{\epsilon} + M\bar{\delta} < a$  in such an evolution the trajectory needs to leave the  $f(a) < d/4$  neighborhood of equilibrium  $\mathbf{x}_k^*$ , and enter the  $f(a)$  neighborhood of equilibrium  $\mathbf{x}_l^*$ .

Let  $t_1$  denote the time instant the trajectory leaves the component of the  $\bar{\epsilon} + M(\delta + \tau)$ -equilibrium set around equilibrium  $\mathbf{x}_k^*$ ,  $t_2$  denote the instant it leaves the  $f(a)$  neighborhood of  $\mathbf{x}_k^*$ ,  $t_3$  denote the instant it enters the  $f(a)$  neighborhood of  $\mathbf{x}_l^*$ , and finally  $t_4$  denote the instant it enters the component of the  $\bar{\epsilon} + M(\delta + \tau)$ -equilibrium set around equilibrium  $\mathbf{x}_l^*$ .

It follows from part (i) of Theorem 4.4 that

$$\phi(\mathbf{x}_{t_2}) - \phi(\mathbf{x}_{t_1}), \phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_3}) \geq 0. \quad (44)$$

Observe that  $\mathbf{x}_t$  is outside the  $f(a)$  neighborhood of equilibria between  $t_2$  and  $t_3$ , and this neighborhood contains  $\mathcal{X}_a$ . Since equilibria are at least  $d$  apart, and  $f(a) < d/4$  we obtain  $\|\mathbf{x}_{t_2} - \mathbf{x}_{t_3}\| \geq d/2$ . Additionally, since  $\tilde{\beta}^m(\mathbf{x}^{-m}), x^m \in \Delta E^m$  for all  $m$ , it follows that  $\|\dot{x}^m\| = \|\tilde{\beta}^m(\mathbf{x}^{-m}) - x^m\| \leq \|x^m\| + \|\tilde{\beta}^m(\mathbf{x}^{-m})\| \leq 2$  and hence  $\|\dot{\mathbf{x}}\| \leq \sum_m \|\dot{x}^m\| \leq 2M$ . Thus, we obtain  $t_3 - t_2 \geq d/4M$ , and hence part (i) of Theorem 4.4 implies that

$$\phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2}) \geq (a - M(\delta + \tau))d/4M. \quad (45)$$

Let  $\bar{\phi}_k = \max_{\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_k^*\| \leq f(\bar{\epsilon} + M(\delta + \tau))\}} \phi(\mathbf{x})$  and define  $\mathbf{y}_k$  as a strategy profile which achieves this maximum. Similarly, let  $\underline{\phi}_l = \min_{\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_l^*\| \leq f(\bar{\epsilon} + M(\delta + \tau))\}} \phi(\mathbf{x})$  and define  $\mathbf{y}_l$  as a strategy profile which achieves this minimum. Since  $\|\mathbf{x}_{t_1} - \mathbf{x}_k^*\|, \|\mathbf{x}_{t_4} - \mathbf{x}_l^*\| \leq f(\bar{\epsilon} + M(\delta + \tau))$ , it follows that  $\|\mathbf{x}_{t_1} - \mathbf{y}_k\|, \|\mathbf{x}_{t_4} - \mathbf{y}_l\| \leq 2f(\bar{\epsilon} + M(\delta + \tau))$ . Thus, by Lipschitz continuity of the potential function it follows that  $\bar{\phi}_k - \phi(\mathbf{x}_{t_1}) \leq 2f(\bar{\epsilon} + M(\delta + \tau))L$  and  $\phi(\mathbf{x}_{t_4}) - \underline{\phi}_l \leq 2f(\bar{\epsilon} + M(\delta + \tau))L$ . From these inequalities we conclude that

$$\begin{aligned} \underline{\phi}_l - \bar{\phi}_k &\geq \phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) - 4f(\bar{\epsilon} + M(\delta + \tau))L \\ &\geq \phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2}) - 4f(\bar{\epsilon} + M(\delta + \tau))L \\ &\geq (a - M(\delta + \tau))d/4M - 4f(\bar{\epsilon} + M(\delta + \tau))L, \end{aligned} \quad (46)$$

where the last two lines follow from Eqs. (44) and (45) and  $\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) = (\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_3})) + (\phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2})) + (\phi(\mathbf{x}_{t_2}) - \phi(\mathbf{x}_{t_1}))$ .

Since  $\bar{\delta} \geq \delta + \tau$ , by step 2 and the fact that  $f$  is weakly increasing in its argument we have

$$\begin{aligned} (a - M(\delta + \tau))d/4M - 4f(\bar{\epsilon} + M(\delta + \tau))L &\geq (a - M\bar{\delta})d/4M - 4f(\bar{\epsilon} + M\bar{\delta})L \\ &= 4L((a - M\bar{\delta})d/16ML - f(\bar{\epsilon} + M\bar{\delta})) \quad (47) \\ &> 0. \end{aligned}$$

Thus, Eqs. (46) and (47) imply that  $\underline{\phi}_l - \bar{\phi}_k > 0$ . Hence, we conclude that if the trajectory leaves the component of the  $\bar{\epsilon} + M(\delta + \tau)$ -equilibrium set around  $\mathbf{x}_k^*$  and enters to that around equilibrium  $\mathbf{x}_l^*$ , the maximum potential in the first set is smaller than the

minimum potential in the second one. Since this is true for arbitrary equilibria  $\mathbf{x}_k^*$  and  $\mathbf{x}_l^*$ , it follows that once trajectories enter the component of approximate equilibrium set around equilibrium  $\mathbf{x}_l^*$ , they cannot revisit the component of the approximate equilibrium set around  $\mathbf{x}_k^*$ . Thus, we conclude that after some time trajectories can visit the  $\bar{\epsilon} + M(\delta + \tau)$ -equilibrium set around at most one equilibrium.

*Step 4.* Let  $\epsilon, \epsilon_1$  be such that  $0 < \epsilon_1 < \epsilon \leq \bar{\epsilon}$ . By Theorem 4.4 it follows that  $\dot{\phi}(\mathbf{x}) \geq \epsilon_1$  for any  $\mathbf{x} \notin \mathcal{X}_{\epsilon_1 + M(\delta + \tau)}$ . Thus, it follows that after any time instant  $T$ , there exists another time instant when the set  $\mathcal{X}_{\epsilon_1 + M(\delta + \tau)}$  is visited. By step 3 it follows that after some time the trajectory visits the component of  $\epsilon + M(\delta + \tau)$ -equilibrium set only around one equilibrium.

Assume that after time instant  $T$ , the component of  $\epsilon + M(\delta + \tau)$ -equilibrium set around only a single equilibrium, say  $\mathbf{x}_k^*$ , is visited, at  $t_1 > t$  the trajectory leaves the  $\epsilon_1 + M(\delta + \tau)$ -equilibrium set, at  $t_2 > t_1$  it leaves the  $f(M(\delta + \tau) + \epsilon)$  neighborhood of  $\mathbf{x}_k^*$ , at  $t_3 > t_2$ , it returns back to this neighborhood, and at  $t_4 > t_3$  it returns to the  $\epsilon_1 + M(\delta + \tau)$ -equilibrium set around  $\mathbf{x}_k^*$ . Let  $d^*$  be the distance the trajectory gets from the  $f(M(\delta + \tau) + \epsilon)$  neighborhood of  $\mathbf{x}_k^*$  (note that this set contains the relevant component of the  $\epsilon_1 + M(\delta + \tau)$ -equilibrium set). Since  $\|\dot{\mathbf{x}}\| \leq 2M$ , as shown in the proof of step 3, it follows that  $t_3 - t_2 \geq \frac{2d^*}{2M} = \frac{d^*}{M}$ . Additionally, since  $\dot{\phi}(\mathbf{x}) \geq \epsilon$  outside the  $M(\delta + \tau) + \epsilon$  equilibrium set, it follows that potential increases at least by  $d^*\epsilon/M$ , when trajectory leaves the  $f(M(\delta + \tau) + \epsilon)$  neighborhood by  $d^*$  and returns back to it, that is,  $\phi(\mathbf{x}_{t_3}) - \phi(\mathbf{x}_{t_2}) \geq d^*\epsilon/M$ . Similarly,  $\dot{\phi}(\mathbf{x}) \geq \epsilon_1 > 0$  outside the  $\epsilon_1 + M(\delta + \tau)$ -equilibrium set, and hence  $\phi(\mathbf{x}_{t_2}) - \phi(\mathbf{x}_{t_1}), \phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_3}) \geq 0$ . Using these, we obtain

$$\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) \geq d^*\epsilon/M. \quad (48)$$

On the other hand, by definition  $\mathbf{x}_{t_4}$  and  $\mathbf{x}_{t_1}$  belong to the  $\epsilon_1 + M(\delta + \tau)$ -equilibrium set around  $\mathbf{x}_k^*$ , and hence  $\|\mathbf{x}_{t_4} - \mathbf{x}_{t_1}\| \leq 2f(\epsilon_1 + M(\delta + \tau))$ . Thus, the Lipschitz continuity of the potential function implies that  $\phi(\mathbf{x}_{t_4}) - \phi(\mathbf{x}_{t_1}) \leq 2f(\epsilon_1 + M(\delta + \tau))L$ . Together with Eq. (48) this implies that

$$d^*\epsilon/M \leq 2f(\epsilon_1 + M(\delta + \tau))L, \quad (49)$$

or equivalently  $d^* \leq 2MLf(\epsilon_1 + M(\delta + \tau))/\epsilon$ . Since the distance between equilibrium  $\mathbf{x}_k^*$ , and any point in the  $f(M(\delta + \tau) + \epsilon)$  neighborhood is at most  $f(M(\delta + \tau) + \epsilon)$ , it follows that the trajectory can become at most  $f(M(\delta + \tau) + \epsilon) + 2MLf(\epsilon_1 + M(\delta + \tau))/\epsilon$  distant from equilibrium  $\mathbf{x}_k^*$ . Since this is true for all  $\epsilon_1 \in (0, \epsilon)$ , taking  $\epsilon_1 \rightarrow 0$ , the result follows from upper semicontinuity of  $f$  proved in step 1.  $\square$

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