2. Convexity and Duality

- Convex sets and functions
- Convex optimization problems
- Standard problems: LP and SDP
- Feasibility problems
- Algorithms
- Certificates and separating hyperplanes
- Duality and geometry
- Examples: LP and and SDP
- Theorems of alternatives

Basic Nomenclature

A set $S \subset \mathbb{R}^n$ is called

- *affine* if $x, y \in S$ implies $\theta x + (1-\theta)y \in S$ for all $\theta \in \mathbb{R}$; i.e., the line through x, y is contained in S
- *convex* if $x, y \in S$ implies $\theta x + (1-\theta)y \in S$ for all $\theta \in [0,1]$; i.e., the line segment between x and y is contained in S.
- *a convex cone* if $x, y \in S$ implies $\lambda x + \mu y \in S$ for all $\lambda, \mu \geq 0$; i.e., the *pie slice* between x and y is contained in S.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called

- *affine* if $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y)$ for all $\theta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$; i.e., f is equals a linear function plus a constant $f = Ax+b$
- *convex* if $f\big(\theta x + (1-\theta)y\big) \leq \theta f(x) + (1-\theta)f(y)$ for all $\theta \in [0,1]$ and $x, y \in \mathbb{R}^n$

Properties of Convex Functions

- $f_1 + f_2$ is convex if f_1 and f_2 are
- $f(x) = \max\{f_1(x), f_2(x)\}$ is convex if f_1 and f_2 are
- $g(x) = \sup_y f(x,y)$ is convex if $f(x,y)$ is convex in x for each y
- convex functions are continuous on the interior of their domain
- $f(Ax+b)$ is convex if f is
- $Af(x)+b$ is convex if f is
- $g(x) = \inf_y f(x,y)$ is convex if $f(x,y)$ is jointly convex
- the α*−sublevel set*

$$\{\, x \in \mathbb{R}^n \mid f(x) \leq \alpha \,\}$$

is convex if f is convex; (the converse is not true)

Convex Optimization Problems

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0 \qquad \text{for all } i = 1, \ldots, m \\ & h_i(x) = 0 \qquad \text{for all } i = 1, \ldots, p \end{aligned}$$

This problem is called a *convex program* if

- the objective function f_0 is convex
- the inequality constraints f_i are convex
- the equality constraints h_i are affine

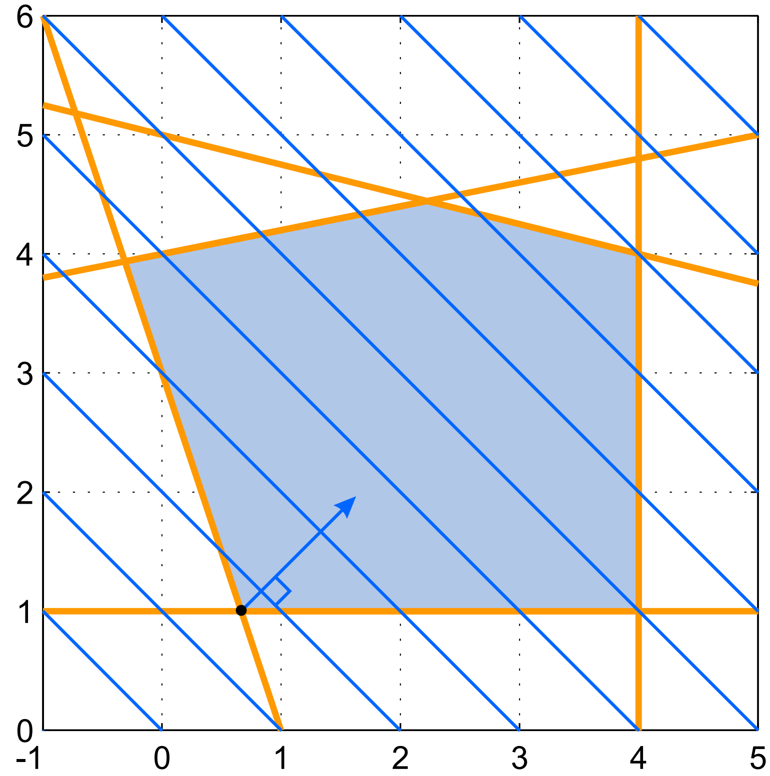
Linear Programming (LP)

In a *linear program*, the objective and constraint functions are affine.

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & Cx \le d \end{array}$$

Example

$$\begin{array}{rr} \text{minimize} & x_1 + x_2 \\ \text{subject to} & 3x_1 + x_2 \ge 3 \\ & x_2 \ge 1 \\ & x_1 \le 4 \\ & -x_1 + 5x_2 \le 20 \\ & x_1 + 4x_2 \le 20 \end{array}$$



Linear Programming

Every linear program may be written in the *standard primal form*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Here $x \in \mathbb{R}^n$, and $x \geq 0$ means $x_i \geq 0$ for all i

- The *nonnegative orthant* $\{\, x \in \mathbb{R}^n \mid x \geq 0 \,\}$ is a *convex cone*.
- This convex cone defines the partial ordering $\geq$ on $\mathbb{R}^n$
- Geometrically, the feasible set is the intersection of an affine set with a convex cone.

Semidefinite Programming

$$\begin{array}{rll} \text{minimize} & \textbf{trace}\, CX & \\ \text{subject to} & \textbf{trace}\, A_i X = b_i & \text{for all } i = 1, \dots, m \\ & X \succeq 0 & \end{array}$$

- The variable X is in the set of $n \times n$ symmetric matrices

$$\mathbb{S}^n = \{\, A \in \mathbb{R}^{n \times n} \mid A = A^T \,\}$$

- $X \succeq 0$ means X is positive semidefinite
- As for LP, the feasible set is the intersection of an affine set with a convex cone, in this case the *positive semidefinite cone*

$$\{\, X \in \mathbb{S}^n \mid X \succeq 0 \,\}$$

Hence the feasible set is convex.

SDPs with Explicit Variables

We can also explicitly parametrize the affine set to give

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where $F_0, F_1, \ldots, F_n$ are symmetric matrices.

The inequality constraint is called a *linear matrix inequality*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

The Feasible Set is Semialgebraic

The *(basic closed) semialgebraic set* defined by polynomials $f_1, \ldots, f_m$ is

$$\left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \ldots, m \right\}$$

The feasible set of an SDP is a semialgebraic set.

Because a matrix $A \succ 0$ if and only if

$$\det(A_k) > 0 \text{ for } k = 1, \ldots, n$$

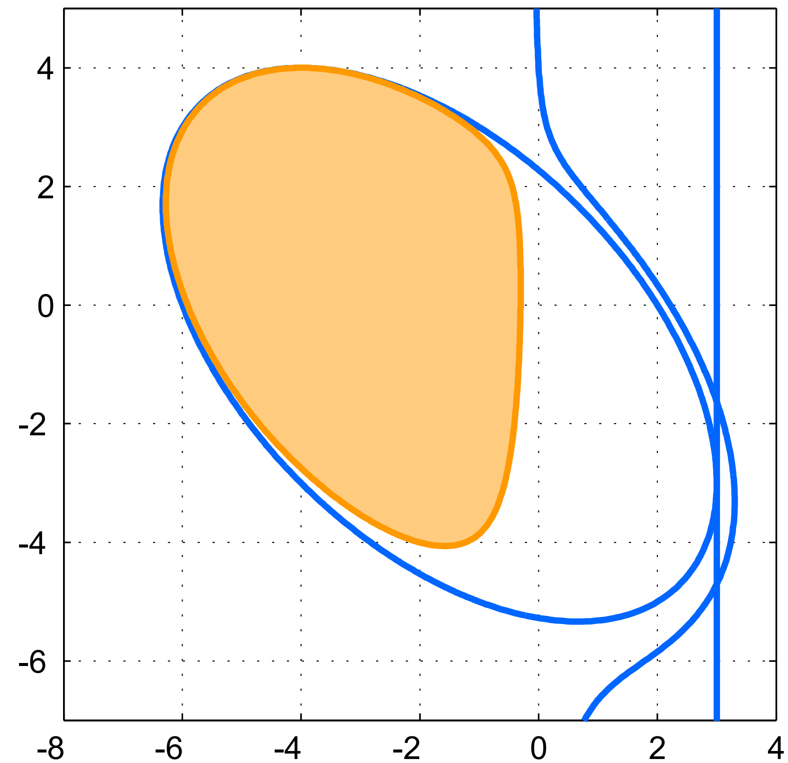
where A_k is the submatrix of A consisting of the first k rows and columns.

The Feasible Set

For example

$$0 \prec \begin{bmatrix} 3 - x_1 & -(x_1 + x_2) & 1 \\ -(x_1 + x_2) & 4 - x_2 & 0 \\ 1 & 0 & -x_1 \end{bmatrix}$$

is equivalent to the polynomial inequalities



$$0 < 3 - x_1$$

$$0 < (3 - x_1)(4 - x_2) - (x_1 + x_2)^2$$

$$0 < -x_1((3 - x_1)(4 - x_2) - (x_1 + x_2)^2) - (4 - x_2)$$

Feasible Sets of SDP

If S is the feasible set of an SDP, then S is defined by polynomials.

$$S = \left\{ x \in \mathbb{R}^m \mid A_0 + \sum_{i=1}^{m} A_i x_i \succeq 0 \right\}$$

In fact, S is the *closure of the connected component containing 0* of

$$C = \left\{ x \in \mathbb{R}^m \mid f(x) > 0 \right\}$$

where $f = \det\bigl(A_0 + \sum_{i=1}^m A_i x_i\bigr)$ and $A_0 \succ 0$

Question: for what polynomials f is C the feasible set of an SDP?

What about $\left\{ (x, y) \mid x^4 + y^4 \leq 1 \right\}$? It is convex and semialgebraic

A Necessary Condition

A simple necessary condition consider the line $x = zt$; then

$$\det\Big(A_0 + t\sum_{i=1}^{m} A_i z_i\Big) = 0$$

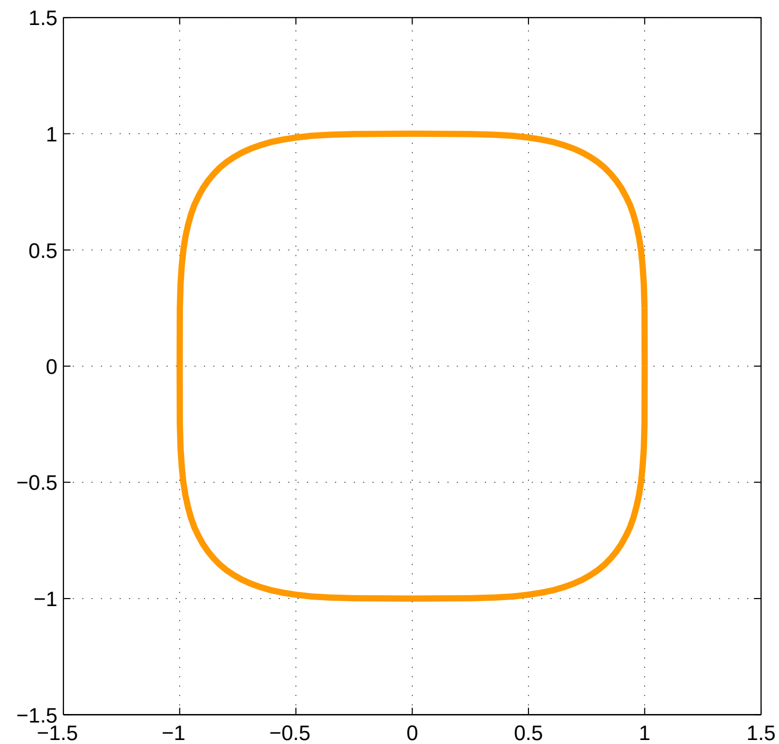
must vanish at exactly n real points

any line through 0 must intersect $\{\, x \mid f(x) = 0 \,\}$ exactly n times

Helton and Vinnikov show this condition is also sufficient (subject to additional technical assumptions)

Example: $x^4 + y^4 < 1$ cannot be represented in the form

$$A_0 + A_1 x + A_2 y \succ 0$$



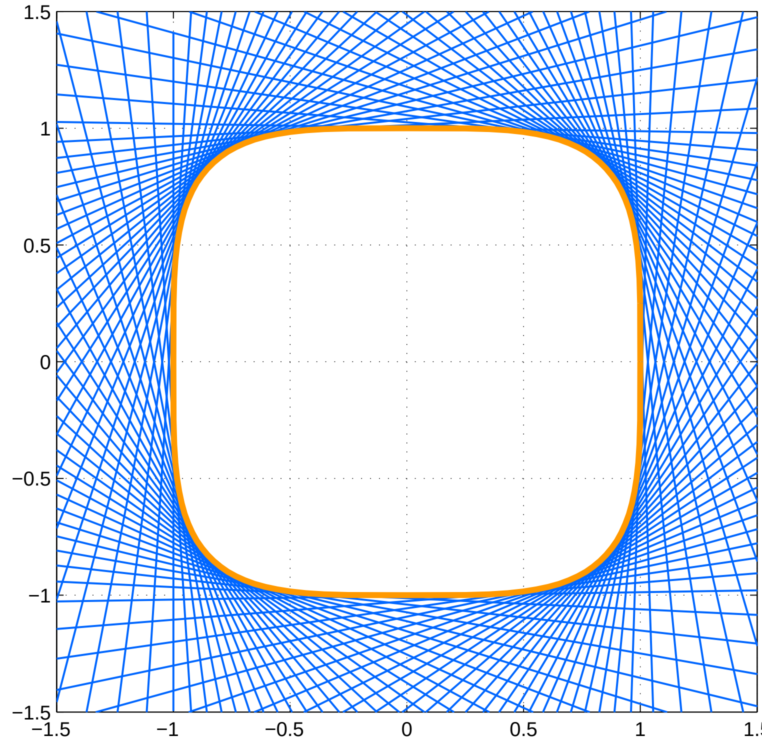
Projections

But $x^4 + y^4 < 1$ is the *projection* of the feasible set of the SDP

$$\begin{bmatrix} 1 & w & z \\ w & 1 & 0 \\ z & 0 & 1 \end{bmatrix} \succ 0$$

$$\begin{bmatrix} w & x \\ x & 1 \end{bmatrix} \succ 0$$

$$\begin{bmatrix} z & y \\ y & 1 \end{bmatrix} \succ 0$$



- Polyehedra are closed under projection. Spectrahedra and basic semi-algebraic sets are not.
- Easy to optimize over the projection.
- How to do this *systematically*? What if the set is not convex?

Convex Optimization Problems

For a convex optimization problem, the *feasible set*

$$S = \{\, x \in \mathbb{R}^n \mid f_i(x) \leq 0 \text{ and } h_j(x) = 0 \text{ for all } i, j \,\}$$

is convex. So we can write the problem as

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & x \in S \end{array}$$

This approach emphasizes the *geometry* of the problem.

For a convex optimization problem, any local minimum is also a global minimum.

Feasibility Problems

We are also interested in *feasibility problems* as follows. Does there exist $x \in \mathbb{R}^n$ which satisfies

$$\begin{aligned} f_i(x) &\leq 0 \qquad \text{for all } i = 1, \dots, m \\ h_i(x) &= 0 \qquad \text{for all } i = 1, \dots, p \end{aligned}$$

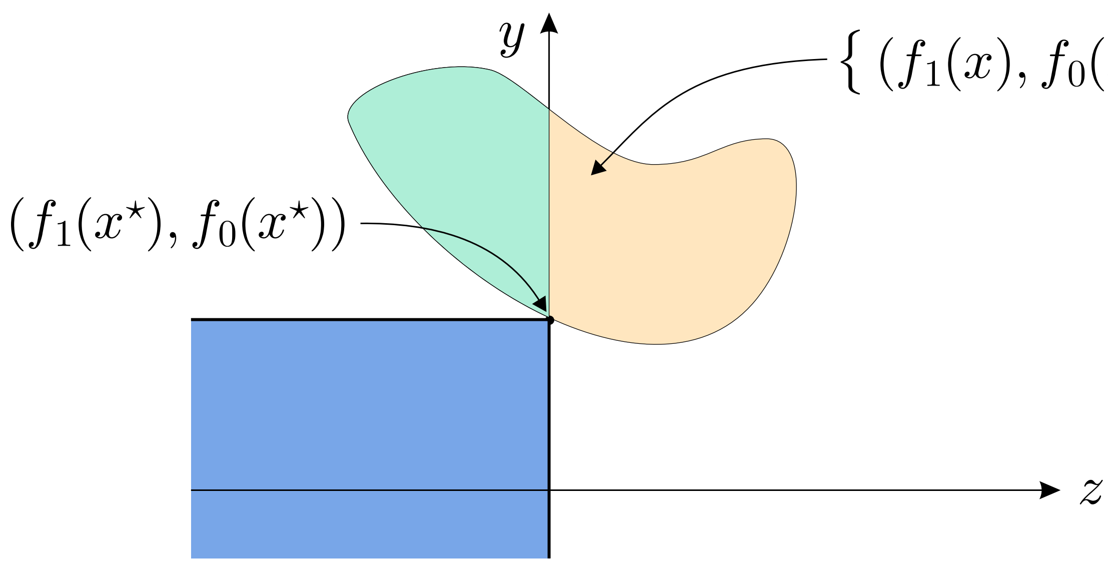
If there does not exist such an x, the problem is described as *infeasible*.

Feasibility Problems

We can always convert an optimization problem into a feasibility problem; does there exist $x \in \mathbb{R}^n$ such that

$$f_0(x) \le t$$
$$f_i(x) \le 0$$
$$h_i(x) = 0$$

Bisection search over the parameter t finds the optimal.



Feasibility Problems

Conversely, we can convert feasibility problems into optimization problems.

e.g. the feasibility problem of finding x such that

$$f_i(x) \leq 0 \qquad \text{for all } i = 1, \dots, m$$

can be solved as

$$\begin{aligned} \text{minimize} \quad & y \\ \text{subject to} \quad & f_i(x) \leq y \qquad \text{for all } i = 1, \dots, m \end{aligned}$$

where there are $n + 1$ variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$

This technique may be used to find an initial feasible point for optimization algorithms

Algorithms

For convex optimization problems, there are several good algorithms

- interior-point algorithms work well in theory and practice
- for certain classes of problems, (e.g. LP and SDP) there is a worst-case time-complexity bound
- conversely, some convex optimization problems are known to be NP-hard
- problems are specified either in *standard form*, for LPs and SDPs, or via an *oracle*

Certificates

Consider the feasibility problem

Does there exist $x \in \mathbb{R}^n$ which satisfies

$$f_i(x) \leq 0 \text{ for all } i = 1, \ldots, m$$

$$h_i(x) = 0 \text{ for all } i = 1, \ldots, p$$

There is a fundamental asymmetry between establishing that

- There exists at least one feasible x
- The problem is infeasible

To show existence, one needs a *feasible point* $x \in \mathbb{R}^n$.

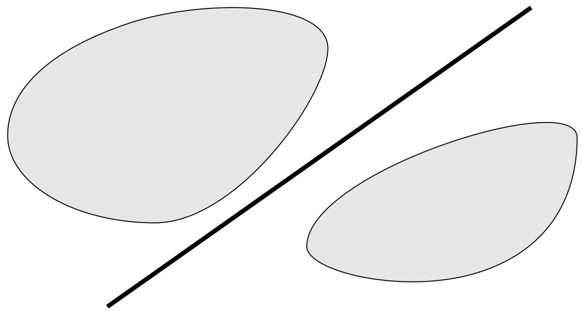
To show emptiness, one needs a a *certificate of infeasibility*; a mathematical proof that the problem is infeasible.

Certificates and Separating Hyperplanes

The simplest form of certificate is a *separating hyperplane*. The idea is that a hyperplane $L \subset \mathbb{R}^n$ breaks $\mathbb{R}^n$ into two half-spaces,

$$H_1 = \left\{ x \in \mathbb{R}^n \mid b^T x \le a \right\} \quad \text{and} \quad H_2 = \left\{ x \in \mathbb{R}^n \mid b^T x > a \right\}$$

If two *bounded, closed and convex* sets are disjoint, there is a hyperplane that separates them.



So to prove infeasibility of

$$f_i(x) \le 0 \qquad \text{for } i = 1, 2$$

we need to *computationally* show that

$$\{ x \in \mathbb{R}^n \mid f_1(x) \le 0 \} \subset H_1 \quad \text{and} \quad \{ x \in \mathbb{R}^n \mid f_2(x) \le 0 \} \subset H_2$$

Even though such a hyperplane exists, the computation may not be easy

Duality

We'd like to solve

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \ldots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \ldots, p \end{array}$$

define the *Lagrangian* for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ by

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

and the *Lagrange dual function*

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu)$$

We allow $g(\lambda, \nu) = -\infty$ when there is no finite infimum

Duality

The *dual problem* is

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

We call λ, ν *dual feasible* if $\lambda \geq 0$ and $g(\lambda, \nu)$ is finite.

- The dual function g is always concave, even if the primal problem is not convex

Weak Duality

For any primal feasible x and dual feasible λ, ν we have

$$g(\lambda, \nu) \leq f_0(x)$$

because

$$\begin{aligned} g(\lambda, \nu) &\leq f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\ &\leq f_0(x) \end{aligned}$$

- A feasible λ, ν provides a *certificate* that the primal optimal is greater than $g(\lambda, \nu)$
- many interior-point methods simultaneously optimize the primal and the dual problem; when $f_0(x) - g(\lambda, \nu) \leq \varepsilon$ we know that x is $\varepsilon-$suboptimal

Strong Duality

- $p^\star$ is the optimal value of the primal problem,
- $d^\star$ is the optimal value of the dual problem

Weak duality means $p^\star \geq d^\star$

If $p^\star = d^\star$ we say *strong duality* holds. Equivalently, we say the *duality gap* $p^\star - d^\star$ is zero.

Constraint qualifications give sufficient conditions for strong duality.

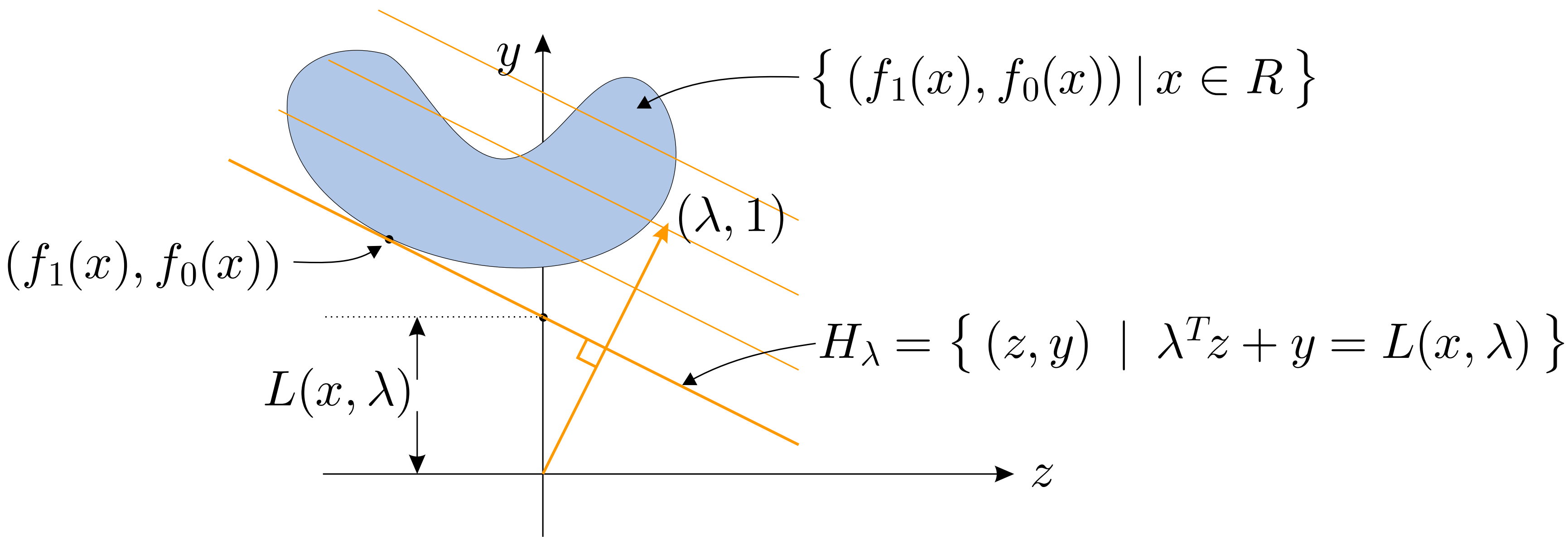
An example is *Slater's condition*; strong duality holds if the primal problem is convex and strictly feasible.

Geometric Interpretations

The Lagrange dual function is

$$g(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda)$$

i.e., the minimum intersection for a given slope $-\lambda$



Example: Linear Programming

minimize	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

The Lagrange dual function is

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} \left(c^T x + \nu^T (b - Ax) - \lambda^T x \right)$$

$$= \begin{cases} b^T \nu & \text{if } c - A^T \nu - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$$

So the dual problem is

maximize	$b^T \nu$
subject to	$A^T \nu \leq c$

Example: Semidefinite Programming

$$\begin{array}{rll} \text{minimize} & \mathbf{trace}\, CX & \\ \text{subject to} & \mathbf{trace}\, A_i X = b_i & \text{for all } i = 1, \ldots, m \\ & X \succeq 0 & \end{array}$$

The Lagrange dual is

$$g(Z, \nu) = \inf_X \left(\mathbf{trace}\, CX - \mathbf{trace}\, ZX + \sum_{i=1}^m \nu_i (b_i - \mathbf{trace}\, A_i X) \right)$$

$$= \begin{cases} b^T \nu & \text{if } C - Z - \sum_{i=1}^m \nu_i A_i = 0 \\ -\infty & \text{otherwise} \end{cases}$$

So the dual problem is to maximize $b^T \nu$ subject to

$$C - Z - \sum_{i=1}^m \nu_i A_i = 0 \qquad \text{and} \qquad Z \succeq 0$$

Semidefinite Programming Duality

The primal problem is

$$\begin{array}{lll} \text{minimize} & \mathbf{trace}\, CX & \\ \text{subject to} & \mathbf{trace}\, A_i X = b_i & \text{for all } i = 1, \dots, m \\ & X \succeq 0 & \end{array}$$

The dual problem is

$$\begin{array}{ll} \text{maximize} & b^T \nu \\ \text{subject to} & \sum_{i=1}^{m} \nu_i A_i \preceq C \end{array}$$

The Fourfold Way

There are several ways of formulating an SDP for its numerical solution.

Because *subspaces* can be described

- Using *generators* or a *basis*; Equivalently, the subspace is the range of a linear map $\{ x \mid x = B\lambda \text{ for some } \lambda \}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 - \lambda_2 \\ 2\lambda_1 + 2\lambda_2 \end{bmatrix}$$

- Through the defining equations; i.e, as the *kernel* $\{ x \mid Ax = 0 \}$

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \quad | \quad 4x_1 - 2x_2 - x_3 = 0\}$$

Depending on which description we use, and whether we write a primal or dual formulation, we have *four* possibilities (two primal-dual pairs).

Example: Two Primal-Dual Pairs

$$\begin{aligned} \text{maximize} \quad & 2x + 2y \\ \text{subject to} \quad & \begin{bmatrix} 1+x & y \\ y & 1-x \end{bmatrix} \succeq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} W \\ \text{subject to} \quad & \mathbf{trace} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} W = 2 \\ & \mathbf{trace} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} W = 2 \\ & W \succeq 0 \end{aligned}$$

Another, *more efficient* fomulation which solves the same problem:

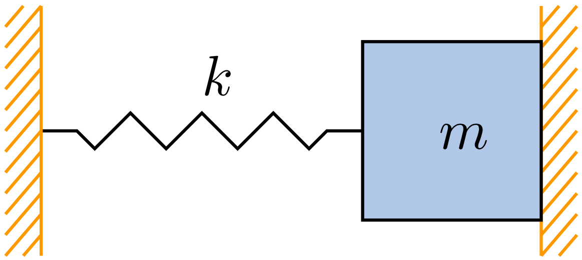
$$\begin{aligned} \text{maximize} \quad & \mathbf{trace} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} Z \\ \text{subject to} \quad & \mathbf{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z = 2 \\ & Z \succeq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 2t \\ \text{subject to} \quad & \begin{bmatrix} t-1 & -1 \\ -1 & t+1 \end{bmatrix} \succeq 0 \end{aligned}$$

Duality

- Duality has many interpretations; via economics, game-theory, geometry.
- e.g., one may interpret Lagrange multipliers as a price for violating constraints, which may correspond to resource limits or capacity constraints.
- Often physical problems associate specific meaning to certain Lagrange multipliers, e.g. pressure, momentum, force can all be viewed as Lagrange multipliers

Example: Mechanics



- Spring under compression
- Mass at horizontal position x, equilibrium at $x = 2$

$$\begin{array}{ll} \text{minimize} & \frac{k}{2}(x-2)^2 \\ \text{subject to} & x \leq 1 \end{array}$$

The Lagrangian is $L(x, \lambda) = \frac{k}{2}(x-2)^2 + \lambda(x-1)$

If λ is dual optimal and x is primal optimal, then $\frac{\partial}{\partial x}L(x, \lambda) = 0$, i.e.,

$$k(x-2) + \lambda = 0$$

so we can interpret λ as a *force*

Feasibility of Inequalities

The *primal feasibility problem* is

> does there exist $x \in \mathbb{R}^n$ such that
> $f_i(x) \geq 0 \qquad$ for all $i = 1, \dots, m$

The *dual function* $g : \mathbb{R}^m \to \mathbb{R}$ is

$$g(\lambda) = \sup_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \lambda_i f_i(x)$$

The *dual feasibility problem* is

> does there exist $\lambda \in \mathbb{R}^m$ such that
> $$g(\lambda) < 0$$
> $$\lambda \geq 0$$

Theorem of Alternatives

If the dual problem is feasible, then the primal problem is infeasible.

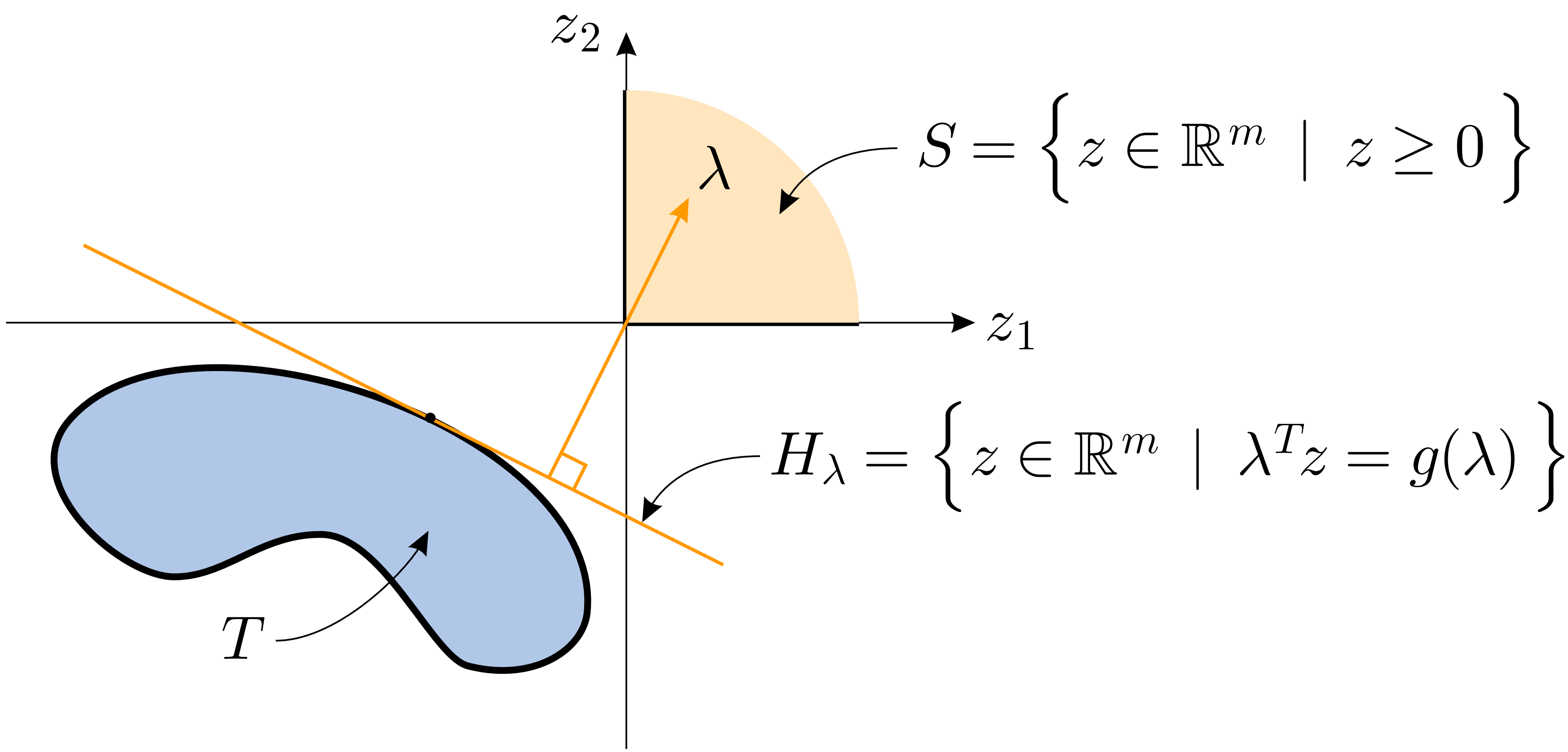
Proof

Suppose the primal problem is feasible, and let $\tilde{x}$ be a feasible point. Then

$$\begin{aligned} g(\lambda) &= \sup_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \lambda_i f_i(x) \\ &\geq \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) \qquad \text{for all } \lambda \in \mathbb{R}^m \end{aligned}$$

and so $g(\lambda) \geq 0$ for all $\lambda \geq 0$.

Geometric Interpretation



if $g(\lambda) < 0$ and $\lambda \geq 0$ then the hyperplane H_λ separates S from T, where

$$T = \left\{ \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \mid x \in \mathbb{R}^n \right\}$$

Certificates

- A dual feasible point gives a *certificate* of infeasibility of the primal.

- If the Lagrange dual function g is easy to compute, and we can show $g(\lambda) < 0$, then this is a *proof* that the primal is infeasible.

- One way to do this is to have an *explicit expression* for

$$g(\lambda) = \sup_x L(x, \lambda)$$

  For many problems, we do not know how to do this

- Alternatively, given λ, we may be able to show directly that

$$L(x, \lambda) < -\varepsilon \qquad \text{for all } x \in \mathbb{R}^n$$

Completion of Squares

If $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$ are symmetric matrices and $B \in \mathbb{R}^{n\times m}$

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x + A^{-1}By)^T A (x + A^{-1}By) + y^T(D - B^T A^{-1} B)y$$

- this gives a test for *global positivity*:

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \succ 0 \quad \Longleftrightarrow \quad A \succ 0 \text{ and } D - B^T A^{-1} B \succ 0$$

- It is a *sum of squares* decomposition
- Applying this recursively, we can *certify* nonnegativity

Quadratic Optimization

The Schur complement gives a general formula for quadratic optimization; if $A \succ 0$, then

$$\min_x \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = y^T (D - B^T A^{-1} B) y$$

and the minimizing x is

$$x_{\text{opt}} = -A^{-1} B y$$