1. Introduction & Research Objectives

My research interests lie in **arithmetic geometry** and **arithmetic statistics**, especially as they relate to the **arithmetic of curves** (both scheme-y and stacky curves). My research has typically leaned towards finiteness results, obtained by applying a mix of techniques from **number theory**, **algebraic geometry**, and **topology**. My current and future work relates to the following topics:

(**Average ranks of elliptic curves, via statistics of Selmer groups**, Section 2.1) In [Ach23], I complete the proof that, over *any* global field, the average rank of elliptic curves is finite. I propose to improve known upper bounds in the function field case (see Problems 1 and 2) as well as to study the theoretical limits of the most commonly used strategy for obtaining such bounds (see Problem 4).

(**Brauer groups of stacky curves**, Section 2.2) I have developed techniques for computing Brauer groups of stacky curves [ABJ+24, Ach24]. I propose to use these to compute the integral étale-Brauer obstruction set for the moduli stack $\mathcal{Y}(1)$ of elliptic curves (see Problem 5) with the hopes of giving a new proof that there is no elliptic scheme over $\mathbb{Z}$.

(**Integral points on higher-dimensional varieties**, Section 2.3) In [AM23], we construct new examples of *irreducible* divisors D on nice varieties X such that $X \setminus D$ has only finitely many integral points. The novelty here is that most such results require D to have many irreducible components.

2. Past Accomplishments, including Results from Prior NSF Support

I would first like to acknowledge that much of my graduate education has been funded by NSF Graduate Research Fellowship grant DGE-2141064. In particular, all the results described in Sections 2.1 and 2.2 were obtained (partially or fully) while I was being supported by this grant.

2.1. Average ranks of elliptic curves, via statistics of Selmer groups.

Arithmetic statistics is a field concerned with understanding the distributions, suitably defined, of various objects of number theoretic interest. Of particular relevance to my own work is the distribution of ranks of elliptic curves. Recall that an elliptic curve E over a field F is a smooth, projective genus 1 curve equipped with chosen point $0 \in E(F)$ defined over F. This choice of point defines a group law on $E(F)$ with 0 serving as the identity. When F is a global field – i.e. a number field ($\mathbb{Q}$ or a finite extension thereof) or a global function field ($\mathbb{F}_q(t)$, for some prime power q, or a finite extension thereof) – the abelian group $E(F)$ is finitely generated and so one can speak of the *rank* of E.

In this context, the main conjectures are as follows.

Conjecture 2.1 (Minimalist Conjecture, folklore)**.** *Fix any global field K. Half of elliptic curves E/K have rank 0 and half have rank 1.*

Most recent progress on this conjecture comes, not from directly studying ranks of elliptic curves, but from studying sizes of their Selmer groups. Here, for every $n \geq 1$ and any elliptic curve E/K, the *n-Selmer group* $\operatorname{Sel}_n(E/K)$ is a specific finite $\mathbb{Z}/n\mathbb{Z}$-module which admits an injection $E(K)/nE(K) \hookrightarrow \operatorname{Sel}_n(E/K)$; consequentially, $n^{\operatorname{rank} E(K)} \leq \# \operatorname{Sel}_n(E/K)$, so Selmer groups give rank bounds. The distribution of sizes of Selmer groups has received much attention. In particular, a conjectural formula for this distribution has been proposed in [BKL+15] (building off of heuristics of Poonen-Rains [PR12]). One major prediction of this conjectural distribution is the following.

Conjecture 2.2 ([BKL+15])**.** *Fix any global field K and any $n \geq 1$. The average size $\mathbb{E}[\# \operatorname{Sel}_n(E/K)]$ of n-Selmer groups of elliptic curves E/K is the sum of the divisors of n.*

Proving Conjecture 2.2 (in addition to understanding the distribution of parities of ranks of elliptic curves) would prove Conjecture 2.1. Evidence for Conjecture 2.2 goes back to work of de Jong [dJ02, Theorem 1.2] who computed that $\mathbb{E}[\#\operatorname{Sel}_3(E/\mathbb{F}_q(t))] \le 4 + o(1)$ as $q \to \infty$. Studying Conjecture 2.2 gained even more popularity after work of Bhargava and Shankar [BS15a, BS15b, BS13a, BS13b] verified that $\mathbb{E}[\#\operatorname{Sel}_n(E/\mathbb{Q})] = \sum_{d|n} d$ for $n = 1, 2, 3, 4, 5$. Since then, many other authors have verified Conjecture 2.2, or variations of it, in a variety of settings, both over number fields and function fields; see e.g. [Sha13, HLHN14, Tho19, Lan21a, FLR23, PW23, EL24]. One common feature of work on this conjecture in the function field case is that authors either usually restrict the characteristics of the function fields over which they work or only work over genus 0 function fields, but neither of these restrictions appear in Conjecture 2.2. In contrast, in my preprint [Ach23], I was able to prove:

Theorem 2.3 ([Ach23, Theorem B])**.** *Let $K = \mathbb{F}_q(B)$ be the function field of a nice[1] curve $B/\mathbb{F}_q$, and let ζ_B be its zeta function. Then,*

$$\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \le 1 + 2\zeta_B(2)\zeta_B(10), \quad \textit{and so} \quad \lim_{n\to\infty} \mathbb{E}[\#\operatorname{Sel}_2(E/\mathbb{F}_{q^n}K)] \le 3.$$

This is the first paper proving such a result for a truly arbitrary global function field. Combined with work of Arul Shankar, this has the following aesthetically pleasing consequence.

Theorem 2.4 ([Ach23, Sha13])**.** *Let K be any global field. Then, $\mathbb{E}[\operatorname{rank} E(K)]$ is finite.*

2.2. **Brauer Groups of Stacky Curves.** *Algebraic stacks* are generalizations of schemes which allows spaces to have stabilizer/automorphism groups attached to their points. They provide a particularly useful context for studying moduli spaces (spaces whose points parameterize other geometric objects of interest) because the objects parameterized often have automorphism. For example, the moduli space $\mathcal{Y}(1)$ of elliptic curves exists as an algebraic stack, but does *not* exist as a scheme because elliptic curve always have at least one non-trivial automorphism (e.g. negation).

Brauer groups $\operatorname{Br}(-) \coloneqq \mathrm{H}^2_{\text{ét}}(-, \mathbb{G}_m)_{\text{tors}}$ of schemes/stacks are particular geometric invariants which are used, e.g., to obstruct points on varieties (see [Poo17, Chapter 8]), and which have received much recent attention in the stacky setting. One can see, for example, the work of Antieau–Meier and others [AM20, Shi19, LP22] computing the Brauer group of $\mathcal{Y}(1)$ and the work of Santens [San23] studying Brauer–Manin obstructions to integral points on stacky curves. In [ABJ$^+$24, Ach24] I have been studying techniques for computing Brauer groups of reasonably general stacky curves. This has allowed me to both extend previous work on specific examples (see [AM20, Mei18, LP22, ABJ$^+$24]) and to compute Brauer groups of fairly general stacky curves over algebraically closed fields.

Theorem 2.5 ([Ach24, Theorem B])**.** *Let S be a noetherian $\mathbb{Z}[1/2]$-scheme.*

(1) $\operatorname{Br}\mathcal{X}(1)_S \simeq \operatorname{Br}\mathbb{P}^1_S \simeq \operatorname{Br} S$*, where $\mathcal{X}(1)$ is the moduli stack of generalized elliptic curves.*
(2) If S is regular, then there is an explicit isomorphism

$$\operatorname{Br}\mathbb{A}^1_S \oplus \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/12\mathbb{Z}) \xrightarrow{\sim} \operatorname{Br}\mathcal{Y}(1)_S.$$

(3) If S is regular, then there is an explicit isomorphism

$$\operatorname{Br}(\mathbb{A}^1_S \setminus \{0\}) \oplus \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/4\mathbb{Z}) \oplus \mathrm{H}^0_{\acute{e}t}(S, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \operatorname{Br}\mathcal{Y}_0(2)_S,$$

where $\mathcal{Y}_0(2)$ is the moduli stack of elliptic curves equipped with a subgroup of order 2.

[1] smooth, projective, and geometrically connected

Theorem 2.6 (See [Ach24, Proposition 7.10])**.** *Let k be an algebraically closed field, and let $\mathcal{X}/k$ be a regular cyclotomic[2] stacky curve with generic stabilizer group G/k. Then,* $\operatorname{Br}\mathcal{X} \simeq \mathrm{H}^1_{\acute{e}t}(X, G^\vee)$.

2.3. **Integral points on higher-dimensional varieties.** The basic problem of arithmetic geometry is understanding integral points on varieties (note that 'integral points' = 'rational points' if the variety is proper). In the case of curves, we have amazing results of Faltings and Siegel.

Theorem 2.7 (Siegel [Sie14], Faltings [Fal83])**.** *Let X be a (not necessarily proper) hyperbolic curve. Then, every set of integral points on X is finite.*

Work of Lang and of Vojta has produced a conjectural generalization. Informally, A smooth quasi-projective variety U is `arithmetically hyperbolic` if it only has finitely many integral points (even after field extension) and calls it `pseudo-arithmetically hyperbolic` if all but finitely many of its integral points belong to some closed subvariety $Z \subsetneq U$ (see e.g. [Jav20] for precise definitions).

Conjecture 2.8 (Lang–Vojta, [Lan86, Conjecture 5.7] and [Voj87, Proposition 4.1.2])**.** *Let X be a nice projective variety over a number field K, and let $D \subset X$ be a normal crossings divisor. If $\omega_X(D)$ is ample, then $X \setminus D$ is pseudo-arithmetically hyperbolic.*

Faltings [Fal91] proved Conjecture 2.8 when X is an abelian variety or subvariety thereof (and Vojta [Voj96, Voj99] obtained analogous results for semiabelian varieites). Beyond these cases, most work on Conjecture 2.8 has focused on cases where the divisor D has many irreducible components (see e.g. [CZ04, Lev09, Aut11, RV20, RV21]). Perhaps the most famous example of a higher-dimensional X (which is not an abelian variety) with irreducible $D \subset X$ for which $X \setminus D$ is *provably* arithmetically hyperbolic is Faltings' [Fal02] construction of such D on $X = \mathbb{P}^2$. In the paper [AM23], Jackson Morrow and I constructed new examples of irreducible divisors D on nice varieties X such that $X \setminus D$ is arithmetically hyperbolic.

Theorem 2.9 ([AM23, Corollary B])**.** *Let X be a smooth projective variety over a number field K. Assume that $\dim X \geq 2$ and that $\pi_1^{\acute{e}t}(X_{\overline{K}})$ is infinite. Then, there exists infinitely many ample irreducible divisors $D \subset X$ such that $X \setminus D$ is arithmetically hyperbolic. If $X(K) \neq \emptyset$, then one can arrange that these D are geometrically irreducible.*

3. Future Work

3.1. **Project 1: Improved average rank bounds over global function fields.** The first project I propose is to improve the elliptic curve average rank bounds I obtained in [Ach23], especially over function fields of characteristic 2. A first very natural continuation of [Ach23] would be to strengthen Theorem 2.3 to state that $\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \leq 3$ over any fixed K. This would mainly involve computing local densities for 'hyper-Weierstrass curves' [Ach23, Definition 4.1.2] which are minimal (have at worst rational singularities) and locally solvable.

Problem 1. *Prove that $\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \leq 3$ for K as in Theorem 2.3.*

A further extension of this would would be to prove analogous bounds for the average size of n-Selmer, at least for $n \leq 5$ (as in the work of Bhargava and Shankar). This should be doable by combining ideas introduced in [Ach23] with those present in [dJ02, BS15b, BS13a, BS13b]. In the case $n = 3$, [dJ02] proves an analogue of Theorem 2.3 for $\mathbb{F}_q(t)$, but the $n \geq 3$ case is still open for higher genus function fields (and even for genus 0 if $n \geq 4$).

[2] An algebraic stack is cyclotomic if all its stabilizer groups of the form μ_n for some n

Problem 2. *Prove an analogue of Theorem 2.3 for $n = 3, 4, 5$. Following [BS13b], use this to prove that $\mathbb{E}[\operatorname{rank} E(K)] \leq 1 - \delta$ for some $\delta > 0$ (independent of K) when K is a global function field.*

Remark 3.1. As explained in [Cow21, Section 4], Problem 2 would give strong evidence in support of Alex Cowan's conjecture [Cow21, Conjecture 1.1] that '100% of elliptic *surfaces* over $\mathbb{Q}$ have rank 0' (viewed as elliptic curves over $\mathbb{Q}(T)$). ∘

In addition to "parameterize-and-count," there is a second strategy for studying Conjecture 2.2 in the function field setting. In brief, one can reduce it to the problem of controlling the cohomology groups of certain moduli spaces; see work of Aaron Landesman and his collaborators [Lan21b, FLR23, EL24]. These works avoid characteristic 2, but it would be interesting to remove this restriction, at least starting with the analogue of [Lan21b]. In this case, as mentioned in [Lan21b, Remark 1.10], the main obstacle is to compute the images of certain (wildly ramified) monodromy representations.

Problem 3. *Extend the main result of [Lan21b] to characteristic 2, say for n-Selmer with n odd.*

3.2. **Project 2: Limits of the "parameterize-and-count" strategy for Selmer averages.** For $n \geq 3$, one most commonly parameterizes n-Selmer elements by representing them as (locally solvable) smooth degree n, genus 1 curves embedded in $\mathbb{P}^{n-1}$. One expects that these objects *cannot* be parameterized once n is large enough, and it would be interesting to prove this for some explicit value of n. Let H_n denote the Hilbert scheme of such curves. In practice, a parameterization of such curves (e.g. a description of the equations needed to cut out such a curve in $\mathbb{P}^{n-1}$) amounts to a dominant rational map $f\colon \mathbb{A}^N \dashrightarrow H_n$, for some N. If H_n is not unirational, there can be no such parameterization, and so the "parameterize–and–count" strategy may be less feasible for such n.

Problem 4. *Find an explicit n such that H_n is not unirational.*

Remark 3.2. Various related moduli spaces have been studied before, e.g. in [dJF08, Section 4] and in [Lan]. Furthermore, rationality problems for moduli spaces of curves have been considered by many authors, including Joe Harris and his collaborators (see e.g. [HM82, Har84, EH84]). The ideas present in these works can serve as a starting point for considering Problem 4. ∘

One can ask an analogue of Problem 4 in the number field counting setting (e.g. in the setting of questions such as "How many S_n-extensions of $\mathbb{Q}$ are there of discriminant bounded?"). Here, one is interested in parameterizing rank n free $\mathbb{Z}$-algebras, and again, one suspects this is not possible for n sufficiently large. The relevant moduli space here has previously been studied in [Poo08].

3.3. **Project 3: An étale-Brauer obstruction to elliptic schemes over $\mathbb{Z}$.** Brauer groups are used to define obstructions to points on varieties (see e.g. [Poo17, Chapter 8]). For stacky curves, Santens [San23] has studied their Brauer–Manin obstructions when the curve in question contains a dense, open subscheme. However, I propose to look at an example which is everywhere stacky: the moduli stack $\mathcal{Y}(1)$ of elliptic curves. I anticipate that its integral étale-Brauer obstruction set is empty. This would give a new proof of the nonexistence of an elliptic scheme over $\mathbb{Z}$. Carrying out such a computation would also serve as a proof-of-concept showcasing that exploiting Brauer groups of moduli stacks can be effective in determining whether certain classes of geometric objects can have everywhere good reduction. It would be my hope that resolving Problem 5 would motivate more people to study Brauer groups of stacks as well as their associated obstructions.

Problem 5. *Compute the integral étale-Brauer obstruction set $\mathcal{Y}(1)(\widehat{\mathbb{Z}} \times \mathbb{R})^{\text{ét-Br}} \supset \mathcal{Y}(1)(\mathbb{Z})$.*

4. Career Development & Choice of Mentor and Host Institution

I would be hard pressed to find a mathematician more well-versed in arithmetic statistics than Melanie Matchett Wood, my proposed host. Her expertise would be invaluable for investigating the problems described in Sections 3.1 and 3.2. While I have so far focused on statistics of Selmer groups, I expect that working with her would also allow me to delve into other aspects of arithmetic statistics as well. In addition to being personally enriching, this would also be especially useful for problems like Problem 4, which exists on both the Selmer and number field counting sides. Additionally, Harvard has Joe Harris who has done much work on computing Kodaira dimensions of moduli space and so who would also be a helpful person to discuss Problem 4 with. Melanie's presence at Harvard draws in many your arithmetic statisticians who would all make great peers. For example, Aaron Landesmann is currently at Harvard and he would be the ideal person to discuss the challenges involved in Problem 3 with. Furthermore, the nearby MIT has Bjorn Poonen, my current PhD advisor. It would be great to still have access to him and his expertise. In particular, he is an expert on Brauer–Manin obstructions and on stacky curves, both directly related to Problem 5.

5. Broader Impacts

I firmly believe that mathematics can be enjoyed and understood by anyone and in the importance of helping students feel comfortable and confident in their abilities and in their place within the mathematical community. I propose to continue my existing pattern of outreach and to make active efforts towards making more students comfortable in Harvard's broader mathematical community.

Every year in grad school, I have served as a mentor for MIT's *directed reading program*, where I have mentored 5 different students as they learned 5 topics of interest to them. Additionally, I have opted to be a teaching assistant for a couple online courses. In the summer of 2021, I served as a teaching assistant (TA) for the undergraduate session of the Park City Math Institute (PCMI), and in the Fall of 2022, I served as a TA for the Preliminary Arizona Winter School (PAWS). Both of these positions were for online courses serving a large, varied collection of undergraduates (and some grad students in the case of PAWS). I am still able to occasionally able to assist with PAWS and the broader, in-person Arizona Winter School (AWS). Last year, I served as one of the study group leaders for AWS and this year I will be on a PAWS panel about navigating grad school. As an undergraduate, I was a TA and residential counselor for the *Stanford University Mathematics Camp* (SUMaC) and a TA for *Euler Circle*, a Bay Area-based math circle. SUMaC served students from accross the globe, and Euler Circle teaches math courses – ranging from general *mathematical thinking*, which serves to bridge the gap between algorithmic and conceptual ways of thinking about math, to more specialized topics like *complex analysis* – to local students.

At Harvard, I would like to join ongoing efforts towards broadening participation in mathematics, such as their *Real Representations*, *Community Committee*, and *Math Includes* programs. I would also like to add to list of efforts by holding frequent "Open Office Hours." In brief, I envision this as a space where, for roughly two hours twice a week, students could come ask me general guidance questions and where I would invite other postdocs to come make themselves available to meet and support undergraduates. This would help increase interaction between undergrads and postdocs (at Harvard, many postdocs do not teach and so do not interact much with undergrads), and I believe that serving as a visibly black entry point into the broader Harvard mathematics community would allow me to draw more underrepresented minorities into the community (who might otherwise be more hesitant to put themselves out there).