MATH-GA 1002.001 Spring 2014
Multivariable Analysis
Midterm Exam
March 24th

Your name (please print): ____
NYU ID: ____

Instructions:

- Please begin by printing your name and student ID in the box above.
- There are 14 short questions each worth 8 points, a total of 100+12 = 112 points. Additional 12 points are for bonus.
- You are allowed 2 hours and 50 mins (170 mins) for this exam. Please pace yourself accordingly.
- This is an in-class, closed-book exam. No books, no notes, no crib sheets, no calculators, no smartphones are allowed.
- The Honor Principle requires that you neither give nor receive any aid on this exam.
- You must justify all of your answers to receive credit. The correct answer without any work will receive little or no credit.
- Please show all your work on this exam sheet. You can also use the backs of the pages. You are not allowed any scratch paper of your own.
- Good Luck!

Question	Points	Score	Question	Points	Score
1a)	8		2a)	8	
1b)	8		2b)	8	
1c)	8		2c)	8	
1d)	8		2d)	8	
1e)	8		2e)	8	
1f)	8		2f)	8	
1g)	8		2g)	8	
Ques 1)	56		Ques 2)	56	

Grade: /100

1. Suppose C and D are nonempty disjoint closed convex sets, i.e. $C \cap D = \emptyset$. Define the distance function between the sets C and D as

$$\operatorname{dist}(C, D) = \inf \{ |u - v| \mid u \in C, v \in D \} .$$

We will assume that

($\mathcal{A}$) there exist points $c \in C$ and $d \in D$ that achieve the minimum distance, i.e., $|c - d| = \operatorname{dist}(C, D)$ and $\operatorname{dist}(C, D) > 0$.

(a) **(8 points)** Show that the set $K_{C,D} = \{x - y \mid x \in C, y \in D\}$ is convex and does not contain the origin. Prove that if C and D are bounded, the assumption ($\mathcal{A}$) is satisfied. Hint: Show that $K_{C,D}$ is closed and then use the fact that the norm function $|\cdot|$ is continuous.

(b) **(8 points)** Are there any closed sets C and D where the above assumption ($\mathcal{A}$) does not hold? If yes, give an example. If no, provide a proof. Hint: Think about unbounded convex sets and their boundary.

(c) **(8 points)** Define

$$a = d - c, \quad b = \frac{|d|^2 - |c|^2}{2}$$

and the affine function

$$f(x) = a^T x - b = (d - c)^T (x - (1/2)(d + c)).$$

Show that for $u \in D$,

$$f(u) = (d - c)^T (u - d) + (1/2)|d - c|^2.$$

(d) **(8 points)** Using part (c), show that if $f(u) < 0$ for some $u \in D$, then there exist a $t > 0$ with $t \leq 1$ such that we have

$$|d + t(u - d) - c| < |d - c|.$$

Hint: Let $g(t) = |d + t(u - d) - c|^2$. Compute $g'(0)$.

(e) **(8 points)** Using part (d), show that f is non-negative on D.

(f) **(8 points)** Show that f is non-positive on C. Hint: Swap C and D, consider $-f$ and use part (e).
Note: This result shows that the hyperplane $\{x | a^T x = b\}$ separates C and D. This hyperplane is perpendicular to the line segment between c and d, and passes through its midpoint. It is called a *separating hyperplane*.

(g) **(8 points)** Define an alternative distance function for sets as

$$\operatorname{dist}_\infty(C, D) = \inf\left\{|u - v|_\infty := \max_i |u_i - v_i| \mid u \in C, v \in D\right\}.$$

If $\operatorname{dist}_\infty(C, D) > 0$, do we always have $\operatorname{dist}(C, D) > 0$? Do we always have $|c - d| = \operatorname{dist}_\infty(C, D)$? If yes, prove it. If no, give a counter-example. Hint: All non-Euclidean norms are equivalent to the Euclidean (standard) norm for Euclidean spaces but they have different level-sets.

2. **(Adapted from a homework problem, Section 2.11 of the book)**
Define the function $f_{\alpha,\beta}(x, y, z) = \alpha x^2 + xy + 4y^2 + \beta|z|^2$ where $\alpha, \beta \in \mathbb{R}$.

   (a) **(8 points)** For which values of α and β, is $f_{\alpha,\beta}$ convex, concave or neither?

(b) **(8 points)** For which values of α and β, is $f_{\alpha,\beta}$ strictly convex, strictly concave?

(c) **(8 points)** Find the critical point(s) of $f_{\alpha,\beta}$. Note that there is a dependence on α and β. Hint: The critical points of a function are where the differential is zero.

(d) **(8 points)** For which values of α and β, is the function $f_{\alpha,\beta}$ bounded below? Find the minimum of $f_{\alpha,\beta}$ if it exists.

(e) **(8 points)** Define the set $K_{\alpha,\beta} = \{(x, y, z) : f_{\alpha,\beta}(x, y, z) \leq 2\}$. For which values of α and β, is $K_{\alpha,\beta}$ the unit ball of a (non-Euclidean) norm?

(f) **(8 points)** Is $f_{\alpha,\beta}$ differentiable? Is $f_{\alpha,\beta}$ continuously differentiable? For which values of α and β, is $f_{\alpha,\beta}$ (real) analytic?

(g) **(8 points)** Does the directional derivative of $f_{\alpha,\beta}$ exist at origin in the direction of $v = (0, 0, 1)$?

0.1 Solutions

1(a) Let $u = c_1 - d_1, v = c_2 - d_2 \in K_{C,D}$ where $c_1, c_2 \in C$ and $d_1, d_2 \in D$. Then for any $t \in [0,1]$, $tu + (1-t)v = (tc_1 + (1-t)c_2) - (td_1 + (1-t)d_2) \in K_{C,D}$ because by the convexity of C and D, we have $(tc_1 + (1-t)c_2) \in C$ and $(td_1 + (1-t)d_2) \in D$.

Since C and D are disjoint, $K_{C,D}$ does not contain the origin. $K_{C,D}$ is clearly bounded as C and D are. We will show that it is also closed. Take a sequence $\{z_n\}$ in $K_{C,D}$ that converges to a point z. It suffices to show that $z \in K_{C,D}$. We can write $z_n = c_n - d_n$ with $c_n \in C$ and $d_n \in D$. Since $\{c_n\}$ is bounded, there exists a subsequence $\{c_{n_j}\}$ that converges to a point c. Furthermore, $c \in C$ as C is closed. This implies that $d_{n_j} \to c - z$, and $c - z \in D$ as D is closed. Thus, $z = c - (c - z) \in K_{C,D}$.

$K_{C,D}$ is compact (closed and bounded), so the Euclidean norm function $|\cdot|$ attains its minimum at a point $z_* = c_* - d_*$, with $|z_*| = |c_* - d_*| = \text{dist}(C, D)$ which is strictly positive as origin is not contained in $K_{C,D}$.

1(b) Assumption does not necessarily hold for unbounded sets. Ex: $C = \{(x,y) \mid y \geq e^{-x}\}$ and $D = \{(x,y) \mid y \geq 0\}$.

1(c)

$$
\begin{aligned}
f(u) &= (d-c)^T(u - (d+c)/2) \\
&= (d-c)^T(u-d) + (d-c)^T(d - (d+c)/2) \\
&= (d-c)^T(u-d) + (1/2)|d-c|^2.
\end{aligned}
$$

1(d) By part c), $f(u) < 0$ implies $g'(0) = 2(d-c)^T(u-d) < 0$. Thus, the function g is decreasing around 0. Since we have $g(0) = |d-c|^2$, there exists a $t > 0$ with $t \leq 1$ such that $g(t) < |d-c|^2$. The results follows.

1(e) Assume $f(u) < 0$ for some $u \in D$. By part (d), there exist a $t > 0$ with $t \leq 1$ such that the point $|d + t(u-d) - c| < |d-c|$. By convexity of D, we have $\tilde{d} = d + t(u-d) = tu + (1-t)d \in D$ and $|\tilde{d} - c| < |d - c| = \text{dist}(C, D)$. Contradiction.

1(f) Swapping C and D (hence swapping d and c), we could define the function $\tilde{f} = (c-d)^T(x - (1/2)(d+c))$. Note that $\tilde{f} = -f$ and $\tilde{f}$ is non-negative by a reasoning similar to part (e).

1(g) Answer is "Yes" to the first question, because all the norms are equivalent in (finite dimensional) Euclidean spaces. Answer is "No" to the second question. Let $C = \{(x,y) \mid x + y = 1\}$ and $D = \{(x,y) \mid x + y = -1\}$. $\text{dist}(C,D) = \sqrt{2}$, however $\text{dist}_\infty(C, D) = 1$.

2(a) The Hessian is

$$
H = \begin{pmatrix} 2\alpha & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & \beta \end{pmatrix}.
$$

Computing the determinant of $k \times k$ sub-matrices of H for $k \in \{1, 2, 3\}$, H is positive semidefinite iff $\alpha \geq 0, 16\alpha - 1 \geq 0$ and $\beta \geq 0$. f is convex iff $\alpha \geq 1/16$ and $\beta \geq 0$. f is concave iff $-f$ is convex. By a similar computation, we get f is concave iff $\alpha \leq 0$, $\alpha \geq 1/16$ and $\beta \leq 0$ which is impossible.So f is never concave. f is neither convex nor concave, iff f is not convex.

2(b) Similar to part 2(a), it suffices to convert inequalities to strict inequalities. f is strictly convex iff $\alpha > 1/16$ and $\beta > 0$. f is never strictly concave.

2(c) A critical point is a solution of the equation $\nabla f_{\alpha,\beta} = (2\alpha x + y, x + 8y, 2\beta z) = 0$ which is equivalent to the system of equations

$$\begin{aligned} x &= -8y \\ y(16\alpha - 1) &= 0 \\ \beta z &= 0. \end{aligned}$$

Case 1: ($\alpha \neq 1/16$ and $\beta \neq 0$) The only solution is $(x, y, z) = 0$.

Case 2: ($\alpha = 1/16$ and $\beta \neq 0$) The solutions are $(-8y, y, 0)$ with $y \in \mathbb{R}$.

Case 3: ($\alpha \neq 1/16$ and $\beta = 0$) The solutions are $(0, 0, z)$ with $y, z \in \mathbb{R}$.

Case 4: ($\alpha = 1/16$ and $\beta = 0$) The solutions are $(-8y, y, z)$ with $y, z \in \mathbb{R}$.

2(d) If $\beta < 0$, $f_{\alpha,\beta}(x_0, y_0, z) \to -\infty$ as $z \to \infty$ for $x_0, y_0 \in \mathbb{R}$ fixed so the function is not bounded below and the minimum does not exist. Similarly, if $\alpha < 0$, $f_{\alpha,\beta}$ is not bounded below. If $\alpha = 0$, it is easy to see that $f_{\alpha,\beta}(x, 1, 0) \to -\infty$ as $x \to \infty$. Assume now $\alpha > 0$ and $\beta \geq 0$. The function $\tilde{f}_{\alpha,\beta}(x, y) = \alpha x^2 + xy + 4y^2$ is a quadratic and is bounded below. Hence, $f_{\alpha,\beta}$ is bounded below and a minimizer exists by continuity. We also have

$$\begin{aligned} f_{\alpha,\beta}(x, y, z) &= \tilde{f}_{\alpha,\beta}(x, y) + \beta z^2 && (1) \\ &= \alpha(x + \frac{y}{2\alpha})^2 + y^2 \frac{16\alpha - 1}{4\alpha} + \beta z^2. && (2) \end{aligned}$$

We see that if $\alpha < 1/16$, $f_{\alpha,\beta}$ is not bounded below over the line $\{(x, y, 0) \in \mathbb{R}^3 \mid x + \frac{y}{2\alpha} = 0\}$. The remaining cases are

Case 1: ($\alpha > 1/16$ and $\beta > 0$) The critical point $(x, y, z) = 0$ is the (global) minimizer with a minimum value 0. In this case, f is strictly convex so there can be at most one minimizer.

Case 2: ($\alpha = 1/16$ and $\beta > 0$) All the critical points of the form $(-8y, y, 0)$ with $y \in \mathbb{R}$ are (global) minimizers with a minimum value 0.

Case 3: ($\alpha > 1/16$ and $\beta = 0$) All the critical points of the form $(0, 0, z)$ with $y, z \in \mathbb{R}$ are (global) minimizers with a minimum value 0.

Case 4: ($\alpha = 1/16$ and $\beta = 0$) All the critical points of the form $(-8y, y, z)$ with $y, z \in \mathbb{R}$ are (global) minimizers with a minimum value 0.

Note that f is convex in all these cases when a minimizer exists (see part 2(c)).

2(e) When f is convex $K_{\alpha,\beta}$ is convex, compact, symmetric with respect to the origin and contains a neighborhood of the origin, hence one can define a norm by the Minkowski functional for which $K_{\alpha,\beta}$ becomes the unit ball.

2(f) For any value of α and β, $f_{\alpha,\beta}$ is a polynomial so it has continuous derivatives up to any order and is real-analytic.

2(g) Yes. The directional derivative exists in any direction. See part 2(f).