Multivariable Analysis HW6 Zhuolun Yang zy451

1.a) Since ω has an integrating factor f, i.e. there exists a function g such that $dg = f\omega$.

Therefore,

$$0 = d(dg) = d(f\omega) = df \wedge \omega + f d\omega$$

$$\Rightarrow d\omega = -\frac{1}{f} df \wedge \omega \quad \text{since } f(x) \neq 0.$$

Then

$$\omega \wedge d\omega = \omega \wedge [-\tfrac{1}{f} df \wedge \omega] = -\tfrac{1}{f}\, \omega \wedge df \wedge \omega = 0.$$

b) For $\frac{dy}{dx} + P(x)y = Q(x)$, if P and Q are continuous, $e^{\int_0^x p(t)dt}$ will be the integrating factor for LHS

Then $d(e^{\int_0^x p(t)dt} y) = e^{\int_0^x p(t)dt}(\frac{dy}{dx} + P(x)y) = e^{\int_0^x p(t)dt} Q(x)$.

$$\Rightarrow y(x) = e^{-\int_0^x p(t)dt}\left[\int_0^x e^{\int_0^s p(t)dt} Q(s)\,ds + C\right]$$

where C is some constant.

2. Since

$$\begin{pmatrix} \omega^1(x) \\ \omega^2(x) \\ \vdots \\ \omega^p(x) \end{pmatrix} = \begin{pmatrix} f_1^1(x) & f_2^1(x) & \cdots & f_p^1(x) \\ f_1^2(x) & f_2^2(x) & \cdots & f_p^2(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^p(x) & f_2^p(x) & \cdots & f_p^p(x) \end{pmatrix} \begin{pmatrix} dg^1(x) \\ dg^2(x) \\ \vdots \\ dg^p(x) \end{pmatrix}$$

and we know that $\omega^1(x), \dots, \omega^p(x)$ are linearly independent for every $x \in D$. Therefore the $p \times p$ matrix $(f_j^i(x))$ must be nonsingular and it has inverse, say $(h_j^i(x))$ for any $x \in D$. Therefore we can define functions h_j^i on D, such that

$$\begin{pmatrix} h_1^1 & h_2^1 & \cdots & h_p^1 \\ h_1^2 & h_2^2 & \cdots & h_p^2 \\ \vdots & \vdots & \ddots & \vdots \\ h_1^p & h_2^p & \cdots & h_p^p \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \vdots \\ \omega^p \end{pmatrix} = \begin{pmatrix} dg^1 \\ dg^2 \\ \vdots \\ dg^p \end{pmatrix}$$

Then

$$\begin{aligned} d\omega^i &= \sum_{j=1}^p df_j^i \wedge dg^j \\ &= df_1^i \wedge dg^1 + df_2^i \wedge dg^2 + \cdots + df_p^i \wedge dg^p \\ &= df_1^i \wedge (\sum_{j=1}^p h_j^1 \omega^j) + df_2^i \wedge (\sum_{j=1}^p h_j^2 \omega^j) + \cdots + df_p^i \wedge (\sum_{j=1}^p h_j^p \omega^j) \\ &= \sum_{j=1}^p h_j^1 df_1^i \wedge \omega^j + \sum_{j=1}^p h_j^2 df_2^i \wedge \omega^j + \cdots + \sum_{j=1}^p h_j^p df_p^i \wedge \omega^j \\ &= \sum_{j=1}^p (\sum_{k=1}^p h_j^k df_k^i) \wedge \omega^j. \end{aligned}$$

If we define 1-forms $\theta_j^i := \sum_{k=1}^p h_j^k df_k^i$, then $d\omega^i = \sum_{j=1}^p \theta_j^i \wedge \omega^j$

3. It suffices to prove that $\beta_0 = \sum_{i=1}^{r} (-1)^{i-1} \beta_i$.

Let e_i denote $X_i - X_0$, then for $k = 1, \cdots, r$,

$\beta_k = \frac{1}{(r-1)!} e_1 \wedge e_2 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_r$.

$$\beta_0 = \frac{1}{(r-1)!} (X_2 - X_1) \wedge (X_3 - X_1) \wedge \cdots \wedge (X_r - X_1)$$
$$= \frac{1}{(r-1)!} (e_2 - e_1) \wedge (e_3 - e_1) \wedge \cdots \wedge (e_r - e_1) \qquad (*)$$

Let I_k denote $(e_k - e_1)$ for $k = 2, \cdots, r$.

Now I want to expand the exterior product in $(*)$ by treating $e_k, -e_1$ as two elements in I_k and choosing one element from each I_k to form an $r-1$ vector by exterior product and sum them up.

(For example, if I choose e_k from each I_k, I will have the term $\frac{1}{(r-1)!} e_2 \wedge e_3 \wedge \cdots \wedge e_r$)

Note that if I choose $-e_1$ simultaneously from two different I_k, the term will make no contribution. Also, since $e_1, \cdots, e_r$ are linearly independent, terms for other cases will make contribution.

Then, if I don't choose $-e_1$ from any I_k, I will have a term

$$\frac{1}{(r-1)!} e_2 \wedge e_3 \wedge \cdots \wedge e_r.$$

If I choose $-e_1$ from I_k, the term that makes contribution will be

$$\frac{1}{(r-1)!} e_2 \wedge \cdots \wedge e_{k-1} \wedge (-e_1) \wedge e_{k+1} \wedge \cdots \wedge e_r.$$

Therefore, $(*) = \frac{1}{(r-1)!} e_2 \wedge e_3 \wedge \cdots \wedge e_r + \sum_{k=2}^{r} \frac{1}{(r-1)!} e_2 \wedge \cdots \wedge e_{k-1} \wedge (-e_1) \wedge e_{k+1} \wedge \cdots \wedge e_r$

If I want to shift $-e_1$ from $k-1$ th place to 1st place, I need to do $k-2$ times interchange.

$$\text{Hence } (*) = \beta_1 + \sum_{k=2}^{r} \frac{1}{(r-1)!} (-1)^{k-1} e_1 \wedge e_2 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_r$$
$$= \beta_2 + \sum_{k=2}^{r} (-1)^{k-1} \beta_k$$
$$= \sum_{k=1}^{r} (-1)^{k-1} \beta_k.$$

$$\Rightarrow \beta_0 = \sum_{i=1}^{r} (-1)^{i-1} \beta_i \quad \Rightarrow \sum_{i=0}^{r} (-1)^i \beta_i = 0.$$

4. a) when $n=4 \quad r=2 \quad \lambda=(1,2)$.

Then we have
$$\begin{cases} g^1(x^1,x^2) = x^1 \\ g^2(x^1,x^2) = x^2 \\ g^3(x^1,x^2) = \phi^1(x^1,x^2) \\ g^4(x^1,x^2) = \phi^2(x^1,x^2). \end{cases}$$

$$g_1(x^1,x^2) = (1,0,\phi_1^1,\phi_1^2) = e_1+\phi_1^1 e_3+\phi_1^2 e_4$$
$$g_2(x^1,x^2) = (0,1,\phi_2^1,\phi_2^2) = e_2+\phi_2^1 e_3+\phi_2^2 e_4.$$

Then
$$\begin{aligned} g_1\wedge g_2 &= e_1\wedge e_2+\phi_2^1 e_1\wedge e_3+\phi_2^2 e_1\wedge e_4+\phi_1^1 e_3\wedge e_2+\phi_1^1\phi_2^2 e_3\wedge e_4 \\ &\quad +\phi_1^2 e_4\wedge e_2+\phi_1^2\phi_2^1 e_4\wedge e_3 \\ &= e_1\wedge e_2+\phi_2^1 e_1\wedge e_3+\phi_2^2 e_1\wedge e_4-\phi_1^1 e_2\wedge e_3-\phi_1^2 e_2\wedge e_4 \\ &\quad +(\phi_1^1\phi_2^2-\phi_1^2\phi_2^1)e_3\wedge e_4. \end{aligned}$$

Therefore
$$\begin{aligned} \mathcal{L}g(x^1,x^2) = |g_1\wedge g_2| &= [1+(\phi_2^1)^2+(\phi_2^2)^2+(\phi_1^1)^2+(\phi_1^2)^2+(\phi_1^1\phi_2^2-\phi_1^2\phi_2^1)^2]^{\frac{1}{2}} \\ &= [1+|\text{grad}\,\phi^1|^2+|\text{grad}\,\phi^2|^2+(\phi_1^1\phi_2^2-\phi_1^2\phi_2^1)^2]^{\frac{1}{2}} \end{aligned}$$

b) In this case, $\phi^1(x^1,x^2)=(x^1)^2-(x^2)^2$, $\phi^2(x^1,x^2)=2x^1x^2$.

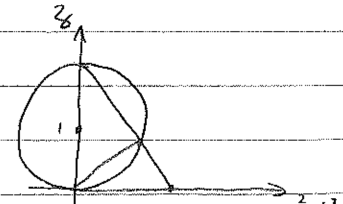
Then
$$\begin{cases} \phi_1^1 = 2x^1 \\ \phi_2^1 = -2x^2 \end{cases} \qquad \begin{cases} \phi_1^2 = 2x^2 \\ \phi_2^2 = 2x^1 \end{cases}$$

Therefore
$$\begin{aligned} \mathcal{L}g(x^1,x^2) &= [1+(2x^1)^2+(2x^2)^2+(2x^2)^2+(2x^1)^2+(4(x^1)^2+4(x^2)^2)^2]^{\frac{1}{2}} \\ &= [1+8(x^1)^2+8(x^2)^2+(4(x^1)^2+4(x^2)^2)^2]^{\frac{1}{2}} \end{aligned}$$

let $x^1=r\cos\theta$, $x^2=r\sin\theta$.

Then
$$\begin{aligned} V_2(A) &= \iint_{(x^1)^2+(x^2)^2\le 1} \mathcal{L}g(x^1,x^2)\,dx^1dx^2 \\ &= \int_0^1\int_0^{2\pi} [1+8r^2+(4r^2)^2]^{\frac{1}{2}}\, r\,d\theta\,dr \\ &= 2\pi\int_0^1 4r^3+r\,dr \\ &= 2\pi\cdot r^4+\frac{r^2}{2}\Big|_0^1 \\ &= 3\pi. \end{aligned}$$

5. a) From some elementary geometry we can see that
$$\begin{cases} \frac{x}{s} = \frac{2-z}{2} \\ \frac{y}{t} = \frac{2-z}{2} \end{cases} \Longrightarrow \begin{cases} x = \frac{2-z}{2}s \\ y = \frac{2-z}{2}t \end{cases} \qquad (*)$$



If we look at the s^2+t^2-z plane

From some similar triangles we can compute that plug z into $(*)$ we will have

$$x = \frac{4s}{4+s^2+t^2} \qquad y = \frac{4t}{4+s^2+t^2}.$$

Define $g(s,t) = \left(\frac{4s}{4+s^2+t^2}, \frac{4t}{4+s^2+t^2}, \frac{2s^2+2t^2}{4+s^2+t^2}\right)$.

Now I want to show g is regular.

- To see g is $\mathscr{C}^1$, let $h(x) = \frac{1}{4+x}$, $\ell(s,t) = s^2+t^2$.

Obviously both $h(x)$ and $\ell(s,t)$ are $\mathscr{C}^1$ for $x \geq 0$

Therefore $\frac{1}{4+s^2+t^2} = h \circ \ell(s,t)$ is $\mathscr{C}^1$

Also, $4s$, $4t$ and $2s^2+2t^2$ are $\mathscr{C}^1$ obviously

Then $\frac{4s}{4+s^2+t^2}$, $\frac{4t}{4+s^2+t^2}$ and $\frac{2s^2+2t^2}{4+s^2+t^2}$ are all $\mathscr{C}^1$

and hence $g(s,t)$ is $\mathscr{C}^1$.

- To see g is univalent, let (s',t') be the point such that

$$\begin{cases} \frac{4s}{4+s^2+t^2} = \frac{4s'}{4+s'^2+t'^2} & \cdots\cdots ① \\ \frac{4t}{4+s^2+t^2} = \frac{4t'}{4+s'^2+t'^2} & \cdots\cdots ② \\ \frac{2s^2+2t^2}{4+s^2+t^2} = \frac{2s'^2+2t'^2}{4+s'^2+t'^2} & \cdots\cdots ③ \end{cases}$$

From ② we know if $t=0$, then $t'=0=t$.

Then ③ becomes $\frac{s^2}{4+s^2} = \frac{s'^2}{4+s'^2} \Rightarrow \frac{4}{4+s^2} = \frac{4}{4+s'^2} \Rightarrow s = \pm s'$

From ① we can see that $s=-s' \neq 0$ is impossible. Hence $s=s'$.

If $t \neq 0$, from ② we know $t' \neq 0$

Then by ①, ② we will have $\frac{s}{t} = \frac{s'}{t'}$, define $k := \frac{s}{t} = \frac{s'}{t'}$

then $s = tk$ and $s' = t'k$.

plug into ③ we will have

$$\frac{(2k^2+2)t^2}{4+(k^2+1)t^2} = \frac{(2k^2+2)t'^2}{4+(k^2+1)t'^2} \Rightarrow \frac{4+(k^2+1)t^2}{t^2} = \frac{4+(k^2+1)t'^2}{t'^2} \Rightarrow \frac{4}{t^2} = \frac{4}{t'^2} \Rightarrow t = \pm t'$$

From ② we can see that $t=-t' \neq 0$ is impossible

Hence $t=t'$ and $s=s'$

Therefore g is univalent.

- Since $g_1(s,t) = \left(\frac{16-4s^2+4t^2}{(4+s^2+t^2)^2}, \frac{-8st}{(4+s^2+t^2)^2}, \frac{16s}{(4+s^2+t^2)^2}\right)$

$g_2(s,t) = \left(\frac{-8st}{(4+s^2+t^2)^2}, \frac{16+4s^2-4t^2}{(4+s^2+t^2)^2}, \frac{16t}{(4+s^2+t^2)^2}\right)$

Therefore $Dg(t)$ has rank 2

In conclusion, g is regular.

Since $F(x,y,z) = (s,t)$, if $F(x',y',z') = (s',t')$ and $(s,t) = (s',t')$.

Then $x = x' = \frac{4s}{4+s^2+t^2}$ $\quad y = y' = \frac{4t}{4+s^2+t^2}$ $\quad z = z' = \frac{2s^2+2t^2}{4+s^2+t^2}$

Therefore F is univalent and hence a coordinate system for M ...

And hence $V_2(M) = \int_0^l \int_0^{2\pi} \rho g(s,t)\, dt\, ds = 2\pi \int_0^l G^2(s) ds = 2\pi \bar{y} l$.
where $\bar{y} = \dfrac{\int_0^l G^2(s)ds}{\int_0^l ds}$, $\bar{x} = \dfrac{\int_0^l G^1(s)ds}{\int_0^l ds}$ and therefore $(\bar{x}, \bar{y})$ is the centroid of γ.

b). Define $g(\varphi,\theta) = (r_1 + r_2\cos\varphi)\cos\theta\, e_1 + (r_1 + r_2\cos\varphi)\sin\theta\, e_2 + r_2 \sin\varphi\, e_3$

let $M = g([0,2\pi) \times [0,2\pi))$.

Then M is the surface of the torus.

let $G^1(\varphi) := r_2 \sin\varphi$, $G^2(\varphi) := r_1 + r_2\cos\varphi$.

Since $G^{1\prime}(\varphi) = r_2\cos\varphi$, $G^{2\prime}(\varphi) = -r_2\sin\varphi$. then $|G'(\varphi)| = r^2$

By part a), we know that

$$\begin{aligned} V_2(M) &= \int_0^{2\pi}\int_0^{2\pi} |G'(\varphi)| \cdot G^2(\varphi)\, d\theta\, d\varphi \\ &= 2\pi \int_0^{2\pi} r_2 (r_1 + r_2\cos\varphi) d\varphi \\ &= 4\pi^2 r_1 r_2 + 2\pi \int_0^{2\pi} r_2^2 \cos\varphi\, d\varphi \\ &= 4\pi^2 r_1 r_2 \end{aligned}$$

6. Denote $\mathcal{O}_{d,2}$ by $\begin{pmatrix} X_1 & X_{d+1} \\ X_2 & X_{d+2} \\ \vdots & \vdots \\ X_d & X_{2d} \end{pmatrix}$. If $(X_1, X_2, \cdots, X_d)$ and $(X_{d+1}, X_{d+2}, \cdots, X_{2d})$ are orthonormal. Then we have 3 contraints.

$$\begin{cases} \Phi^1 = X_1 X_{d+1} + X_2 X_{d+2} + \cdots + X_d X_{2d} = 0 \\ \Phi^2 = X_1^2 + X_2^2 + \cdots + X_d^2 - 1 = 0 \\ \Phi^3 = X_{d+1}^2 + X_{d+2}^2 + \cdots + X_{2d}^2 - 1 = 0 \end{cases}$$

Then $\begin{cases} \nabla\Phi^1 = (X_{d+1}, X_{d+2}, \cdots, X_{2d}, X_1, X_2, \cdots, X_d) \\ \nabla\Phi^2 = (2X_1, 2X_2, \cdots, 2X_d, 0, 0, \cdots, 0) \\ \nabla\Phi^3 = (0, 0, \cdots, 0, 2X_{d+1}, 2X_{d+2}, \cdots, 2X_{2d}). \end{cases}$

From the constraints we can know that $X_1, \cdots, X_d$ cannot be all zero. Same for $X_{d+1}, \cdots, X_{2d}$. Hence $D\Phi$ will always have rank 3.

And $\mathcal{O}_{d,2}$ is a manifold with dimension $2d-3$.

And the tangent space is the kernel of $D\Phi$.

b) Since $Dg(s,t)$ form a basis for tangent space.

Assume $h_1 = Dg(s,t)\cdot\begin{pmatrix}a_1\\a_2\end{pmatrix}$ $\quad h_2 = Dg(s,t)\cdot\begin{pmatrix}b_1\\b_2\end{pmatrix}$

By Inverse Function Theorem, We know $DF(x,y,z) = [Dg(s,t)]^{-1}$

Therefore $k_1 = DF(x,y,z)\,h_1 = \begin{pmatrix}a_1\\a_2\end{pmatrix}$ $\quad k_2 = DF(x,y,z)\,h_2 = \begin{pmatrix}b_1\\b_2\end{pmatrix}$.

Since $g_1(s,t) = \frac{4}{(4+s^2+t^2)^2}\,(4-s^2+t^2,\ -2st,\ 4s)$

$\qquad g_2(s,t) = \frac{4}{(4+s^2+t^2)^2}\,(-2st,\ 4+s^2-t^2,\ 4t)$.

By direct computation we can see that

$$g_1(s,t)\cdot g_2(s,t) = 0 \quad \text{and} \quad |g_1(s,t)| = |g_2(s,t)|.$$

$$h_1 = \begin{pmatrix} g_1^1 & g_2^1\\ g_1^2 & g_2^2\\ g_1^3 & g_2^3\end{pmatrix}\cdot\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}a_1g_1^1+a_2g_2^1\\ a_1g_1^2+a_2g_2^2\\ a_1g_1^3+a_2g_2^3\end{pmatrix} \qquad h_2 = \begin{pmatrix} g_1^1 & g_2^1\\ g_1^2 & g_2^2\\ g_1^3 & g_2^3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}b_1g_1^1+b_2g_2^1\\ b_1g_1^2+b_2g_2^2\\ b_1g_1^3+b_2g_2^3\end{pmatrix}$$

Then $(h_1,h_2) = (a_1g_1^1+a_2g_2^1)(b_1g_1^1+b_2g_2^1) + (a_1g_1^2+a_2g_2^2)(b_1g_1^2+b_2g_2^2) + (a_1g_1^3+a_2g_2^3)(b_1g_1^3+b_2g_2^3)$

$$= a_1b_1\cdot|g_1|^2 + a_2b_2\cdot|g_2|^2 + (a_1b_2+a_2b_1)\cdot\cancelto{0}{(g_1\cdot g_2)}$$

$$= (a_1b_1+a_2b_2)\cdot|g_1|^2$$

$$|h_1|^2 = (a_1g_1^1+a_2g_2^1)^2 + (a_1g_1^2+a_2g_2^2)^2 + (a_1g_1^3+a_2g_2^3)^2$$

$$= a_1^2\cdot|g_1|^2 + 2a_1a_2\cdot\cancelto{0}{(g_1\cdot g_2)} + a_2^2\cdot|g_2|^2$$

$$= (a_1^2+a_2^2)\cdot|g_1|^2$$

Similarly $|h_2|^2 = (b_1^2+b_2^2)\cdot|g_1|^2$.

let $\langle h_1,h_2\rangle$ denote the angle between h_1 and h_2, then

$$\cos\langle h_1,h_2\rangle = \frac{(h_1,h_2)}{|h_1|\,|h_2|} = \frac{a_1b_1+a_2b_2}{\sqrt{a_1^2+a_2^2}\cdot\sqrt{b_1^2+b_2^2}}$$

Also, $\cos\langle k_1,k_2\rangle = \frac{(k_1,k_2)}{|k_1|\cdot|k_2|} = \frac{a_1b_1+a_2b_2}{\sqrt{a_1^2+a_2^2}\cdot\sqrt{b_1^2+b_2^2}}$

Therefore, the angle between k_1 and k_2 equals the angle between h_1 and h_2.

7. a). Since $g(s,t) = G^1(s)\,e_1 + G^2(s)\cos t\,e_2 + G^2(s)\sin t\,e_3$

Then $g_1(s,t) = G^{1'}(s)\,e_1 + G^{2'}(s)\cos t\,e_2 + G^{2'}(s)\sin t\,e_3$

$\qquad g_2(s,t) = -G^2(s)\sin t\,e_2 + G^2(s)\cos t\,e_3$

$$g_1\wedge g_2 = -G^{1'}(s)G^2(s)\sin t\ e_1\wedge e_2 + G^{1'}(s)G^2(s)\cos t\ e_1\wedge e_3$$
$$+ G^{2'}(s)G^2(s)\cos^2 t\ e_2\wedge e_3 - G^{2'}(s)G^2(s)\sin^2 t\ e_3\wedge e_2$$
$$= -G^{1'}(s)G^2(s)\sin t\ e_1\wedge e_2 + G^{1'}(s)G^2(s)\cos t\ e_1\wedge e_3$$
$$+ G^{2'}(s)G^2(s)\ e_2\wedge e_3.$$

Therefore $\mathcal{L}g(s,t) = |g_1\wedge g_2| = [(G^{1'}(s)G^2(s))^2 + (G^{2'}(s)G^2(s))^2]^{\frac{1}{2}}$

$$= [\,|G'(s)|^2(G^2(s))^2\,]^{\frac{1}{2}} = G^2(s) \qquad \text{since } |G'(s)| = 1$$