4.7.1 Let $F(x,y)=c$. When $c<e^2$, the level set doesn't exist. When $c=e^2$, the level set is a point and when $c>e^2$ the level sets are ellipses. The ellipses are 1-manifolds. A point is not a 1-manifold.

4.7.2 (a) $c\neq 0$: let $\Phi(x,y,z)=x^2+y^2-4z^2-c$. Then $D\Phi(x,y,z)=2xe_1+2ye_2-8ze_3$, which is nonzero and has rank 1 when $(x,y,z)\neq(0,0,0)$. Thus for all $x\in M_c$ where $M_c:=\{(x,y,z):x^2+y^2-4z^2=c\}$, pick U be a neighborhood of x that does not contain $(0,0)$. This shows M_c is a 2-manifold.

$c=0$: suppose M_0 is a 2-manifold. Then there exists some neighborhood U around the origin and Φ such that $D\Phi(x,y,z)\neq 0$ in U and $M_0\cap U=\{(x,y,z)\in U:\Phi(x,y,z)=0\}$. Note that if we take any path to the origin along the cone, the image of the path under Φ equals 0. Hence we have a linearly independent set of directions, along which the derivatives of Φ are all 0. Then $D\Phi(0)=0$, a contradiction.

(b) Let $F=x^2+y^2-4z^2-1$. Then M_1 is a 2-manifold determined by F. $\nabla F(x,y,z)=2xe_1+2ye_2-8ze_3$. And $\nabla F(2,-1,1)=(4,-2,-8)$. Hence the tangent plane is

$$4(x-2)-2(y+1)-8(z-1)=0.$$

6.2.5 (a) Let $u=1/t$. Then as $u\to+\infty$

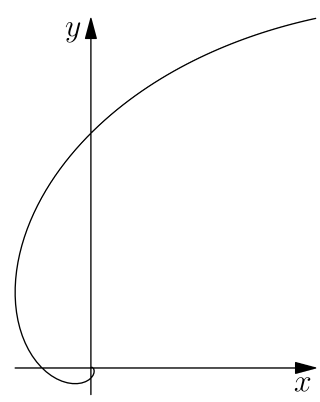
$$[\cos(1/t)e^{-1/t}]'=\frac{1}{t^2}\left(\sin(1/t)+\cos(1/t)\right)e^{-1/t}=u^2(\sin u+\cos u)e^{-u}\leq 2|u^2|e^{-u}\to 0.$$

Similarly,

$$[\sin(1/t)e^{-1/t}]'=\frac{1}{t^2}\left(\sin(1/t)-\cos(1/t)\right)e^{-1/t}=u^2(\sin u-\cos u)e^{-u}\leq 2|u^2|e^{-u}\to 0.$$

The derivatives of g^1 and g^2 are continuous at 0 and thus they are C^1 functions.

(b) No, because $g'(0)=0$.



6.3.3 (a) ω is exact with $f=\frac{1}{2}x^2y+C$.

(b) ω is not closed ($\frac{\partial\omega_1}{\partial y}=0\neq z=\frac{\partial\omega_2}{\partial x}$) and thus ω cannot be exact.

(c) ω is not closed because $\frac{\partial\omega_1}{\partial y}=1\neq 0=\frac{\partial\omega_2}{\partial x}$, and thus ω is not exact.

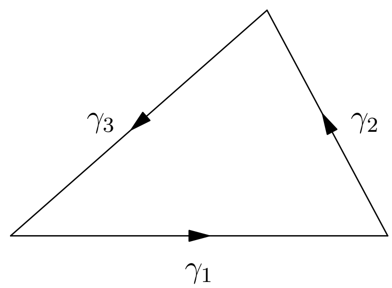
(d) ω is exact with $f=\frac{x^2-y^2}{xy}+C$.

6.4.1 (a) Let $\gamma_1 = ate_1, \gamma_2 = [a+(b-a)t]e_1 + cte_2, \gamma_3 = b(1-t)e_1 + c(1-t)e_2$ for $t \in [0,1]$. Note that $\gamma = \gamma_1 \cap \gamma_2 \cap \gamma_3$.

$$\begin{aligned}
\int_{\gamma_1} x\,dy - y\,dx &= \int_0^1 (at \cdot 0 - 0 \cdot a)\,dt0, \\
\int_{\gamma_2} x\,dy - y\,dx &= \int_0^1 [(a+t(b-a))c - ct(b-a)]\,dt = \int_0^1 ac\,dt = ac \\
\int_{\gamma_3} x\,dy - y\,dx &= \int_0^1 [b(1-t)(-c) + c(1-t)b)\,dt = 0.
\end{aligned}$$

Thus

$$\frac{1}{2}\int_\gamma x\,dy + y\,dx = \frac{ac}{2}.$$



(b)

$$\frac{1}{2}\int_\gamma x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos tb\cos t + b\sin ta\sin t)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.$$

6.4.2 (a) Note $g_1(1) = (1,1), g_1(2) = (2,3), g_2(0) = (1,1), g_2(1) = (2,3)$. Thus

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega = \left[\frac{1}{2}x^2y + C\right|_{(1,1)}^{(2,3)} = \frac{11}{2}$$

(c)

$$\begin{aligned}
\int_{\gamma_1} y\,dx &= \int_1^2 (2t-1)\,dt = \left[t^2 - t\right|_1^2 = 2 \\
\int_{\gamma_2} y\,dx &= \int_0^1 (t^2+t+1)\,dt = \left[\frac{t^3}{3} + \frac{t^2}{2} + t\right|_0^1 = \frac{11}{6}
\end{aligned}$$

(d)

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega = \left[\frac{x^2-y^2}{xy} + C\right|_{(1,1)}^{(2,3)} = -\frac{5}{6}$$

7.1.3 (a) $(e^1 + e^2) \wedge (e^1 - 3e^3) = -3e^1 \wedge e^3 + e^2 \wedge e^1 - 3e^2 \wedge e^3$.

(b) $(dy - x\,dz) \wedge (xy\,dx + 3\,dy + z\,dz) = xy\,dy \wedge dx + (z+3x)\,dy \wedge dz + x^2y\,dx \wedge dz$.

7.1.4 (a) $d(x^2y\,dy - xy^2\,dx) = 2xy\,dx \wedge dy - 2xy\,dy \wedge dx = 4xy\,dx \wedge dy$.

(b) $d(x\,dy + y\,dx) = dx \wedge dy + dy \wedge dx = 0$.

(c) $d(f(x)\,dx + g(y)\,dy) = f'(x)\,dx \wedge dx + g'(y)\,dy \wedge dy = 0$.

(d) $d(f(x,y)dy) = \frac{\partial f}{\partial x}\,dx \wedge dy$.

7.2.2 $M(e_4, e_1 - e_3, e_2 + e_3) = M(e_4, e_1, e_2) + M(e_4, e_1, e_3) - M(e_4, e_3, e_2) - M(e_4, e_3, e_3) = \omega_{412} + \omega_{413} - \omega_{432} = \omega_{124} + \omega_{134} + \omega_{234} = -1$.

7.2.3 (a) If there is no integer repeated in μ and λ and λ is obtain from μ by p interchanges ($p \geq 0$), that means μ can be obtained from λ by p interchanges as well. Thus $\delta^\mu_\lambda = \delta^\lambda_\mu = (-1)^p$. In other cases, $\delta^\lambda_\mu = \delta^\mu_\lambda = 0$.

(b) If $\delta^\lambda_\nu = 0$, then either λ, ν has repeated integers or λ, ν are not permutations of each other. In the first case, the equality holds because $\sum_{[\mu]} \delta^\lambda_\mu \delta^\mu_\nu$ also equals zero. In the second case, μ cannot be permutations of λ and ν thus the sum is zero.

Otherwise, assume that λ, ν are permutations of each other. In the sum $\sum_{[\mu]} \delta^\lambda_\mu \delta^\mu_\nu$, exactly one increasing μ is obtained by permuting λ. Let μ denote the increasing permutation. Suppose it takes p interchanges to get from λ to μ and q interchanges to get from μ to ν. Then it takes $p + q$ interchanges to get from λ to ν. Hence

$$\delta^\lambda_\nu = (-1)^{p+q} = (-1)^p(-1)^q = \delta^\lambda_\mu \delta^\mu_\nu.$$

Note that the above equalities hold when μ is not increasing.

For the second equality, note that there are $\binom{n}{r}$ increasing r-tuples, and there are $\frac{n!}{(n-r)!}$ non-repeated r-tuples. So there are $r!$ permutations of each increasing r-tuple that occur in the sum $\sum_{[\mu]} \delta^\lambda_\mu \delta^\mu_\nu$.

(c) Let $e_\nu = (e_{\nu_1}, \ldots, e_{\nu_r})$ be an arbitrary r-tuple of the n standard basis vectors. Note that $e^\lambda(e_\nu) = \delta^\lambda_\nu$ and $e^\mu(e_\nu) = \delta^\mu_\nu$. When applied to e_ν, part (c) reduces to part (b). As a multilinear form is determined by its action on the basis vectors, the proof is complete.

7. (a) Let $p \in \mathcal{C}$ and let $r = \max\{|z| : p(z) = 0\}$. Then the infimum must be smaller than r. Let D be the subset of $\mathcal{C}$ whose roots of max norm are all at most r. From complex analysis, we know the norm of the root of maximum norm of a polynomial is a continuous function of the coefficients. Hence D is closed. Furthermore, by Vieta's formulas, we know the coefficients in D are bounded, and thus D is compact. Thus the max root norm function (continuous in the coefficients) attains its minimum on the compact set D.

(b) This result comes directly from Theorem 1 in "Explicit Solutions for Root Optimization of a Polynomial Family with One Affine constraint".