Homework 4

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4.2.4 a. Let $f(t) = L_f(t) + x_f$ and $g(t) = L_g(t) + x_g$. Then $f(g(t)) = L_f(L_g(t)) + L_f(x_g) + x_f$ which is again an affine transformation since $L_f(L_g(t))$ is linear and $L_f(x_g) + x_f$ is a constant.

b. Suppose $f(x) = f(y)$ means $x = y$, and write f as $f(x) = L(x) + x_0$ for some fixed x_0. Then $f(x) = f(y)$ if and only if $L(x) = L(y)$.

4.3.3 a. Let $x(s,t) = t\cos 2\pi s$, $y(s,t) = t\sin 2\pi s$ and $z(s,t) = 1 - t$. Then $x(s,t)^2 + y(s,t)^2 = (z(s,t) - 1)^2$. This defines a cone with vertex at $(0,0,1)$.

b. The image of the square is the cone with vertex at $(0,0,1)$ and base the unit circle in the xy-plane.

c. To find the pre-image of $\{e_1\}$ we solve $(0,0,1) = (t\cos 2\pi s, t\sin 2\pi s, 1 - t)$. This tells us that the pre-image is $\{(s,t) : t = 0, s \in \mathbb{R}\}$. The pre-image of $\{e_3\}$ is given by $(1,0,0) = (t\cos 2\pi s, t\sin 2\pi s, 1 - t)$. So the pre-image is $\{(s,t) : t = 1, s = (1/2)k, k \in \mathbb{Z}\}$.

4.4.5 By the chain rule,

$$f_{22}(x,y) = c^2(\phi''(x - cy) + \psi''(x - cy)) = c^2 f_{11}(x,y)$$

4.5.1 a. We have $Jg(t) = \det(I) = 1$, $g(\Delta) = E^n$, and $g^{-1}(y) = y - x_0$.

b. We have $g(\Delta) = E^2$. The Jacobian is given by

$$\det\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = -3$$

The inverse is given by

$$g^{-1}(x,y) = \left(\frac{x + 2y}{3}, \frac{x - y}{3}\right)$$

c. The Jacobian of $g(s,t)$ is given by

$$\det\begin{pmatrix} 2s - 1 & 0 \\ 0 & 3 \end{pmatrix} = 6s - 3$$

So it is non-zero except when $s = 1/2$. $g(s,t)$ is not invertible since $g(2,0) = g(-1,0) = (0,0)$, and $g(\Delta) = \{(s,t) : s \geq -9/4, t \in \mathbb{R}\}$.

d. The Jacobian of $g(s,t)$ is

$$\det \begin{pmatrix} 2s & -2t \\ t & s \end{pmatrix} = 2(t^2 + s^2)$$

Since the domain of g does not include $(s,t) = (0,0)$, the Jacobian is non-zero on the domain of g. We have $g(\Delta) = E^2$, and that it is not invertible since $g(1,1) = g(-1,-1) = (0,1)$.

e. The Jacobian of $g(s,t)$ is

$$\det \begin{pmatrix} \frac{1}{s} & \frac{1}{t} \\ -\frac{2s}{(s^2+t^2)^2} & -\frac{2t}{(s^2+t^2)^2} \end{pmatrix} = \frac{2(s^2 - t^2)}{st(s^2+t^2)^2} \neq 0$$

for every (s,t). Next we show that g is invertible. For suppose $g(u,v) = g(x,y)$. Then $\log(xy) = \log(uv)$ so $xy = uv$. Also we must have $x^2 + y^2 = u^2 + v^2$. Therefore $(x-y)^2 = (u-v)^2$ and by the definition of Δ we have $x - y = u - v$. We also have $(x+y)^2 = (u+v)^2$, so $x+y = u+v$. It follows that $(x,y) = (u,v)$. Therefore g is invertible. Now let

$$x = \log(st), \quad y = \frac{1}{s^2 + t^2}$$

Therefore $e^{-x} = 1/(st)$. Again by the definition of Δ we have

$$0 < \frac{1}{s^2 + t^2} < \frac{1}{st} = \frac{1}{2}e^{-x}$$

This shows that $y < (1/2)e^{-x}$. On the other hand since $s > t > 0$, $s^2 + t^2 < 2s^2$ so $y > 1/(2s^2)$. So we have

$$g(\Delta) = \{(x,y) : x \in \mathbb{R}, 0 < y < (1/2)e^{-x}\}$$

Finally we calculate the inverse of g. From our definitions of x, y above we have

$$(s-t)^2 = \frac{1}{y} - 2e^x$$
$$(s+t)^2 = 2e^x - \frac{1}{y}$$

By the definition of Δ we have

$$s - t = \sqrt{\frac{1}{y} - 2e^x}$$
$$s + t = \sqrt{2e^x - \frac{1}{y}}$$

So

$$s = \frac{\sqrt{\frac{1}{y} - 2e^x} + \sqrt{2e^x - \frac{1}{y}}}{2}$$
$$t = \frac{\sqrt{2e^x - \frac{1}{y}} - \sqrt{\frac{1}{y} - 2e^x}}{2}$$

4.5.5 a. We have
$$\det \begin{pmatrix} e^s \cos t & -e^s \sin t \\ e^s \sin t & e^s \cos t \end{pmatrix} = e^{2s} \neq 0$$
for all s. $g(s,t)$ is not invertible since $g(s,t) = g(s, t + 2\pi)$.

b. We compute the inverse. This also gives its existence. Let $x = e^s \cos t$, and $y = e^s \sin t$. Then $x^2 + y^2 = e^{2s}$ so $s = (1/2)\log(x^2 + y^2)$. Next $e^s \cos t = x$ shows $t = \cos^{-1}\left(x/\sqrt{x^2 + y^2}\right)$. So

$$g^{-1}(x, y) = \left(\frac{1}{2}\log(x^2 + y^2), \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right)\right)$$

c. Since $g(\Delta) = E^2$ we have $g(E^2) = E^2$.

d. We have
$$Dg(t) = e^s \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix}$$
And the matrix here is a a rotation matrix so g is conformal.

4.6.3 a. We have $\Phi(x_0) = (0, 0)$ and

$$D\Phi(x_0) = \begin{pmatrix} -2 & -2 & -2 & 2 \\ 3 & 3 & -3 & 3 \end{pmatrix}$$

Therefore
$$\tilde{J}\Phi(x_0) = \det \begin{pmatrix} -2 & -2 \\ 3 & -3 \end{pmatrix} = 12 \neq 0$$
Therefore Φ satisfies the hypotheses of the implicit function theorem.

b. First note that by taking $(x_2, x_4) = (-1, 1)$ in the given equations, we find $\phi(-1, 1) = 1$ and $\psi(-1, 1) = 1$. Now taking ∂_1 of both equations we have

$$0 = 2x_2 - 2\phi_1(x_2, x_4)\psi(x_2, x_4) - 2\phi(x_2, x_4)\psi_1(x_2, x_4)$$
$$0 = 3x_2^2 + 3\phi^2(x_2, x_4)\phi_1(x_2, x_4) - 3\psi^2(x_2, x_4)\psi_1(x_2, x_4)$$

It follows that $\phi_1(-1, 1) = -1$ and $\psi_1(-1, 1) = 0$. Taking ∂_2 of our equations gives $\phi_2(-1, 1) = 0$ and $\psi_2(-1, 1) = 1$.

4.6.4 a. We have $\Phi(2,1,-4) = 0$ and $D\Phi(x,y,z) = (2x, 8y-2z, -2y-2z)$. At x_0 this gives $(4,16,6)$, so we satisfy the hypotheses of the implicit function theorem.

b. Setting $\phi_3(x,y,z) = 0$ gives $-2y-2z$ so $z = -y$. The distance from $(2,1,-4)$ to $y = -z$ is given by

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (4-y)^2}$$

Taking partials with respect to x, y we find that $(x,y,z) = (2, 5/2, -5/2)$ is a critical point and that $d(2, 5/2, -5/2) = 3/\sqrt{2}$.

c. The only critical point of f is given by $(2x, 8y-2z, -2y-2z) = (0,0,0)$, which gives $(x,y,z) = (0,0,0)$. The distance from $(2,1,-4)$ to $(0,0,0)$ is $\sqrt{21}$.

4.8.1 Let $g(x,y,z) = x + y + z - 1$. Then note that for any h such that $h \cdot \delta g(x,y,z) = 0$, $f(h) \geq 0$, since $f(h) \geq 0$ for any h. Therefore, the constrained minimum we find is an absolute minimum. To find it, we use Lagrange multipliers. This gives us the system of equations

$$\begin{aligned} 0 &= 6x - \lambda \\ 0 &= 6y - \lambda \\ 0 &= 2z - \lambda \end{aligned}$$

With the constraint $x + y + z = 1$, this system of equations has the solution $(x,y,z) = (1/5, 1/5, 3/5)$ and $f(1/5, 1/5, 3/5) = 3/5$.

4.8.2 Applying as above the method of Lagrange multipliers, we have the system of equations

$$\begin{aligned} 0 &= 2(x-c) + \lambda \\ 0 &= 2y(1-\lambda) \end{aligned}$$

Combining these two equations, we have that either $y = 0$ or $x = c - 1/2$. Since $x = y^2$, we know $x \geq 0$, so we consider two cases. If $c \geq 1/2$ then $f(x,y) = c - 1/4$, so $d = \sqrt{c - 1/4}$. If $c \leq 1/2$, then $x = y = 0$, so $d = |c|$.