Multivariable Analysis HW3 Zhuolun Yang 84451

Problem 3.4.

2. a) $f_1(x,y) = -x^{-2}\cos y$

$f_2(x,y) = -x^{-1}\sin y$

$f_{11}(x,y) = 2x^{-3}\cos y$

$f_{12}(x,y) = f_{21}(x,y) = x^{-2}\sin y$

$f_{22}(x,y) = -x^{-1}\cos y$

Hence $f(x,y) = f(1,0) + f_1(1,0)(x-1) + f_2(1,0)y + R_2(x,y)$

$= 1 - (x-1) + R_2(x,y)$

$= 2 - x + R_2(x,y)$.

$R_2(x,y) = \frac{1}{2}\cdot 2\,[1+s(x-1)]^{-3}\cos(sy)\cdot(x-1)^2$

$+ \frac{1}{2}\cdot 2\,[1+s(x-1)]^{-2}\sin(sy)\cdot(x-1)y$

$-\frac{1}{2}[1+s(x-1)]^{-1}\cos(sy)\cdot y^2$, where $0\le s\le 1$.

Hence $|R_2(x,y)| \le (\min\{x,1\})^{-3}\,[(x-1)^2 + (x-1)y + \frac{1}{2}y^2]$

b) The partial derivatives w.r.t x contribute a factorial and x^{-n} for some n.

Hence $|f_{i_1\ldots i_q}(x,y)| \le q!\cdot 2^q$ for (x,y) in a small neighborhood of $(1,0)$.

$\Rightarrow |R_q(x,y)| \le q!\cdot 2^q\cdot 2^{\frac{q}{2}}\cdot |h|^q/q! = 2^q\cdot 2^{\frac{q}{2}}\cdot|h|^q = (2\sqrt{2}\cdot|h|)^q$

Where $h = (x-1, y)$. (by section 1.2 Problem 4)

Therefore if I choose (x,y) in a small neighbourhood of $(1,0)$ s.t. $|h| < \frac{1}{2\sqrt{2}}$, then $|R_q(x,y)| \to 0$ as $q\to\infty$

6 a) If $k=0$, $f(x) = \sin(\frac{1}{x})$. $\lim_{x\to 0} f(x)$ does not converge while $f(0)=0$. Hence f is discontinuous at 0.

b) If $k=1$, $f(x) = x\sin(\frac{1}{x})$, then $|f(x)| \le x \to 0$ as $x\to 0$.

Hence $\lim_{x\to 0} f(x) = 0 = f(0)$. $\Rightarrow f$ is continuous at 0.

For $x\ne 0$, $f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$ exists.

Therefore $f(x)$ is $C^{(0)}$

But if we want to find the derivative of f at 0.

we will see $\lim_{x\to 0}\frac{x\sin(\frac{1}{x})}{x} = \lim_{x\to 0}\sin(\frac{1}{x})$ does not exists.

c) If $k=2$, compute $f'(x)$ directly by chain rule, we will have

$f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$ when $x \neq 0$

and $f'(0) = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = 0$

Hence f is differentiable, but $\lim_{x\to 0} f'(x) \neq 0$ since $\cos(\frac{1}{x})$ gives oscillation.
Therefore f is not $C^{(1)}$

d) For $k \geq 3$, we only need to look at the term that has least order of x after differentiating. From chain rule we can see that differentiating one time will kill 2 order of x.

Hence for $f^{(p)}(x)$, we only need to look at $x^{(k-2p)}\sin(\frac{1}{x})$ or $x^{(k-2p)}\cos(\frac{1}{x})$ to determine its differentiability.

- when k is an even number, the term we care will turn into $\sin(\frac{1}{x})$ or $\cos(\frac{1}{x})$, which is simlar as c). Then we know f is $\frac{k}{2}$-th differentiable but not of class $C^{(\frac{k}{2})}$. And f is $C^{(\frac{k}{2}-1)}$ since its $\frac{k}{2}$ times derivative exists everywhere.
- when k is an odd number, if we differentiate $\frac{k-1}{2}$ times we can see the term we care turns to be $x\sin(\frac{1}{x})$ or $x\cos(\frac{1}{x})$. Therefore the $\frac{k-1}{2}$ times derivative is continuous since x will kill the oscillation created by trigonometric function when $x \to 0$. Hence f is $C^{(\frac{k-1}{2})}$. But f is not $\frac{k+1}{2}$ times differentiable since $\lim_{x\to 0} \frac{x\sin(\frac{1}{x})}{x}$ does not exist.

7. a) $f(x,y) = xy(x^2-y^2)(x^2+y^2)^{-1}$

$f_1(x,y) = y(x^2-y^2)(x^2+y^2)^{-1} + 2x^2y(x^2+y^2)^{-1} - 2x^2y(x^2-y^2)(x^2+y^2)^{-2}$

$f_2(x,y) = x(x^2-y^2)(x^2+y^2)^{-1} + 2xy^2(x^2+y^2)^{-1} - 2xy^2(x^2-y^2)(x^2+y^2)^{-2}$

Hence $f_{12}(x,y) = (x^2-y^2)(x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-1} - 2y^2(x^2-y^2)(x^2+y^2)^{-2}$
$+ 2x^2(x^2+y^2)^{-1} - 4x^2y^2(x^2+y^2)^{-2}$
$- 2x^2(x^2-y^2)(x^2+y^2)^{-2} + 4x^2y^2(x^2+y^2)^{-2} + 8x^2y^2(x^2-y^2)(x^2+y^2)^{-3}$

$f_{21}(x,y) = (x^2-y^2)(x^2+y^2)^{-1} + 2x^2(x^2+y^2)^{-1} - 2x^2(x^2-y^2)(x^2+y^2)^{-2}$
$-2y^2(x^2+y^2)^{-1} + 4x^2y^2(x^2+y^2)^{-2}$
$-2y^2(x^2-y^2)(x^2+y^2)^{-2} - 4x^2y^2(x^2+y^2)^{-2} + 8x^2y^2(x^2-y^2)(x^2+y^2)^{-3}$

By comparing the terms we can see $f_{12}(x,y) = f_{21}(x,y)$.

b) Note that $f_1(x,y)$ and $f_2(x,y)$ is continuous at $(x,y) \neq (0,0)$.
By problem 4, $f_1(0,0) = \lim_{(x,y)\to(0,0)} f_1(x,y) = 0$, $f_2(0,0) = \lim_{(x,y)\to(0,0)} f_2(x,y) = 0$
since the order of numerater is greater than denominater.
Hence f is of class $C^{(1)}$

c) $f_{12}(0,0) = \lim_{y\to 0} \frac{f_1(0,y) - f_1(0,0)}{y} = \lim_{y\to 0} \frac{-\frac{y^3}{y^2} - 0}{y} = -1$

$f_{21}(0,0) = \lim_{x\to 0} \frac{f_2(x,0) - f_2(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^3}{x^2} - 0}{x} = 1$

For $f_{12}(x,y)$, let (x,y) approach $(0,0)$ in the direction s.t. $y = x$
$\lim_{x\to 0} f_{12}(x,x) = \lim_{x\to 0} -2x^2\cdot(2x^2)^{-1} + 2x^2(2x^2)^{-1} = 0$ which implies that
$f_{12}(x,y)$ is not continuous.
Similarly we can verify that $f_{21}(x,y)$ is not continuous.
Therefore this result does not contradict Theorem 3.3.

Section 3.5

Q2. a) $\nabla f(x,y) = (3x^2+1-4y,\ -4x-4y)$, let $\nabla f(x,y) = 0$
we will get two critical points $(-1,1)$ and $(-\frac{1}{3}, \frac{1}{3})$.
Looking at the Hessian matrix $f_{ij} = \begin{pmatrix} 6x & -4 \\ -4 & -4 \end{pmatrix}$

$f_{11} = 6x < 0$ $\quad f_{22} = -4 < 0$ $\quad f_{11}\cdot f_{22} - f_{12}^2 = -24x - 16 < 0$ for $(-\frac{1}{3}, \frac{1}{3})$ and > 0 for $(-1,1)$.
Therefore $(-1,1)$ is a local maximum and $(-\frac{1}{3}, \frac{1}{3})$ is a saddle point.

b) $\nabla f(x,y) = (y+1-2xy,\ x-x^2)$, let $\nabla f(x,y) = 0$.
we will get two critical points $(0,-1)$ and $(1,1)$
Looking at the Hessian matrix $f_{ij} = \begin{pmatrix} -2y & 1-2x \\ 1-2x & 0 \end{pmatrix}$

$f_{11}\cdot f_{22} - f_{12}^2 = -(1-2x)^2 < 0$ for both $(0,-1)$ and $(1,1)$
Therefore both $(0,-1)$ and $(1,1)$ are saddle points.

c) $\nabla f(x,y) = [-\sin x \cosh y,\ \cos x \sinh y]$, let $\nabla f(x,y) = 0$ we will have
critical points at $(k\pi, 0)$ where $k \in \mathbb{Z}$.
Looking at the Hessian matrix $f_{ij} = \begin{pmatrix} -\cos x \cosh y & -\sin x \sinh y \\ -\sin x \sinh y & \cos x \cosh y \end{pmatrix}$.

Since $f_{11} \cdot f_{22} - f_{12}^2 = -4 < 0$.
Therefore $(k\pi, 0)$, $k \in \mathbb{Z}$ are all saddle points.

4. $\nabla f(x,y,z) = (2x-2y+2z,\ 6y-2x,\ 4z+2x)$, therefore 0 is indeed a critical point.

The Hessian matrix is $\begin{pmatrix} 2 & -2 & 2 \\ -2 & 6 & 0 \\ 2 & 0 & 4 \end{pmatrix}$.

$d_1(f) = 2 > 0$. $d_2(f) = \begin{vmatrix} 2 & -2 \\ -2 & 6 \end{vmatrix} = 8 > 0$. $d_3(f) = \begin{vmatrix} 2 & -2 & 2 \\ -2 & 6 & 0 \\ 2 & 0 & 4 \end{vmatrix} = 8 > 0$

Hence 0 is a local minimum.

Since the determinent of Hessian matrix is not zero.
Therefore the row vectors are independent, which means the only vector (x, y, z) such that

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 6 & 0 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{is } (x,y,z) = (0,0,0)$$

which means that $\nabla f(x,y,z) = 0 \Leftrightarrow (x,y,z) = (0,0,0)$.
Therefore $(0,0,0)$ is the only critical point. And f is convex
Hence $f(0,0,0) = 0$ is the min. value of f.

9. Assume x be another critical point in U, a small neighbourhood of x_0.
Then we have $0 = f_i(x)$ and $0 = f_i(x_0)$ for all i from 1 to n.
$\Rightarrow 0 = f_i(x) - f_i(x_0)$
By mean value theorem, $0 = \sum_{j=1}^{n} f_{ij}(y_i)(x^j - x_0^j)$ for all i
where $y_i \in U$ for all i.
Since x_0 is a nondegenerate critical points, then $\det(f_{ij}(x_0)) \neq 0$.
$\Leftrightarrow$ the row vectors of $f_{ij}(x_0)$ are linearly independent.
Since f_{ij} is continuous for all i, j, we can firstly chose a neighbourhood U_1 of x_0 s.t. the first row vector takes from this U_1 are all linearly independent with the rest row vectors. Then chose U_2 s.t. all possible second row vectors are independent with first row vector in U_1 and the rest row vectors. Continuously doing n times, we will have a neighbourhood $U = \bigcap_{k=1}^{n} U_k$ s.t. all $y_i \in U$ and $\det(f_{ij}(y_i)) \neq 0$ for all i.
Therefore $f_{ij}(y_i) \cdot (x - x_0)^T = 0 \Rightarrow x = x_0$, which gives a contradiction.

Section 3.6

2. a) $\mathrm{Hess}(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -8 \end{pmatrix}$ it has both positive and negative eigenvalue. Hence it is neither convex nor concave.

b) $\mathrm{Hess}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ it is negative semidefinite. Hence f is concave, but not strictly concave.

c) • for $p=0$ $f(x,y)=1$ on K, therefore f is convex and concave.
• for $p=1$ $f(x,y)=x+y+1$ on K. for any $t\in[0,1]$ $(x_1,y_1),(x_2,y_2)\in K$

$f(tx_1+(1-t)x_2,\ ty_1+(1-t)y_2) = tx_1+ty_1+t+(1-t)x_2+(1-t)y_2+(1-t)$
$= tf(x_1,y_1)+(1-t)f(x_2,y_2)$

Therefore f is convex and concave.
• for $p\neq 0$ and $p\neq 1$. $\mathrm{Hess}(f) = \begin{pmatrix} p(p-1)(x+y+1)^{p-2} & p(p-1)(x+y+1)^{p-2} \\ p(p-1)(x+y+1)^{p-2} & p(p-1)(x+y+1)^{p-2} \end{pmatrix}$

We can see that $\mathrm{Hess}(f)$ is a singular matrix, then 0 is an eigenvalue. another eigenvalue will be the trace $2p(p-1)(x+y+1)^{p-2}$ which >0 when $p>1$ or $p<0$, <0 when $0<p<1$.

Therefore. f is convex when $p>1$ or $p<0$
f is concave when $0\le p\le 1$.

d)
$$\mathrm{Hess}(f) = e^{x^2+xy+y^2+z^2}\begin{pmatrix} 2+(2x+y)^2 & 1+(2x+y)(2y+x) & (2x+y)2z \\ 1+(2x+y)(2y+x) & 2+(2y+x)^2 & (2y+x)2z \\ (2x+y)2z & (2y+x)2z & 2+4z^2 \end{pmatrix}$$

Since $e^{x^2+xy+y^2+z^2}>0$, we don't need to consider it. let UL_n denote the upper-left $n\times n$ matrix. $\det(UL_1)=2+(2x+y)^2>0$.

$\det(UL_2) = 4+2(2y+x)^2+2(2x+y)^2+(2x+y)^2(2y+x)^2-1-2(2x+y)(2y+x)-(2x+y)^2(2y+x)^2$
$= 3+(2y+x)^2+(2x+y)^2+[(2y+x)-(2x+y)]^2>0.$

To compute $\det(UL_3)$, do some row operation to simplify the calculation
$\begin{pmatrix} \text{row }1 - \text{row }3\cdot\frac{2x+y}{2z} \\ \text{row }2 - \text{row }3\cdot\frac{2y+x}{2z} \end{pmatrix}$ we will have $\begin{pmatrix} 2 & 1 & -(2x+y)\frac{2}{2z} \\ 1 & 2 & -(2y+x)\frac{2}{2z} \\ (2x+y)2z & (2y+x)2z & 2+4z^2 \end{pmatrix}$

Then $\det(UL_3) = 8+16z^2-4(2y+x)(2x+y)+4(2x+y)^2+4(2y+x)^2-2-4z^2$
$= 6+12z^2+[2(2x+y)-2(2y+x)]^2+2(2x+y)^2+2(2y+x)^2>0$

Therefore the Hessian matrix of f is positive definite

$\Rightarrow f$ is strictly convex.

e) $\text{Hess}(f) = \begin{pmatrix} y^2 e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2 e^{xy} \end{pmatrix}$

$\det(\text{Hess}(f)) = e^{2xy}[x^2y^2 - 1 - 2xy - x^2y^2] = e^{2xy}[-1-2xy]$. $\begin{cases} >0 & \text{when } xy < \frac{-1}{2} \\ <0 & \text{when } xy > \frac{-1}{2} \end{cases}$

Therefore the eigenvalues are not always with the same sign.

$\Rightarrow f$ is neither convex nor concave.

f) a) Let $f(x) = \log x$. $f'(x) = \frac{1}{x}$ is strictly decreasing, therefore $f(x)$ is concave. By Problem 8, for $x_1, \cdots, x_m$ positive, $t^j \geq 0$ and $t^1 + \cdots + t^m = 1$

$\log\left(\sum_{j=1}^{m} t^j x_j\right) \geq \sum_{j=1}^{m} t^j \log(x_j) = \log\left(\prod_{j=1}^{m} x_j^{t_j}\right)$

Since $f'(x) = \frac{1}{x} > 0$ for x positive, $\log(x)$ is increasing when $x > 0$.

Therefore $\sum_{j=1}^{m} t^j x_j \geq \prod_{j=1}^{m} x_j^{t^j}$

b) By part a), let $t^j = \frac{1}{m}$ for all j

Then $\frac{x_1 + \cdots + x_m}{m} \geq (x_1 \cdots x_m)^{\frac{1}{m}}$

13. First I want to show $\frac{j}{2^k} f(x_1) + (1 - \frac{j}{2^k}) f(x_2) \geq f(\frac{j}{2^k} x_1 + (1 - \frac{j}{2^k}) x_2)$. $(*)$

for all $k \in \mathbb{N}$ and $j = 0, 1, \cdots, 2^k$. Prove by induction.

when $k = 1$, we have $f(x_2) \geq f(x_2)$ $(j=0)$ $f(\frac{1}{2}(x_1+x_2)) \leq \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2)$ $(j=1)$

and $f(x_1) \geq f(x_1)$ $(j=2)$. Therefore $(*)$ is true for $k=1$

Assume $(*)$ is true for $k=n$, we need to show

$\frac{j}{2^{n+1}} f(x_1) + (1 - \frac{j}{2^{n+1}}) f(x_2) \geq f(\frac{j}{2^{n+1}} x_1 + (1 - \frac{j}{2^{n+1}}) x_2)$. for $j = 0, 1, \cdots, 2^{n+1}$

If j is even number, the situation will reduce to the case when $k=n$, which is true by assumption. Then we only need to verify when j is an odd number.

$f\left(\frac{j}{2^{n+1}} x_1 + (1 - \frac{j}{2^{n+1}}) x_2\right) = f\left(\frac{1}{2}\left\{\left[\frac{\frac{j-1}{2}}{2^n} x_1 + (1 - \frac{\frac{j-1}{2}}{2^n}) x_2\right] + \left[\frac{\frac{j+1}{2}}{2^n} x_1 + (1 - \frac{\frac{j+1}{2}}{2^n}) x_2\right]\right\}\right)$

$\leq \frac{1}{2} f\left(\frac{\frac{j-1}{2}}{2^n} x_1 + (1 - \frac{\frac{j-1}{2}}{2^n}) x_2\right) + \frac{1}{2} f\left(\frac{\frac{j+1}{2}}{2^n} x_1 + (1 - \frac{\frac{j+1}{2}}{2^n}) x_2\right)$ (by $k=1$ case)

$\leq \frac{\frac{j-1}{2}}{2^{n+1}} f(x_1) + (\frac{1}{2} - \frac{\frac{j-1}{2}}{2^{n+1}}) f(x_2) + \frac{\frac{j+1}{2}}{2^{n+1}} f(x_1) + (\frac{1}{2} - \frac{\frac{j+1}{2}}{2^{n+1}}) f(x_2)$ (by $k=n$ case).

$= \frac{j}{2^{n+1}} f(x_1) + (1 - \frac{j}{2^{n+1}}) f(x_2)$

Therefore $(*)$ is true for all $k \in \mathbb{N}$ and $j = 0, 1, \cdots, 2^k$.

Since the sequence $\{t_{kj} = \frac{j}{2^k}, k \in \mathbb{N}, j = 0, 1, \cdots, 2^k\}$ is dense in $[0,1]$ and f is continuous on K

Therefore for any $x_1, x_2 \in K$, $t \in [0,1]$, we have $f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$

Hence f is convex on K.

15. Since $f(x)$ is bounded above on K, $\sup_{x \in K} f(x)$ exists.

Now we need to prove $\sup_{x \in K} f(x) = f(x_0)$ for some $x_0 \in K$.

~~To~~ Let $c = \sup_{x \in K} f(x)$, by the definition of supremum, one can find a sequence, say $c_n = f(x_n)$ increasingly converges to c.

Consider the set $K_{c_n} = \{x : f(x) \ge c_n\}$.

When $c > c_n$, $K_{c_n} \cap K_c \ne \phi$. By convexity.

Since K is closed, $K_c \cap K \ne \phi$

Therefore when $x_0 \in K_c \cap K$, $f(x_0) = \sup_{x \in K} f(x)$

Now by our assumption on f we can bound this by

$$\frac{1}{2}\left(f\left(\frac{n}{2^{m-1}}x_1 + \left(1 - \frac{n}{2^{m-1}}\right)x_2\right) + f\left(\frac{n+1}{2^{m-1}}x_1 + \left(1 - \frac{n+1}{2^{m-1}}\right)x_2\right)\right)$$

since by the induction hypothesis we are sure that the points $\frac{n}{2^{m-1}}x_1 + \left(1 - \frac{n}{2^{m-1}}\right)x_2$ and $\frac{n+1}{2^{m-1}}x_1 + \left(1 - \frac{n+1}{2^{m-1}}\right)x_2$ are in K. Applying the induction hypothesis to the two terms in the above establishes (1). Now fix $t \in [0,1]$ and choose $\delta > 0$. Then for k sufficiently large and some j as above, we have

$$\left|t - \frac{j}{2^k}\right| < \delta$$

So fix such j, k and now choose $\epsilon > 0$. Since f is continuous, for δ sufficiently small, we have

$$\begin{aligned} f(tx_1 + (1-t)x_2) &\leq f\left(\frac{j}{2^k}x_1 + \left(1 - \frac{j}{2^k}\right)x_2\right) + \epsilon \\ &\leq \frac{j}{2^k}f(x_1) + \left(1 - \frac{j}{2^k}\right)f(x_2) + \epsilon \\ &\leq (t+\delta)f(x_1) + (1-(t+\delta))f(x_2) + \epsilon \end{aligned}$$

Since this inequality holds for arbitrary δ, ϵ, f is convex.

3.6.15: We will prove the claim by induction on the dimension, n, of the polytope. In the case $n = 1$, the claim follows since f is bounded. So assume we have the claim in $n-1$ dimensions, and consider the problem of finding the maximum of f in an n dimensional polytope. Now let $\{x_i\}$ be a sequence in K converging to the supremum of f in K. For each x_i, we know that there is some vector α so that if we shift x_i in the direction α that $f(x_i)$ will increase maximally. Since f is bounded, this means by repeating this process, we must eventually reach a point x_i^* so that x_i^* lies on a wall of the polytope. Now assume that x_i is an infinite sequence, for if the sequence is finite, then there is nothing to prove. Then the sequence $\{x_i^*\}$ is an infinite sequence, and each x_i^* lies on some wall of the polytope. Since the sequence is infinite, there is some wall of the polytope with infinitely many points. But now we apply the induction hypothesis and know that f attains its supremum on this wall. Since the sequence on this wall is a subsequence of the original sequence, on this sequence f converges to its supremum.