Homework 2

§3.1.4 (a)

$$f_1(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)-0}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$
$$f_2(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)-0}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Let $\mathbf{v} = (u,v)$ be a vector not in the direction of $\pm\mathbf{e}_1$ or $\pm\mathbf{e}_2$, i.e. $u \neq 0$ and $v \neq 0$.

$$f_{\mathbf{v}}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(u,v)) - f(0,0)}{t} = \lim_{t\to 0} \frac{(tu \cdot tv)^{1/3}}{t} = \lim_{t\to 0} \frac{u^{1/3}v^{1/3}}{t^{1/3}}$$

As the limit does not exist, the derivative does not exist, either.

(b) Claim that the limit of f at $(0,0)$ is 0. Let $V = B_r(0)$ be a neighborhood of 0 with radius r. Take $U = B_r(0,0) \setminus \{(0,0)\}$, the punctured neighborhood of $(0,0)$ with radius r. For each $(x,y) \in U$, we have $x, y \leq r$ and thus $f(x,y) = (xy)^{1/3} \leq (r^2)^{1/3} = r^{2/3} < r$, i.e. $f(x,y) \in V$. Hence $f(U) \subset V$, and f has limit 0 at $(0,0)$. Now we have

$$f(0,0) = 0 = \lim_{(x,y)\to(0,0)} f(x,y)$$

It follows that f is continuous at $(0,0)$.

§3.1.5 Let $\mathbf{v} = (u,v,w)$ with $u^2 + v^2 + w^2 = 1$.

$$\begin{aligned} f_{\mathbf{v}}(x_0,y_0,z_0) &= \lim_{t\to 0} \frac{f((x_0,y_0,z_0)+t(u,w,w)) - f(x_0,y_0,z_0)}{t} \\ &= \lim_{t\to 0} \frac{|(x_0+tu)+(y_0+tv)+(z_0+tw)| - |x_0+y_0+z_0|}{t} \\ &= \lim_{t\to 0} \frac{|tu+tv+tw|}{t} = \lim_{t\to 0} \frac{|t||u+v+w|}{t} \end{aligned}$$

If $u+v+w \neq 0$, then $f_{\mathbf{v}}(x_0,y_0,z_0) = \lim_{t\to 0} \pm|u+v+w|$, and the derivative fails to exist since the limit does not exist. If $u+v+w = 0$, then $f_{\mathbf{v}}(x_0,y_0,z_0) = \lim_{t\to 0} 0/t = 0$. Therefore, the directional derivative exists along any direction that is on the plane $x_0 + y_0 + z_0 = 0$.

§3.2.3 (a) Because L is linear, there exists an $\mathbf{a} = \{a_1, \ldots, a_n\}$ such that $L(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x} = c$. If $\mathbf{a} \neq \mathbf{0}$, then $\{\mathbf{x} : L(\mathbf{x}) = c\} = \{\mathbf{x} : \mathbf{a}\cdot\mathbf{x} = c\}$, which is a hyperplane. Otherwise, $L(\mathbf{x}) = 0$ for all $\mathbf{x} \in E^n$, i.e. $\{\mathbf{x} : L(\mathbf{x}) = c\} = E^n$.

(b) Because L is continuous and the hyperplane is the preimage of the closed set $\{c\}$, the hyperplane is closed. A closed half-space takes the form $\{\mathbf{x} : L(\mathbf{x}) \geq c\}$, which means it is the preimage of the closed set $[c, \infty)$ under the liner function L. So closed half-spaces are closed.

§3.2.5 (a) i. Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$. Because $\|\mathbf{x}\| = 1$ is the surface of a unit ball, it passes through all the hyperoctant. That is, there must be some $\mathbf{x}$ such that, for each i, a_i and x_i have the same sign. For this $\|\mathbf{x}\|$, $\mathbf{a} \cdot \mathbf{x} = \sum_{i=1}^{n} a_i x_i \geq 0$. Hence, the maximum of $\mathbf{a} \cdot \mathbf{x}$ is non-negative, i.e. $\|\mathbf{a}\|$ is non-negative.

ii. Let $d = \|\mathbf{a}\|$. Then the maximum of $\mathbf{a} \cdot \mathbf{x}$ is d for $\|\mathbf{x}\| = 1$. As $(c\mathbf{a}) \cdot \mathbf{x} = c(\mathbf{a} \cdot \mathbf{x})$, the maximum of $(c\mathbf{a}) \cdot \mathbf{x}$ is cd for $\|\mathbf{x}\| = 1$ and $c \in \mathbb{R}$. So $\|c\mathbf{a}\| = cd = c\|\mathbf{a}\|$.

iii. Let $\mathbf{b} \in E^n$. For any $\mathbf{x}$ with $\|\mathbf{x}\| = 1$, we have

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{x} + \mathbf{b} \cdot \mathbf{x} \leq \max\{\mathbf{a} \cdot \mathbf{x}\} + \max\{\mathbf{b} \cdot \mathbf{x}\}$$

For the $\mathbf{x}$ such that $(\mathbf{a}+\mathbf{b}) \cdot \mathbf{x}$ reaches its maximum, this inequality should also hold. Hence,

$$\begin{aligned}\|\mathbf{a} + \mathbf{b}\| &= \max\{(\mathbf{a} + \mathbf{b}) \cdot \mathbf{x} : \|\mathbf{x}\| = 1\} \\ &\leq \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{x}\| = 1\} + \max\{\mathbf{b} \cdot \mathbf{x} : \|\mathbf{x}\| = 1\} = \|\mathbf{a}\| + \|\mathbf{b}\|\end{aligned}$$

Therefore, the dual norm satisfies the properties specifies in Section 2.9.

(b) Denote $\max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{a}\| = 1\}$ as $\|x\|_{**}$. Write the dual norm as

$$\|\mathbf{a}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{x}\| = 1\} = \max\{\mathbf{a} \cdot \mathbf{x}/\|\mathbf{x}\| : \mathbf{x} \neq 0\}$$

It implies that, for any $\mathbf{x} \neq 0$ and $\mathbf{a} \neq 0$,

$$\|\mathbf{a}\| \geq \mathbf{a} \cdot \mathbf{x}/\|\mathbf{x}\| \quad \Rightarrow \mathbf{a} \cdot \mathbf{x} \leq \|\mathbf{a}\|\|\mathbf{x}\| \quad \Rightarrow \mathbf{a} \cdot \mathbf{x}/\|\mathbf{a}\| \leq \|\mathbf{x}\|$$

Hence,

$$\|\mathbf{x}\|_{**} = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{a}\| = 1\} = \max\{\mathbf{a} \cdot \mathbf{x}/\|\mathbf{a}\| : \mathbf{a} \neq 0\} \leq \|\mathbf{x}\|$$

Conversely, consider the ball of radius $\|\mathbf{x}\|$ centered at $\mathbf{0}$ for some $0 < \|\mathbf{x}\| < 1$. The supporting hyperplane of this ball passing through $\mathbf{x}$ is $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|$. Because there are points of the unit ball $\|\mathbf{a}\| = 1$ on both sides of the hyperplane, there exists some $\mathbf{a}$ such that $\mathbf{x} \cdot \mathbf{a} \geq \|\mathbf{x}\|$. Thus, $\|\mathbf{x}\|_{**} = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{a}\| = 1\} \geq \|\mathbf{x}\|$ in this case. For $\mathbf{x}$ with norm larger than 1, we can shrink it to some $\mathbf{x}/c$ where $c > \|\mathbf{x}\|$, and apply the same argument to find $\|\mathbf{x}/c\|_{**} \geq \|\mathbf{x}/c\|$, which is equivalent to $\|\mathbf{x}\|_{**} \geq \|\mathbf{x}\|$. So $\|\mathbf{x}\|_{**} \geq \|\mathbf{x}\|$ for all $\mathbf{x} \neq 0$.

Combining the results above, we get $\|\mathbf{x}\|_{**} = \|\mathbf{x}\|$ for $\mathbf{x} \neq 0$. Since $\mathbf{0}$ multiplied by any vector is 0,

$$\|\mathbf{0}\|_{**} = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{a}\| = 1\} = 0 = \|\mathbf{0}\|$$

Hence, $\|\mathbf{x}\|_{**} = \|\mathbf{x}\|$ for all $\mathbf{x}$.

§3.3.1

$$\begin{aligned}f_1(x, y) &= 6xy + 2y^2 \qquad & f_1(1, -2) &= 6 \cdot 1 \cdot (-2) + 2 \cdot (-2)^2 = -4 \\ f_2(x, y) &= 3x^2 + 4xy \qquad & f_2(1, -2) &= 3 \cdot 1^2 + 4 \cdot 1 \cdot (-2) = -5\end{aligned}$$

The tangent plane is

$$z = f(1, -2) + f_1(1, -2)(x - 1) + f_2(1, -2)(y + 2) = 2 - 4(x - 1) - 5(y + 2) = -4x - 5y - 4,$$

§3.3.8 Let $\phi(t) = f(t\mathbf{x})$. By Proposition 3.2, $\phi'(t) = df(t\mathbf{x}) \cdot \mathbf{x}$. On the other hand,

$$\phi'(t) = \frac{\partial}{\partial t} f(t\mathbf{x}) = \frac{\partial}{\partial t} t^p f(\mathbf{x}) = pt^{p-1} f(\mathbf{x})$$

Hence $df(t\mathbf{x}) \cdot \mathbf{x} = pt^{p-1} f(\mathbf{x})$. Take $t = 1$. we have $df(\mathbf{x}) \cdot \mathbf{x} = pf(\mathbf{x})$.

Conversely, let $\psi(t) = \phi(t)t^{-p} - f(\mathbf{x})$. Then

$$\begin{aligned}
\psi'(t) &= \phi'(t)t^{-p} + \phi(t)(-p)t^{-p-1} = t^{-p}\frac{\partial}{\partial t} f(t\mathbf{x}) - pt^{-p-1} f(t\mathbf{x}) \\
&= t^{-p} \cdot df(t\mathbf{x}) \cdot \mathbf{x} - pt^{-p-1} f(t\mathbf{x}) = t^{-p-1} \cdot df(t\mathbf{x}) \cdot t\mathbf{x} - pt^{-p-1} f(t\mathbf{x}) \\
&= t^{-p-1} \cdot pf(t\mathbf{x}) - pt^{-p-1} f(t\mathbf{x}) = 0
\end{aligned}$$

This shows that $\psi(t)$ is constant. Since $\psi(1) = \phi(1) - f(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}) = 0$, $\psi(t) = 0$ for all t. Thus,

$$\phi(t)t^{-p} - f(\mathbf{x}) = 0 \quad \Rightarrow \phi(t) - t^p f(\mathbf{x}) = 0 \quad \Rightarrow f(t\mathbf{x}) = \phi(t) = t^p f(\mathbf{x})$$

§3.3.9

$$\begin{aligned}
df(\mathbf{x}) &= \sum_{k=1}^{n} f_k(x)e^k = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(Q(\mathbf{x}))^{p/2} e^k = \sum_{k=1}^{n} \frac{p}{2}(Q(\mathbf{x}))^{p/2-1} \frac{\partial}{\partial x_k} Q(\mathbf{x}) e^k \\
&= \sum_{k=1}^{n} \frac{p}{2}(Q(\mathbf{x}))^{p/2-1} \left(\sum_{j=1}^{n} C_{kj}x^j + \sum_{j=1}^{n} C_{jk}x^j \right) \\
&= \sum_{k=1}^{n} \frac{p}{2}(Q(\mathbf{x}))^{p/2-1} \left(\sum_{j=1}^{n} C_{kj}x^j + \sum_{j=1}^{n} C_{kj}x^j \right) e^k \\
&= \sum_{k=1}^{n} p(Q(\mathbf{x}))^{p/2-1} \sum_{j=1}^{n} C_{kj}x^j e^k = p(Q(\mathbf{x}))^{p/2-1} \sum_{j,k=1}^{n} C_{kj}x^j e^k
\end{aligned}$$

$$df(\mathbf{x}) \cdot \mathbf{x} = p(Q(\mathbf{x}))^{p/2-1} \sum_{j,k=1}^{n} C_{kj}x^j x^k = p(Q(\mathbf{x}))^{p/2-1} Q(x) = p(Q(\mathbf{x}))^{p/2} = pf(\mathbf{x})$$