Homework 1

§1.5.1 (a) The vertices of the hexagon are $(1,0)$, $(1/2, \sqrt{3}/2)$, $(-1/2, \sqrt{3}/2)$, $(-1,0)$, $(-1/2, -\sqrt{3}/2)$, and $(1/2, -\sqrt{3}/2)$. Let $L_1 = \{\mathbf{x} : \mathbf{z}_1 \cdot \mathbf{x} = c_1\}$ be the line passing through $(1,0)$ and $(1/2, \sqrt{3}/2)$. Let $\mathbf{z}_1 = (z_{11}, z_{12})$, and require $|\mathbf{z}_1| = 1$. Then

$$(z_{11}, z_{12}) \cdot (1,0) = (z_{11}, z_{12}) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \Rightarrow \mathbf{z}_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), c_1 = \frac{\sqrt{3}}{2}$$

Let P_1 be the closed half-plane below L_1. We have $P_1 = \{\mathbf{x} : (\sqrt{3}/2, 1/2) \cdot \mathbf{x} \leq \sqrt{3}/2\} = \{\sqrt{3}x + y \leq \sqrt{3}\}$. Similarly, from other edges of the hexagon, we get

$$P_2 = \{\mathbf{x} : (0,1) \cdot \mathbf{x} \leq 1\} = \{(x,y) : y < 1\}$$
$$P_3 = \left\{\mathbf{x} : \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \cdot \mathbf{x} \leq \frac{\sqrt{3}}{2}\right\} = \{(x,y) : -\sqrt{3}x + y \leq \sqrt{3}\}$$
$$P_4 = \left\{\mathbf{x} : \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \cdot \mathbf{x} \geq -\frac{\sqrt{3}}{2}\right\} = \{(x,y) : \sqrt{3}x + y \geq -\sqrt{3}\}$$
$$P_5 = \{\mathbf{x} : (0,1) \cdot \mathbf{x} \geq -1\} = \{(x,y) : y \geq -1\}$$
$$P_6 = \left\{\mathbf{x} : \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \cdot \mathbf{x} \geq -\frac{\sqrt{3}}{2}\right\} = \{(x,y) : -\sqrt{3}x + y \geq -\sqrt{3}\}$$

The hexagon is thus $P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5 \cap P_6$. Because all P_i's are closed and convex, the hexagon is convex, too.

(b) The points of this set are inside the region enclosed by three lines: $y = 1$, $y = x$, and $y = -x$. Let $P_1 = \{(x,y) : y \leq 1\}$, $P_2 = \{(x,y) : x - y \leq 0\}$, and $\{(x,y) : x + y \geq 0\}$. Then this set can be expressed as the intersection of three closed half-planes $P_1 \cap P_2 \cap P_3$, and is therefore closed and convex.

(c) Let L_α be the tangent line of $\log x$ at $x = \alpha > 0$. Since $\frac{d}{dx} \log x\big|_{x=\alpha} = \frac{1}{\alpha}$, the tangent line is $y = x/\alpha + \log \alpha - 1$. Let P_α be the closed half-plane below L_α. Then $P_\alpha = \{(x,y) : y \leq x/\alpha + \log \alpha - 1\}$. The intersection of P_α for all $\alpha > 0$ gives the set all points on below $y = \log x$ for $x > 0$. That is,

$$\bigcap_{\alpha > 0} P_\alpha = \{(x,y) : y \leq \log x, x > 0\}$$

Hence, $\{(x,y) : y \leq \log x, x > 0\}$ is closed and convex.

(d) Let L_α be the tangent line of $\sin x$ at $x = \alpha$ for $0 \leq \alpha \leq \pi$. Then $L_\alpha = \{(x,y) : y = x \cos \alpha - \alpha \cos \alpha + \sin \alpha\}$. Let P_α be the closed half-plane below L_α, i.e. $P_\alpha = \{(x,y) :$

$y \le x\cos\alpha - \alpha\cos\alpha + \sin\alpha\}$, and let $P = \{(x, y) : y \le 0\}$, which is the closed half-plane above the x-axis. Then

$$P \cap \bigcap_{0\le\alpha\le\pi} P_\alpha = \{(x, y) : 0 \le y \le \sin x, 0 \le x \le \pi\}$$

Hence, $\{(x, y) : 0 \le y \le \sin x, 0 \le x \le \pi\}$ is closed and convex.

§1.5.6 If $K = \varnothing$, then trivially $\operatorname{cl} K = \operatorname{int} K = \varnothing$ are both convex.

Let $x, y \in \operatorname{int} K$, and $z = tx + (1-t)y$ for $t \in [0, 1]$. There is an $r_x > 0$ such $B_{r_x}(x) \subset \operatorname{int} K$, and also $r_y > 0$ such that $B_{r_y}(y) \subset \operatorname{int} K$. Take $r = \min\{r_x, r_y\}$. Then $B_r(x)$ and $B_r(y)$ are both subsets of $\operatorname{int} K$. For any w with $|w| < r$, we have $x + w \in B_r(x) \subset \operatorname{int} K$ and $y + w \in B_r(y) \subset \operatorname{int} K$. Consider $z' = t(x + w) + (1-t)(y + w) = tx + (1-t)y + w = z + w$, which is in K by convexity. It follows that $B_r(z)$ is a subset of K, which means z is an interior point of K, i.e. $z \in \operatorname{int} K$. Therefore, the segment joining any x and y in $\operatorname{int} K$ is contained in $\operatorname{int} K$, and $\operatorname{int} K$ is convex.

Let $x, y \in \operatorname{cl} K$, and $z = tx + (1-t)y$ for $t \in [0, 1]$. There exist a sequence $\{x_n\}$ in K that converges to x and a sequence $\{y_n\}$ in K that converges to y. Let $\{z_n\}$ be a sequence such that $z_n = tx_n + (1-t)y_n$ with $t \in [0, 1]$ for all n. Then $z_n \to z$ as $n \to \infty$. Due to convexity, each z_n is in K. So z is also a limit point of K, which means that $z \in \operatorname{cl} K$. It implies that the segment joining any x and y in $\operatorname{cl} K$ is contained in $\operatorname{cl} K$, and $\operatorname{cl} K$ is hence convex.

§1.5.9 (a) Let $x = \sum_{k=1}^m t^k x_k$, where $x_k \in \hat{S}$ for all k. If all x_k are in S, then $x \in \hat{S}$ by definition of $\hat{S}$. If there is some $x_k \notin S$, then such $x_k = \sum_{k=1}^m t^k y_k$ for some y_k's in S. By Problem 8, $x = \sum_{k=1}^p t^p z_p$, where z_p are selected from the set of all y_k's. It means that x is also a convex combination of points in S, i.e. $x \in \hat{S}$. Thus, any convex combination of points in $\hat{S}$ is a point of $\hat{S}$, and $\hat{S}$ is convex.

(b) Any $x \in \hat{S}$ can be written as $x = \sum_{k=1} t^k x_k$ for some $x_1, \dots x_m \in S$ and $t^1 + \cdots + t^m = 1$. Because these x_k's are also points of K, x becomes a convex combination of points of K. By Proposition 1.6, $x \in K$ since K is convex. Thus $\hat{S} \subset K$.

§2.11.2 (a) By Theorem 2.13, for norm

$$\|x\| = \frac{1}{\max\{t : t\mathbf{x} \in K\}},$$

K is a closed unit 2-ball.

(b) $\|\mathbf{e}_1 - \mathbf{e}_2\| = \|(2, 0)\|$. Observe that, when $t = \frac{1}{2}$, $t(\mathbf{e}_1 - \mathbf{e}_2) = (1, 0)$, which is a point on the boundary of K (Take $(x, y) = (1, 0)$, $x^2 + xy + 4y^2 = 1^2 + 1 \cdot 0 + 4 \cdot 0^2 = 1$). If $t > \frac{1}{2}$, this point gets further away from K. So $\max\{t : t\mathbf{x} \in K\} = \frac{1}{2}$, and $\|\mathbf{e}_1 - \mathbf{e}_2\| = \frac{1}{1/2} = 2$.

§2.11.4 (a) Let $\mathbf{x}, \mathbf{y} \in K$, and consider $\mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$ for $0 \le t \le 1$. For z^i as the ith component of $\mathbf{z}$,

$$|z^i|^p = |tx^i + (1-t)y^i|^p \le t|x^i|^p + (1-t)|y^i|^p$$

by Problem 3. Therefore,

$$\sum_{i=1}^{n}|z^i|^p \leq \sum_{i=1}^{n}(t|x^i|^p + (1-t)|y^i|^p) = \sum_{i=1}^{n} t|x^i|^p + \sum_{i=1}^{n}(1-t)|y^i|^p$$
$$= t\sum_{i=1}^{n}|x^i|^p + (1-t)\sum_{i=1}^{n}|y^i|^p$$
$$\leq t + (1-t) = 1$$

Hence, $\mathbf{z} \in K$, and K is convex.

(b) i. K is compact because it is closed and bounded in E^n (under p-norm).

ii. Part (a) shows that K is convex.

iii. Let $\mathbf{x} \in K$. For $-\mathbf{x}$,

$$\sum_{i=1}^{n}|-x^i|^p = \sum_{i=1}^{n}|x^i|^p \leq 1 \quad \Rightarrow -\mathbf{x} \in K$$

So K is symmetric about $\mathbf{0}$.

iv. Let $r = \frac{1}{2n}$, and consider $B_r(\mathbf{0})$. For any $\mathbf{x} = (x^1, \dots, x^n) \in B_r(\mathbf{0})$, $|x_i| \leq r$ for all n. Thus,

$$\sum_{i=1}^{n}|x^i|^p \leq \sum_{i=1}^{n}\left(\frac{1}{2n}\right)^p \leq \sum_{i=1}^{n}\frac{1}{2n} = \frac{1}{2} \quad \Rightarrow \mathbf{x} \in K$$

So $B_r(\mathbf{0})$ is a Euclidean neighborhood of $\mathbf{0}$ in K.

(c) For any $\mathbf{x}$ and $t > 0$, $t\mathbf{x} \in K$ if $\sum_{i=1}^{n}|tx^i|^p = \sum_{i=1}^{n} t^p|x^i|^p = t^p\sum_{i=1}^{n}|x^i|^p \leq 1$. Note that, as t grows larger, $t^p\sum_{i=1}^{n}|x^i|^p$ grows larger as well. It follows that, when reaching its maximum, t satisfies $t^p\sum_{i=1}^{n}|x^i|^p = 1$. Therefore,

$$\max\{t : t\mathbf{x} \in K\} = \left(\frac{1}{\sum_{i=1}^{n}|x^i|^p}\right)^{1/p} = \frac{1}{(\sum_{i=1}^{n}|x^i|^p)^{1/p}}$$
$$\|\mathbf{x}\| = \frac{1}{\max\{t : t\mathbf{x} \in K\}} = \left(\sum_{i=1}^{n}|x^i|^p\right)^{1/p}$$

3. (a) Let x^k be a component of $\mathbf{x}$ such that $|x_k| = \|\mathbf{x}\|_\infty = \max\{x^1, \dots, x^n\}$. Then

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n}|x^i|^p\right)^{1/p} \geq (|x_k|^p)^{1/p} = |x_k| = \|\mathbf{x}\|_\infty$$

On the other hand,

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} = \|\mathbf{x}\|_\infty\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{\|\mathbf{x}\|_\infty}\right)^p\right)^{1/p} \leq \|\mathbf{x}\|_\infty\left(\sum_{i=1}^{n}1\right)^{1/p} = \|\mathbf{x}\|n^{1/p}$$

So we have $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_\infty n^{1/p}$. As $n \to \infty$, $n^{1/p} \to 1$. By the squeeze theorem,

$$\lim_{p\to\infty}\|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$$

(b) If $\|\cdot\|$ is a norm, then $\|5\mathbf{e}_1\|_0 = 5\|\mathbf{e}_1\|_0 = 5$. However, $\|5\mathbf{e}_1\|_0 = 1$, as there is only 1 non-zero entry in $5\mathbf{e}_1$. So $\|\cdot\|$ is not a norm.