Dominant Eigenvalues and Directed Graphs

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1 Preliminaries

Definition 1

A **block matrix** is a matrix with non-trivial partitions on its rows and columns, and the resulting smaller matrices are called blocks. Then, a **block upper-triangular matrix** is a block matrix such that all blocks below the main diagonal are blocks with only 0s as entries, and that all blocks on the main diagonal are square.

Example 2

If we define matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

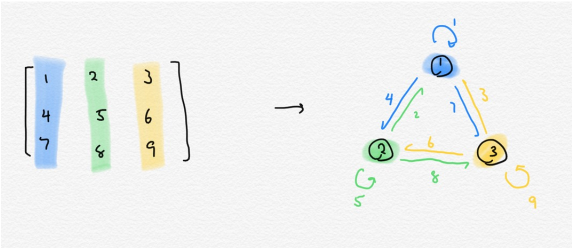
$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix},\ B = \begin{pmatrix} 5 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} 4 \end{pmatrix},$$

then M is a block upper-triangular matrix.

Definition 3

An irreducible matrix is a square matrix that is not similar, via permutation, to a block upper triangular matrix.

For an n by n matrix A, we consider a corresponding directed graph with n vertices such that for each entry $a_{ij} \in A$ there is an edge from vertex j to vertex i with weight a_{ij}, if $a_{ij} \neq 0$. Unique up to relabeling.



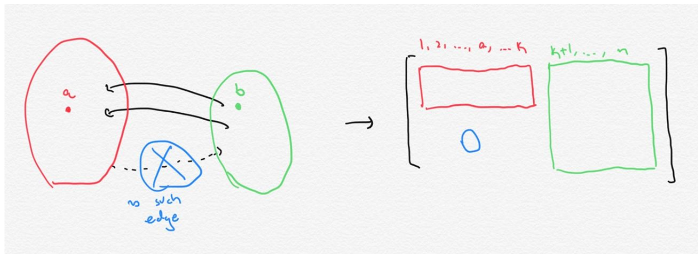
Definition 4

A directed graph is strongly connected if, for any two vertices v_1, v_2, there exists a path from v_1 to v_2.

Proposition 5

A matrix is irreducible $\Rightarrow$ the matrix's associated graph is strongly connected.

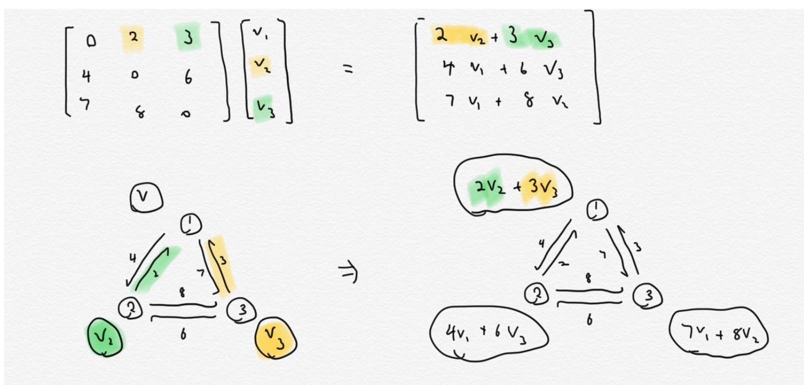
Proof. Take the contrapositive. If a graph is not strongly connected, there exists two vertices a and b where there is no path from a to b. Then we can partition the vertices into two partitions: the vertices reachable from a, and those that aren't. It follows that there are no edges from the first partition to the second, thus this graph can be represented with a reducible matrix.



□

Question. What is the multiplication of a matrix and a vector, in terms of directed graphs?

We can think of matrix multiplication as pushing weights along edges. Suppose we multiply matrix A with vector v, and assign v_i as a value to vertex i. Then $(Av)_i$ can be calculated by the sum of the values of its in-neighbors times the weight of those edges. This is because the ith row of the matrix consist of the edge weights in-neighbors of vertex i, and $(Av)_i$ is the dot product of the ith row of A and the vector v.



Fact 6

If J is any matrix in the Jordan Normal form, then $||J^\ell v||$ is $O((\rho)^\ell \cdot poly(\ell))$. Since any square matrix A can be written in the Jordan Normal form (i.e. $\exists S$ s.t. $A = SJS^{-1}$) we know that $||A^\ell v|| = ||SJ^\ell S^{-1} v||$ is $O((\rho)^\ell \cdot poly(\ell))$.

(ρ is the spectral radius, which is the magnitude of a dominant eigenvalue which are the eigenvalues with the greatest magnitude.)

2 Proof

Theorem 7 (Perron-Frobenius)

For any nonnegative irreducible matrix, A,

- There exists an eigenvalue λ_1 with the largest magnitude is real and positive.
- There is corresponding eigenvector to λ_1, v with all positive entries.
- The eigenspace of λ_1 is one-dimensional.

Lemma 8

Any nonnegative v such that $Av \geq \rho v$ (defined entry-wise) must satisfy $Av = \rho v$.

Proof. Let the graph associated with A be G. For the sake of contradiction, suppose there exist a v such that $Av \geq \rho v$, but $Av \neq \rho v$. In particular, there exists some i where $(Av)_i > \rho v_i$. Now, consider vector $w = v + me_i$, where e_i is the ith standard basis vector and $m > 0$. Then, since

$$Aw = A(me_i + v) = mAe_i + Av, \tag{1}$$

we can write that for all j,

$$(Aw)_j = ma_{ji} + (Av)_j, \tag{2}$$

and since

$$\rho w = \rho me_i + \rho v, \tag{3}$$

which means for $j \neq i$,

$$(\rho w)_j = \begin{cases} \rho m + \rho v_j & j = i \\ \rho v_j & j \neq i. \end{cases} \tag{4}$$

If we consider matrix multiplication as "pushing weights along edges" on graph G, we can see that for all out-neighbors of vertex i, $ma_{ji} > 0$ since a_{ji} represents the weight on an edge from vertex i. This means that, with eq. (2) and (4), noting that $(Av)_j \geq (\rho v)_j$, we have $(Aw)_j > (\rho w)_j$ when j is an out-neighbor of i. Further, since $(Av)_i - \rho v_i$ is a positive constant due to our initial assumption, our m could have been chosen to be sufficiently small such that

$$m(\rho - a_{ii}) < (Av)_i - \rho v_i, \tag{5}$$

and after rearranging, we have

$$\rho m + \rho v_i < ma_{ii} + (Av)_i, \tag{6}$$

and from eq. (2) and (4), we have $(Aw)_i > (\rho w)_i$.

Now, since $(Aw)_j > (\rho w)_j$ where j is an out-neighbor of i, we can repeat the same process at vertex j, finding a new vector at each vertex. Since G is strongly connected, we can continue to repeat this process until we find a vector w' such that $(Aw')_j > (\rho w)_j$ for all vertices j, or $Aw' > \rho w'$; we can choose $c > 1$ such that $Aw' \geq c\rho w'$.

For positive integer ℓ, we have $A^\ell(Aw') \geq c\rho A^\ell w' \geq ... \geq c^{\ell+1}\rho^{\ell+1}w'$. This implies $||A^\ell w'|| \geq ||c^\ell \rho^\ell w'||$. However, note that the left-hand side is $O((\rho)^\ell \cdot poly(\ell))$ whereas the right-hand side is $O(c^\ell \rho^\ell)$, and noting that $c > 1$, we have a contradiction. □

Theorem (Pt. 1)

There exists a dominant eigenvalue λ_1 of A that is real and positive.

Proof. Suppose λ is the eigenvalue with corresponding v, and let $\rho = |\lambda|$. Since $Av = \lambda v$, we have $|(Av)_j| = \rho|v_j|$. Further, we have $|(Av)_j| = |\sum_i a_{ji} v_i| \leq \sum_i a_{ji}|v_i|$ by the triangular inequality. Let w be the vector such that $w_j = |v_j|$. Thus we have $\rho w \leq Aw$, but by lemma 2, $\rho w = Aw$. □

Problem 1

Prove that there exists an eigenvector of λ_1 with all nonnegative entries.

Theorem (Pt. 2)

There is corresponding eigenvector to λ_1, v with all positive entries.

Proof. Now, we would like to show that v is strictly positive, assuming that it is nonnegative. Suppose for the sake of contradiction that there is a 0 in the nth entry in v, and since v is an eigenvector, the nth position of Av must also be zero. Returning to our graphical process of multiplying matrices, vertex n is assigned the value 0 after the multiplication, or $(Av)_n = 0$. This implies that all of vertex n's in-neighbors s must have been assigned 0 before the multiplication, or $v_s = 0$, since all edge weights must be positive. But since $Av = \lambda_1 v$, we have $0 = \lambda_1 v_s = (Av)_s$, or all of vertex n's in-neighbors must be 0 after the multiplication process. We can repeat this argument at each of n's in-neighbors, and since the graph is strongly connected, we can repeat the argument at each entry so that we must have $Av = 0 \implies v = 0$. Contradiction, as v is an eigenvector. Thus, v must have all positive entries. □

Problem 2

Prove that the eigenspace of λ_1 is one-dimensional, or, in other words, for two eigenvectors v and v', there is some k such that $v = kv'$. (You can use similar reasoning as above.)

3 Applications

Problem 3 (Leontiev Input/Output Economic Model)

We have seen at the beginning of today's class that the following model

	Ag.	Indust.	Serv.	Consumer	Total prod.
Ag.	$0.3x_1$	$0.2x_2$	$0.3x_3$	4	x_1
Indust.	$0.2x_1$	$0.4x_2$	$0.3x_3$	5	x_2
Serv.	$0.2x_1$	$0.5x_2$	$0.1x_3$	12	x_3

reduces to a vector equation

$$Ax + b = x,$$

so the question becomes: when does this vector equation have a nonnegative solution $x \geq 0$ for $b \geq 0$? Please give the conditions on the spectral radius/dominant eigenvalue.

HINT: think about how you can write $(I - A)^{-1}$.

(If you're interested in the solution of this problem or want to see more applications of the theorem, see https://epubs.siam.org/doi/pdf/10.1137/S0036144599359449.)