STEINBERG SYMBOLS: A BRIEF SURVEY OF ALGEBRAIC K-THEORY

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ABSTRACT. In this paper, we explore the definition of the Steinberg symbol and its significance. To do this, we first providing the motivation and background about projective modules for studying K-theory. Then, we define K_0 as a nicer object to work with, obtained from the category of projective modules over a ring. We then further define K_1 and K_2, motivated by finding objects which fits an analog of the Mayer-Vietoris sequence. After we introduce these definitions, we provide some tools with which we can calculate these rings, and we demonstrate the nice structure on $K_2(R)$ encoded by the Steinberg symbol. We then venture out to find the next right definitions for higher K-groups, introducing Milnor K-theory and topological constructions.

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1. Overview

The Steinberg symbol is a pairing function on fields that plays a role in the algebraic K-theory of fields, which provides formalism for discussing and generalizing the Hilbert symbol. It was introduced following Steinberg's 1962 Brussels conference paper [6], where he worked out generators and relations of simple adjoint algebraic groups over arbitrary fields in order to study non-adjoint groups and how projective modular representations of the adjoint groups lifted to universal groups. Matsumoto's thesis [2] in 1969 and Moore's paper that appeared in the same year [4] fleshed out the connections of Steinberg symbols to classical symbols and reciprocity laws [1]. In 1971, Milnor published *Introduction to Algebraic K-theory* [3], which serves a summative exposition of the above developments.

Algebraic K-theory was developed at a concurrent time by topologists to study vector bundles and obstructions in homotopy theory. Matsumoto's theorem, which prompted the definition of Steinberg symbols in order to clarify the structure of K_2, has the following immediate corollary:

Corollary 1.1. *Suppose A is an abelian group and F is a field, and let F^* denote the multiplicative group $F - \{0\}$. Given any bimultiplicative map*

$$c : F^* \times F^* \to A$$

satisfying the identity

$$c(x, 1-x) = 1,$$

there exist a unique homomorphism from $K_2(F)$ to A which sends the symbol $\{x, y\}$ to $c(x, y)$ for all x and y.

In other words, $K_2(F)$ is the universal object for the identity $c(x, 1-x) = 1$. Any such c satisfying the conditions of the corollary is defined to be a *Steinberg symbol* over the field F. In fact, there are several classical pairing maps which turn out to be Steinberg symbols, including, most notably, the Hilbert symbol. In this paper we intend to given a basic overview of algebraic K-theory, with the goal of demonstrating the significance of the Steinberg symbol. In section 2, we introduce projective modules, which are the motivating objects of K-theory. In section 3, we define K_0, K_1, and K_2 groups, and describe some tools with which we can study the structures of these K-groups. In section 4, we explore Steinberg symbols, which essentially control the structure of the K_2 group of fields. Finally, in section 5, we talk briefly about a way in which we can extend the idea of K-groups into higher degrees.

For background, we assume that the reader knows about modules. The bulk of the content in this paper is sourced from *The K-Book* [7], with nods to Milnor's classic *Introduction to Algebraic K-theory* [3].

2. Projective Modules

In this section we aim to define projective modules over a ring R, which are the main objects of study in algebraic K-theory, and we give some well-known results about the stability of projective modules.

2.1. **Free and stably free modules.** A module M over ring R is *free* if it has a basis, i.e. if there exists a subset $\{e_i\}_{i\in I} \subset M$ such that for all $m \in M$, there is a unique way to write $m = \sum r_i e_i$, with $r_i \in R$. In particular, any module with torsion cannot be free, but a torsion-free module is not necessarily free: consider the ideal $(2, x)$ in the ring $\mathbb{Z}[x]$ as a $\mathbb{Z}[x]$-module, which is not free since it is not a principal ideal.

Notice that if $R = F$ is a field, then any F-module is a vector space, so it always has a basis and is therefore free. In this case, the vector spaces F^s and F^t for $s \neq t$ are necessarily not isomorphic. This is not true of a general ring; we say that a ring R satisfies the *invariant basis property* if $R^s \cong R^t$ if and only if $s = t$. For a ring R that satisfies this property, there is a definite notion of rank for finitely generated free R-modules. Notice also that all commutative rings automatically satisfy this property, since any basis of M lifts to a basis of $M \otimes_R (R/\mathfrak{m})$, a $R/\mathfrak{m}$-vector space, via the map $R \to R/\mathfrak{m}$.

We say that an R-module P is *stably free* of rank $n-m$ if $P \oplus R^m \cong R^n$ for some m and n. Notice that if R satisfies the invariant basis property, the rank is invariant under the choice of m and n. A way to produce stably free modules is by looking at the kernels K of surjective maps $R^n \to R^m$, since the exact sequence

$$0 \to K \to R^n \to R^m \to 0$$

splits since R^m is free, and clearly we can engineer this short exact sequence given a stably free module, so this is a correspondence. The natural next question is whether stably free R-modules M are always free; unfortunately this is not true, even over commutative rings. Here is a classic example:

Example 2.1. Take the ring $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, and consider the map

$$\sigma : R^3 \to R$$

$$(a, b, c) \mapsto ax + by + cz.$$

We would like to write $R^3 = R \oplus \ker(\sigma)$. Notice that every element of R^3 represents a continuous vector field on $S^2 \subset \mathbb{R}^3$, and under this interpretation, σ is taking the dot product of (a, b, c) with (x, y, z), the radial vector field. Hence $(a, b, c) \in \ker(\sigma)$ are tangent vector fields on S^2. If $\ker(\sigma)$ were free, its basis would represent two tangent vector fields which are linearly independent at every point on S^2. But this is impossible by the hairy ball theorem.

The following theorem provides a description of the "stable range" where stably free R-modules are in fact free.

Theorem 2.2 (Bass Cancellation). *Suppose R is a commutative Noetherian ring of Krull dimension d. Then every stably free R-module of rank $> d$ is free.*

Notice that Example 2.1 shows this bound is sharp.

2.2. **Projective modules.** We may be curious about the further generalization of stably free R-modules where we consider those modules P for which there exists some (not necessarily free) module Q such that their direct sum $P \oplus Q$ is free. These modules are said to be *projective*. As before, we'd like a similar characterization of these as kernels of maps, but since we don't know whether Q is free, the splitting is nontrivial. To this end, we can rephrase our definition of projective as follows: P is projective if there is some free module F and a surjective map $\phi : F \to P$ such that the exact sequence

$$0 \to Q = \ker(\phi) \to F \xrightarrow{\phi} P \to 0$$

splits. By the splitting lemma, it suffices to have a right inverse of ϕ. In other words, it suffices to have a morphism f such that

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[d, "\wr"] & \\ F \arrow[r, "\phi"] & P \arrow[r] & 0 \end{tikzcd}$$

commutes. We make this correspondence precise in the following stronger statement.

Proposition 2.3 (projective lifting property). *An R-module P is projective if and only if for all surjective R-module maps $\phi : M \to N$ and for every map $g : P \to N$ there exists a map $f : P \to M$ such that*

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[d, "g"] & \\ M \arrow[r, "\phi"] & N \arrow[r] & 0 \end{tikzcd}$$

commutes.

Proof. The discussion above gives the reverse direction: since there's always a surjective morphism into P from some free module, we could always construct the exact sequence. Hence it suffices to show that any projective module P satisfies the projective lifting property. First notice that if P is free, constructing f amounts to lifting the image of a basis, which we could always do. Then this is true for all projective modules P by extending g to a map from $Q \oplus P$, a free module, from which we can construct f by restricting the domain. □

Notice that all stably free modules are projective. As before, we have the following theorem which provides a description of the "stable range" where projective R-modules can be described nicely. We say that two R-modules M, N are *stably isomorphic* if $M \oplus R^m \sim N \oplus R^n$ for some m, n.

Theorem 2.4 (Bass-Serre Cancellation). *Suppose R is a commutative Noetherian ring of Krull dimension d, and P is a projective R-module of constant rank $n > d$. Then:*

(1) *$P \simeq P_0 \oplus R^{n-d}$ for a projective module P_0 of constant rank d. (Serre)*

(2) *If P and P' are stably isomorphic, then $P \simeq P'$. (Bass)*

(3) *If $P \oplus M$ is stably isomorphic to M', then $P \oplus M \simeq M'$ for any M, M'. (Bass)*

We will now focus our attention on the category $\mathbf{P}(R)$ of finitely-generated projective R-modules, whose only nontrivial part consists of projective R-modules of rank $\leqslant d$.

3. K_0, K_1 AND K_2

In this section, we will define the groups K_0, K_1, and K_2. We first define K_0 as the group completion of $\mathbf{P}(R)$, and then obtain K_1 and K_2 through an analog of the Mayer-Vietoris sequence.

3.1. Grothendiek K_0 Group.

To figure out what kind of object $\mathbf{P}(R)$ is, first notice that it is closed under direct sum: if P, P' are both projective R-modules, we have $P \oplus Q$ and $P' \oplus Q'$ free, so clearly $(P \oplus P') \oplus (Q \oplus Q')$ is a free R-module. Hence direct sum gives us a natural notion of addition, with additive identity being the zero module. A priori, it isn't clear that there is a notion of inverse.

3.1.1. *Group completion of an abelian monoid.* In this section, we will describe $K_0(R)$ as the group completion of $\mathbf{P}(R)$ by artificially including inverses.

We call categories like $\mathbf{P}(R)$ abelian monoids. More precisely, an *abelian monoid* $(M, +)$ is a set M equipped with a binary operation $+$ which is commutative, associative, and has an identity 0. Notably, abelian groups are abelian monoids. If M, N are abelian monoids, then the map $f : M \to N$ is a morphism of monoids if f respects addition; in other words, if $f(0) = 0$ and $f(m_1 + m_2) = f(m_1) + f(m_2)$.

The *group completion* of an abelian monoid M is an abelian group $M^{-1}M$ with a monoid map $[] : M \to M^{-1}M$ which is universal: for any map $M \to A$ with A an abelian group, there is a unique morphism of abelian groups $M^{-1}M \to A$ making

$$\begin{tikzcd} M \arrow[r] \arrow[d] & A \\ M^{-1}M \arrow[ur, dashed] & \end{tikzcd}$$

commute. The group completion $A^{-1}A$ of any abelian group A is itself, and if M is a submonoid of A then $M^{-1}M$ is the subgroup in A generated by M. For example, the group completion of $\mathbb{N}$ is $\mathbb{Z}$.

Every abelian monoid M has a group completion $M^{-1}M$: we can construct it by forming the free abelian group $F(M)$ whose symbols $[m]$ come from elements $m \in M$, then quotienting out the relation $[m+n]-[m]-[n]$. The structure map $M \to M^{-1}M$ is simply $m \mapsto [m]$. This construction is in fact universal, since given a map $f : M \to A$ there is exactly one way to determine where a symbol is mapped to under $M^{-1}M \to A$: $[m]$ is mapped to $f(m)$ and $-[m]$ is mapped to $f(m)^{-1}$.

To write $M^{-1}M$ explicitly in terms of M without the artificial addition of inverses, notice that every element of $M^{-1}M$ can be written as $[m]-[n]$ for $m, n \in M$. In addition, it is easy to verify that $[m]=[n]$ if and only if $m+p=n+p$ for some $p \in M$. This means that if we consider the surjective map

$$M \times M \to M^{-1}M$$
$$(m,n) \mapsto [m]-[n]$$

we see that $M^{-1}M$ can be written explicitly as the quotient of $M \times M$ by the relation $(m,n) \sim (m+p, n+p)$.

Now, since each abelian monoid has a group completion, and since if we have a monoid map $M \to N$, the map $M \to N \to N^{-1}N$ extends uniquely to a map $M^{-1}M \to N^{-1}N$, group completion is actually a functor from abelian monoids to abelian groups. In fact, it is the left adjoint to the forgetful functor since we have

$$\mathrm{Hom}(M, A) \simeq \mathrm{Hom}(M^{-1}M, A)$$

by definition. Now we're ready to state the definition of the K_0 group.

Definition 3.1. The *Grothendieck Group* of a ring R, denoted $K_0(R)$, is the group completion $\mathbf{P}^{-1}\mathbf{P}$ of the category of finitely-generated projective R-modules, $\mathbf{P}(R)$.

In other words, $K_0(R)$ is the group with generators $[P]$ where P is a projective R-module, subject to the relation $[P]+[Q]=[P \oplus Q]$, Sand K_0 is a functor from rings to abelian groups. In the case when R is commutative, notice that $K_0(R)$ is also closed under the tensor product $\otimes_R$, since $\otimes$ distributes over $\oplus$. Using $[R]=1$, we see that tensor product makes $K_0(R)$ a commutative ring. In fact, in this case K_0 is actually a functor from commutative rings to commutative rings, since for some ring map $R \to S$, the tensor product $\otimes_R S$ induces a monoid map $\mathbf{P}(R) \to \mathbf{P}(S)$, which is made into a map of abelian groups $K_0(R) \to K_0(S)$ via group completion, which is in fact a ring map if R, S commutative since

$$(P \otimes_R Q) \otimes_R S \simeq (P \otimes_R S) \otimes_R (Q \otimes_R S).$$

From our previous discussion about group completions abelian monoids, we have the following simple reduction, which is just a rephrasing of the definitions:

Lemma 3.2. *Consider the monoid map $\mathbb{N} \to \mathbf{P}(R)$ sending n to $[R^n]$. This induces a morphism of abelian groups, $\mathbb{Z} \to K_0(R)$. Then,*

- *$\mathbb{Z} \to K_0(R)$ is injective if and only if R satisfies the invariant basis property, and*
- *supposing that R satisfies the invariant basis property, $K_0(R) \simeq \mathbb{Z}$ if and only if every finitely generated projective R-module is stably free.*

Notably, if R is commutative, $\mathbb{Z}$ is a summand of $K_0(R)$. This is because the surjective map $R \to R/\mathfrak{m}$ induces the ring map $K_0(R) \to K_0(R/\mathfrak{m}) = \mathbb{Z}$ which sends $[R]$ to 1.

Now, for some ring R, we have the following inclusion $GL_n(R) \subset GL_{n+1}(R)$: for $g \in GL_n(R)$, we can send it to $\begin{pmatrix} g & \\ & 1 \end{pmatrix}$. Then we can define $GL(R) = \varinjlim GL_n(R)$. Now suppose we have a ring map $f : R \to S$ where the ideal $I \subset R$ is mapped isomorphically to an ideal $I \subset S$. Then we have the following pullback diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d] \\ R/I \arrow[r] & S/I \end{tikzcd}$$

which we call a *Milnor square*. Given a Milnor square, we can construct an R-module M from an S-module M_1, an R/I-module M_2, and an ismorphism $g : M_2 \otimes_{R/I} S/I \simeq M_1/IM_1$. This process is called *Milnor patching*. Then we can define the following maps

$$\partial_n : GL_n(S/I) \to K_0(R)$$
$$g \mapsto [P] - [R^n]$$

where P is obtained by Milnor patching together copies of free modules. We can assemble them together into a map $\partial : GL(S/I) \to K_0(R)$. Then we have the following theorem.

Theorem 3.3 (Mayer-Vietoris)**.** *The sequence*

$$GL(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \to K_0(S/I)$$

is exact.

See [7, II.2.9] for more details.

A natural question which arises is whether one could extend this exact sequence to the left. In particular, the K_0 part of this sequence looks rather similar to the topological cohomology long exact sequences. In order to study this, we now introduce the notion of the K_1 group.

3.2. **Whitehead Group K_1.** We now define $K_1(R)$ as $GL(R)/[GL(R), GL(R)]$, where $[GL(R), GL(R)]$ denotes the commutator subgroup of $GL(R)$, emulating the cohomology long exact sequence from topology. (In fact, the analogous construction of K-groups on vector bundles gives a generalized cohomology theory.) Later, in Section 5, we will see that $K_1(R)$ is actually $\pi_1(K(R))$ where $K(R)$ is some topological space built out of $\mathbf{P}(R)$ whose fundamental group is $K_0(R)$.

From the above definition of $K_1(R)$, it is easy to see that $K_1(R)$ is universal with regards to the property that every homomorphism from $GL(R)$ to an abelian group A

$$\begin{tikzcd} GL(R) \arrow[r] \arrow[d] & A \\ K_1(R) \arrow[ur, dashed] & \end{tikzcd}$$

factors through the quotient map $GL(R) \to K_1(R)$.

In order to compute K_1, we must find the commutator subgroup of $GL(R)$. We will now show that the commutator is the subgroup $E(R)$ generated by *elementary matrices*.

Definition 3.4. An elementary matrix $e_{ij}(r) \in GL(R)$ with $i \neq j$ and $r \in R$ is a square matrix whose diagonal entries are 1 and whose (i, j)-th entry is r.

We will denote the subgroup containing $n \times n$ elementary matrices as $E_n(R)$, so that $E(R) = \varinjlim E_n(R)$. Notice that the matrix $e_{ij}(r)$ acts on matrices on the left by adding r copies of the j-th row to the i-th row. Using this description, we see that $E(R)$ consists of those matrices from which we can obtain the identity matrix via this row operation.

Lemma 3.5 (Whitehead). *The commutator subgroup of $GL(R)$ is $E(R)$, and so $K_1(R) = GL(R)/E(R)$.*

Proof. First we compute the commutators of elementary matrices. Noting that $e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$ and in particular $e_{ij}(-r)$ is the inverse to $e_{ij}(r)$, the following cases are easy to describe:

$$[e_{ij}(r), e_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ e_{il}(rs) & \text{if } j = k \text{ and } i \neq l \\ e_{kj}(-sr) & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

From this we see that the commutators of $E(R)$ is the entire $E(R)$ by taking i, j, k distinct and noting $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$. and so $E(R) \subset [GL(R), GL(R)]$. In addition, for each commutator in $GL_n(R)$, we can write it as a product in $GL_{2n}(R)$ in the following way:

$$[g, h] = \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & h^{-1} \end{bmatrix} \begin{bmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{bmatrix}.$$

Each of the terms is clearly in $E(R)$ since we can get to identity via elementary row operations. Hence $K_1(R) = GL(R)/E(R)$. □

Notice that as an immediate consequence of this lemma, if F is a field, then the usual linear algebra applies and it is well-known in this case that $E(F) = SL(F)$, so $K_1(F) = F^\times$.

As before, K_1 for a Milnor square fits into an exact sequence.

Theorem 3.6 (Mayer-Vietoris). *Given a Milnor square, the following sequence*

$$K_1(R) \xrightarrow{\Delta} K_1(S) \oplus K_1(R/I) \to K_1(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \to K_0(S/I)$$

is exact.

The proof amounts to chasing some definitions and can be found at [7, III.2.6].

3.3. K_2 **of a Ring and the Steinberg Group.** In the spirit of further extending the Mayer-Vietoris sequence, we are interested in the next group, K_2, that would further resolve our exact sequence in the same manner. Milnor found the right definition for this group following a 1962 paper by Steinberg on Universal Central Extensions of Chevellay groups.

In order to define this group, we first present the definition of the Steinberg group, $St(R)$, and then we will define K_2 as the kernel of a map in a certain exact sequence, and then we will show that K_2 is the center of $St(R)$.

Definition 3.7. For $n \geqslant 3$, the *Steinberg group* $St_n(R)$ of a ring R, is a group with generators x_{ij} with i, j distinct integers, $i, j \in [1, n]$, and $r \in R$, and relations ("Steinberg relations"):

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s),$$

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ x_{il}(rs) & \text{if } j = k \text{ and } i \neq l \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

Notice that each elementary matrix in $E_n(R)$ satisfy the relations of $St_n(R)$, we have the surjective homomorphism

$$\phi_n : St_n(R) \to E_n(R)$$
$$x_{ij}(r) \mapsto e_{ij}(r).$$

Also notice that each St_n includes naturally into St_{n+1}, we can similarly defining $St(R) = \varinjlim St_n(R)$. Then the maps ϕ_n induce the surjective map $\phi : St(R) \to E(R)$. Since the definitions for St_n and E_n look almost identical, we may wonder whether they are the same group. To capture this information, we consider the kernel of ϕ.

Definition 3.8. The group $K_2(R)$ of a ring R is defined to be the kernel of the map $\phi : St(R) \to E(R)$ as above.

From the definition, we see that K_2 actually fits into the following exact sequence of groups:

$$1 \to K_2(R) \to St(R) \to GL(R) \to K_1(R) \to 1.$$

In fact, K_2 is an abelian group, which we know from the following theorem:

Theorem 3.9 (Steinberg). *$K_2(R)$ is the center of $St(R)$.*

Proof. First suppose x is in the center of $St(R)$. Then x commutes with every element of $St(R)$, and since ϕ is a surjective homomorphism, it follows that $\phi(x)$ commutes with every element of $E(R)$. But for elementary matrices $e_{ij}(r)$, taking matrix A to be the matrix where $A_{kj} \neq 0$, for any j. Then $e_{ij}(r)A \neq Ae_{ij}(r)$ for any $r \neq 0$. From this we see that the center of $E(R)$ is trivial, and hence x is in the kernel of the map ϕ, so it must be a member of $K_2(R)$.

Now suppose x is a member of $K_2(R)$, so $\phi(x) = 1$. Then, for every $p \in St(R)$, we must have

$$\phi(pxp^{-1}x^{-1}) = \phi(p)\phi(x)\phi(p)^{-1}\phi(x)^{-1} = \phi(p)\phi(p)^{-1} = 1.$$

Choose an integer n such that x can be written using generators $x_{ij}(r)$ where $i, j < n$. Write P_n to denote the subgroup of $St(R)$ generated by the symbols x_{in} with $i < n$. Then clearly $pxp^{-1}x^{-1}$ is in P_n. But P_n also injects into E_n by the map ϕ, so $\phi(p)\phi(x)\phi(p)^{-1}\phi(x)^{-1} = 1$ means $pxp^{-1}x^{-1} = 1$, so x commutes with all generators x_{in} with $i < n$, and by the same argument x also commutes with all generators x_{nj} with $j < n$. Hence x is in the center of $St(R)$. ☐

From this, we also see that since $K_2(R)$ is the kernel of the map from $St(R)$ to $E(R)$, $K_2(R)$ collects the data of which elements of $E(R)$ commute with each other, i.e. the pairs of elements whose commutators $[e_{ij}, e_{kl}] = 1$.

We have now seen that $K_2(R)$ is a measure of how much $E(R)$ commutes. We can make this measurement specific by taking two matrices A, B which commute in $E(R)$, lifting them to $St(R)$, and taking their commutator. We call this operation $\star$. In other words, we define, for $A, B \in E(R)$ such that $AB = BA$,

$$A \star B = \bar{A}^{-1}\bar{B}^{-1}\bar{A}\bar{B},$$

for some choice of $\bar{A} \in \phi^{-1}(A)$, $\bar{B} \in \phi^{-1}(B)$. This definition doesn't depend on the choice of $\bar{A}, \bar{B}$: for some other choice, we can write those as $\bar{A}C_1, \bar{B}C_2$, for $C_1, C_2 \in K_2(R)$, so C_1, C_2 are in the center. Then clearly $[\bar{A}C_1, \bar{B}C_2] = \bar{A}^{-1}\bar{B}^{-1}\bar{A}\bar{B}$. Also, notice that we chose elements $\bar{A}, \bar{B}$ so that their commutator is sent to 1, so in fact $A \star B$ is an element of $K_2(R)$.

There are several easy properties of the $\star$ operation, as follows.

Proposition 3.10. *Suppose $A, B \in E(R)$ such that $AB = BA$. Then the following hold:*

- $(P^{-1}AP) \star (P^{-1}BP) = A \star B$, *(invariant under conjugation)*
- $(A \star B)(B \star A) = 1$, *(skew-symmetric)*
- $A_1A_2 \star B = (A_1 \star B)(A_2 \star B)$ *(bilinear).*

Proof. The invariance under conjugation follows from the definition of commutators and the fact that $[\bar{A}, \bar{B}]$ is in the center of $St(R)$, so $\bar{P}^{-1}[\bar{A}, \bar{B}]\bar{P} = [\bar{A}, \bar{B}]$. The fact that $\star$ is skew-symmetric follow directly from the definition of commutators. To see bilinearity, notice that by the following commutator identity and the fact that $[\overline{A_1}, \bar{B}], [\overline{A_2}, \bar{B}]$ are in the center,

$$[\overline{A_1}\overline{A_2}, \bar{B}] = [\overline{A_1}, [\overline{A_2}, \bar{B}]][\overline{A_2}, \bar{B}][\overline{A_1}, \bar{B}] = [\overline{A_2}, \bar{B}][\overline{A_1}, \bar{B}] = [\overline{A_1}, \bar{B}][\overline{A_2}, \bar{B}].$$

□

Using the $\star$ operation, we can now define a map from a subset of $R^* \times R^*$ to $K_2(R)$ which gives us insight into the structure of $K_2(R)$.

Definition 3.11. If r, s are units in R which commute, we define the symbol $\{r, s\} \in K_2(R)$ to be

$$\{r, s\} = \begin{bmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{bmatrix} \star \begin{bmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{bmatrix}.$$

Since that $\star$ operation is skew-symmetric and bilinear, so is the $\{-, -\}$ symbol. There is one more nice property of $\{-, -\}$ which is important in the characterization of Steinberg symbols, as we will see later.

Lemma 3.12. *If $r, = 1 - r \in R$ are both units, then $\{r, 1 - r\} = 1$, and $\{r, -r\} = 1$.*

The proof amounts to chasing definitions cleverly; see [7, III.5.10.2].

To characterize $K_2(R)$ in terms of the $\{-, -\}$ symbol, we have the following powerful result due to Milnor [3].

Theorem 3.13 (Milnor)**.** *If R is a field, division ring, or local ring, $K_2(R)$ is generated by the symbols $\{r, s\}$.*

From this theorem we see that we could study the structure of K_2 for certain rings by just looking at $R^* \times R^*$. In the situation of fields, the characterization is even simpler. We explore this more in the next section.

4. The Steinberg Symbol

The definition of the symbol in Definition 3.11 seems quite artificial, and it involved a choice of a lift in defining the $\star$ operation. There is, in fact, a cleaner and more universal way to describe it. We are now ready to recall Matsumoto's theorem, which we stated in the beginning of this document.

Theorem 4.1 (Matsumoto 1969, [2])**.** *If F is a field then $K_2(F)$ is the abelian group with the following presentation: its generators being the set of symbols $\{x, y\}$ where $x, y \in F^*$, subject to relations*

(1) $\{xx', y\} = \{x, y\}\{x', y\}, \{x, yy'\} = \{x, y\}\{x, y'\}$, *(bilinearity) and*

(2) $\{x, 1 - x\} = 1$ *for* $x \neq 0, 1$. *(Steinberg relation)*

For a self-contained proof, see [3, p.109]. We can restate this theorem in the following way.

Corollary 4.2. *Suppose A is an abelian group and F is a field, and let F^* denote the multiplicative group $F - \{0\}$. Given any bilinear map*

$$c : F^* \times F^* \to A$$

satisfying the identity

$$c(x, 1 - x) = 1,$$

there exist a unique homomorphism from $K_2(F)$ to A which sends the symbol $\{x, y\}$ to $c(x, y)$ for all x and y.

In other words, $K_2(F)$ is the universal object for maps c satisfying the identity $c(x, 1-x) = 1$. We give this class of maps a special name.

Definition 4.3 (Steinberg symbol)**.** For some abelian group A, we call any bilinear map

$$c : F^* \times F^* \to A$$

satisfying the identity

$$c(x, 1 - x) = 1$$

a *Steinberg symbol.*

Interestingly but perhaps unsurprisingly, there is a number of maps satisfying the Steinberg identity which appeared previously elsewhere in mathematics. One notable example is the Hilbert symbol.

Example 4.4 (Hilbert symbol). Take the field $\mathbb{Q}_p$ of p-adic numbers. The Hilbert symbol is defined by setting $c(x, y) \in \{1, -1\}$ depending on whether $ax_1^2 + bx_2^2 = 1$ has a solution. The bilinearity is classical; see [5, 3.14.5] for a proof, and it clearly satisfies the identity: setting $x_1 = x_2 = 1$ gives the desired result.

From this, we see that Steinberg symbols are useful in terms of encoding the information about K_2 in the case of fields, and we can probe at the K_2 group by instead studying the Steinberg symbols.

4.1. **Milnor K-Theory.** Now that we have several tools to study the K_2 group, we wonder if there is a way to directly describe all the K-groups including those of higher degree, without appealing to the long exact sequence. In 1970 Milnor introduced a way to do so for fields, which has applications in Galois cohomology and in the theory of the Grothendieck-Witt ring of quadratic forms.

Let $T(F^*)$ denote the tensor algebra of the group F^*, i.e.

$$T(F^*) = \mathbb{Z} \oplus F^* \oplus (F^*)^{\otimes 2} \oplus \cdots .$$

Let $l(x)$ denote, for $x \in F^*$, the x in the degree 1 piece of $T(F^*)$. Then, motivated by Matsumoto's theorem, Milnor defined the *Milnor K-group* $K_n^M(F)$ as the degree n-piece of the quotient of $T(F^*)$ by the ideal generated by $l(x) \otimes l(1 - x)$ where $x \neq 0, 1$. This is clearly rigged to agree with the lower-degree K-groups we have defined previously: $K_0(F) = \mathbb{Z}$, $K_1(F) = F^*$, and $K_2(F)$ satisfies Matsumoto's theorem.

Unfortunately, it isn't clear how to generalize this notion to rings, as we don't have an analogous result to Matsumoto's theorem. It also isn't immediately clear if these groups still extend the Mayer-Vietoris sequence. In the next section, we present a different, topological construction which agree on lower degrees.

5. The BGL^+ Construction of K-theory

This section is a brief sketch of Quillen's topological notion of K-groups as the homotopy groups of some space, constructed to agree with our previous ad hoc definitions. We assume familiarity with basic algebraic topology.

Recall that for a group G, the topological space BG is the geometric realization of the nerve of the category with one object whose morphisms are elements of the group, and that its fundamental group is G. Then, taking G to be $GL(R)$ of a ring R, we can define the following CW complex.

Definition 5.1. We denote a CW complex X as $BGL^+(R)$ if there is a distinguished map $BGL(R) \to BGL^+(R)$ such that

(1) $\pi_1(BGL^+(R)) \simeq K_1(R)$,
(2) the induced map $GL(R) \simeq BGL(R) \to \pi_1(BGL^+(R)) \simeq K_1(R)$ is surjective with kernel $E(R)$, and
(3) the induced map on homology $H_*(BGL(R), M) \to H_*(BGL^+(R), M)$ is an isomorphism for every $K_1(R)$-module M.

We claim that this CW complex is well-defined up to homotopy and that $\pi_n(BGL^+(R)) = K_n(R)$ for n small; $\pi_n(BGL^+(R)) = K_n(R)$ essentially follows from the fact that small K-group agree with the $\mathbb{Z}$-homology of R. See precise proofs and various ways of concretely constructing BGL^+ in [7, IV.1]. Then we can define higher K-groups as $K_n(R) = \pi_n(BGL^+(R))$, and we can get Mayer-Vietoris from the homotopy long exact sequence.

We can also describe a higher-degree analog for the $\{-,-\}$ symbol which appeared in our discussion of Steinberg symbols. To do this, note that we have a product map

$$K_p(R) \times K_q(R) \to K_{p+q}(R)$$

analogous to the cohomology cup product. In $K_2(R)$ this definition agrees with $\{-,-\}$, and we can generalize the notion to n-fold symbols by taking the product of n elements in $K_1(R)$. Then, when F is a field, from the universality of Milnor K-theory, there is a map $K_n^M(R) \to K_n(R)$. However, this map isn't an injection, since the symbol $\{-1,-1,-1,-1\}$ vanishes in $K_4(\mathbb{Q})$ but not $K_4^M(\mathbb{Q})$. For more details, see [7, IV.1].

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