Correlation functions for a one-dimensional Fermi system with long-range interaction (Tomonaga model)

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The correlation functions (Green functions) are found for a one-dimensional system of Fermi particles with long-range interaction (Tomonaga model). It is shown that such a system does not behave like Fermi liquid since the Green function does not possess a pole and a "Fermi step" is absent from the momentum distribution of the particles.

1. INTRODUCTION

In recent years there has been intensive study of conductors (or semiconductors) of a special type, in which the electrons execute an essentially one-dimensional motion. An example is organic materials containing TCNQ. In these materials the molecules form threads separated by such large distances that practically no jumps of electrons occur from one thread to another. The structure of the electronic spectrum in such quasi-unidimensional systems differs significantly from the spectrum of a three-dimensional system, especially in the case of a "quasiunidimensional metal": that is, when the number of electrons associated with a single molecule of the chain is less than two. Therefore it is especially important to investigate exactly solvable models of a one-dimensional metal.

There are two groups of such models: the Fermi-gas model with δ-function interaction, which was investigated in the papers of Gaudin, of Yang, and of Lieb and Wu[1], and the Fermi-gas model with long-range interaction, first considered by Tomonaga[2] and in a somewhat different form by Lieb and Mattis[3] ([2,3] are reproduced also in the book of Lieb and Mattis[4]). The exact solutions made it possible to calculate the energy of the ground state, the energy of the Bose branches of the spectrum, and also some characteristics of the one-electron spectrum, for example the difference between the chemical potentials corresponding to addition or removal of one particle: $\mu_+ - \mu_-$. But the exact wave functions are so complicated that to calculate the correlation functions or to solve the problem of the presence or absence of long-range order (in the ground state; at finite temperatures, of course, there is no long-range order) by means of them is practically impossible. The only exception so far is Lieb and Mattis's[3] calculation of the distribution n(p) of particles with respect to momentum.

At the same time, the correlation functions and the long-range order parameters[1)] have been calculated in[5] and in our previous paper[6] by perturbation theory, within the framework of the so-called logarithmic approximation. Although these correlation functions are not only in qualitative, but also with exponential accuracy in quantitative agreement with the results of the exact theory[6], nevertheless the possibility of extending them to the case of strong interaction seems very doubtful. The fact is that such an extension implies retention in the one-dimensional Fermi system of all the properties of a Landau Fermi liquid, and most importantly of the pole characteristics of the single-time Green function[2)]:

$$G(p)=\frac{a}{\varepsilon - v_F(|p|-p_0)+i\delta\cdot}. \qquad (1)$$

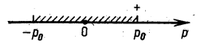


FIG. 1

Here v_F is the velocity on the "Fermi surface", which in the one-dimensional case is the two points $p = \pm p_0$ (see Fig. 1); a ($0 < a < 1$) is a constant that determines, as is well known (see[7], Sec. 7), the jump of the particle distribution function n(p) at the Fermi surface $p = p_0$.

Consideration of the problem by perturbation theory has already shown that formula (1) certainly does not hold in the one-dimensional case with the highest (nonlogarithmic) accuracy (for details see Sec. 7). It is therefore important to find G(p) in one of the exactly solvable models mentioned above, and to compare the result with formula (1).

It has been shown that such a calculation can be carried out in the Tomonaga[2] model. The Tomonaga model is a gas of Fermi particles with density n (Fermi momentum $p_0 = \pi n/2$), mass m (velocity $v_F = p_0/m$), and interaction potential $\lambda(|x|)$, whose Fourier components $\lambda(k)$ are different from zero only in an extremely narrow momentum interval $|k| \lesssim \Lambda \ll p_0$. The corresponding Hamiltonian has the form

$$\hat{H}=\sum_p \frac{p^2}{2m} a_p^+ a_p + \frac{1}{2}\sum_{pp'k} \lambda(k)\, a_p^+ a_{p'}^+ a_{p'-k} a_{p+k}. \qquad (2)$$

Tomonaga showed[2] that in the case of extremely long-range interaction, that is with neglect of all quantities of the type $\Lambda/p_0 \to 0$, the spectrum of the Hamiltonian (2) coincides with the spectrum of a system of bosons b_k, b_k^+:

$$\hat{H}=\sum_k v(k)\, b_k^+ b_k, \quad v^2(k)=v_F^2+\frac{2v_F}{\pi}\lambda(k). \qquad (3)$$

As is shown below, the Green function retains the form (1) in the region $|p - p_0| \gg \Lambda$ but has a quite different form near the Fermi surface, $|p - p_0| \lesssim \Lambda$.

2. BASIC EQUATIONS; WARD'S IDENTITY

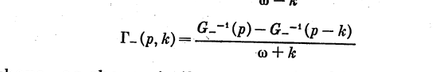
We shall calculate the Green function near the right and left "Fermi points": $G_+(p)$ and $G_-(p)$ (see Fig. 1). For the free gas[3)]

$$G_+^{(0)}(p)=\frac{1}{\varepsilon - p + p_0 + i\delta}, \quad G_-^{(0)}(p)=\frac{1}{\varepsilon + p + p_0 + i\delta}; \qquad (4)$$

it will be seen below that formulas (4) remain valid for $|\,|p| - p_0| \gg \Lambda$. In the region $|\,|p| - p_0 \gg p_0$ we can consider the particles near the right Fermi point and the left Fermi point essentially different fermions (+particles) and −particles), since in the limit $\Lambda/p_0 \to 0$ the interaction cannot convert one to another. This means, in particular, that, to the same accuracy, the values of $p - p_0$ for + particles and of $p + p_0$ for − particles can vary over the range $-\infty$ to $+\infty$.

We shall calculate also the effective potential $\lambda_{eff}(k)$ (denoted hereafter by the symbol D(k) and by a broken line on the diagrams, by analogy with the Green function of phonons) and the three-point vertex $\Gamma(p, k)$. For the same reason, $\Lambda/p_0 \to 0$, at all vertices the important $k \ll p_0$; therefore both Green functions of fermions entering Γ must lie on the same "Fermi point" + or − (symbols $\Gamma_+(p, k)$ and $\Gamma_-(p, k)$).

Finally, we shall consider a certain generalization of the Tomonaga model, which, while it has no direct physical meaning, is nevertheless convenient for comparison with the results of other authors. Namely, we shall suppose that the interactions of particles of the same "sign" (+ or −) and of different "signs" (+ −) are different:



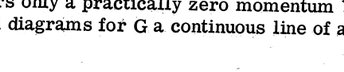
Therefore the functions λ(k) and D(k) will also be provided with indexes:

$$\lambda_{++}=\lambda_{--}=\lambda_1, \quad \lambda_{+-}=\lambda_2, \quad D_{++}=D_{--}, D_{+-},$$

and Dyson's equations for D will take the form

$$D_{++}=\lambda_1+\lambda_1\Pi_+ D_{++}+\lambda_2\Pi_- D_{-+},$$
$$D_{-+}=\lambda_2+\lambda_2\Pi_+ D_{++}+\lambda_1\Pi_- D_{-+}. \qquad (5)$$

For the polarization operators Π the usual graphical expressions hold:



(6)

Dyson's equation for G retains the standard form (see, for example,[7], Sec. 10):

$$G_\pm^{-1}=\varepsilon \mp p + p_0 - \Sigma_\pm,$$

(7)

In the usual problems of statistical physics and of quantum field theory, the vertex part Γ occurring in (5)–(7) is determined by an infinite series of diagrams, and therefore such problems can be solved only approximately. But in the Tomonaga model there are simple exact (in the limit $\Lambda/p_0 \to 0$) relations between Γ and G, and this makes it possible to solve the problem. These relations are the usual Ward identities (see, for example,[7], Sec. 19 or[8], Sec. 105), which in our case takes the form

$$\Gamma_+(p,k)=\frac{G_+^{-1}(p)-G_+^{-1}(p-k)}{\omega - k},$$
$$\Gamma_-(p,k)=\frac{G_-^{-1}(p)-G_-^{-1}(p-k)}{\omega + k} \qquad (8)$$

where, as above, in the arguments of the functions we denote by k the pair (k, ω).

The relations (8) can be derived directly by considering the diagrams of different orders and by using the identity, which follows from (4),

$$G_\pm^{(0)}(p)G_\pm^{(0)}(p+k)=\frac{1}{\omega \mp k}\left(G_\pm^{(0)}(p)-G_\pm^{(0)}(p+k)\right).$$

Here it is necessary to remember the following two facts. First, since the interaction (the broken line D) transfers only a practically zero momentum $\lesssim \Lambda$, there is in all diagrams for G a continuous line of a particle

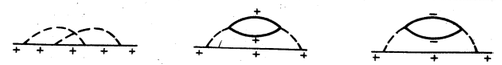


FIG. 2

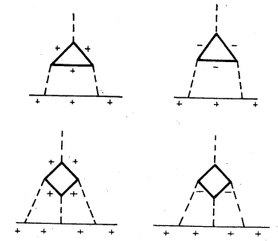


FIG. 3

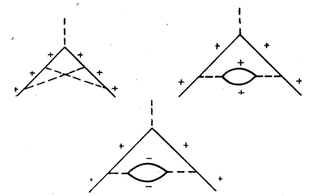


FIG. 4

of one sign, transferring external momentum p (see Fig. 2, where typical diagrams for G_+ are shown). Second, by using the same long-range property and the specific form of the free Green function (4), it can be shown that all diagrams containing closed loops consisting of more than two fermion lines are zero. (More accurately, what vanishes is their appropriately symmetrized sum; see below, formula (11)). Therefore the diagrams of Fig. 3 for Γ_+ vanish, and there remain only diagrams of the type of Fig. 4.

This theorem rests on the law of conservation of the number of particles, which follows from (4). In fact, the equation for the free operators of "right" (+) particles has the form

$$i\frac{\partial \psi_+}{\partial t}+i\frac{\partial \psi_+}{\partial x}=0,$$

whence follows the law of conservation of number of particles (charge)

$$\frac{\partial \rho_+}{\partial t}+\frac{\partial j_+}{\partial x}=0, \quad \rho_+ = j_+ = \psi_+^+\psi_+. \qquad (9)$$

For left particles

$$i\frac{\partial \psi_-}{\partial t}-i\frac{\partial \psi_-}{\partial x}=0;$$
$$\frac{\partial \rho_-}{\partial t}+\frac{\partial j_-}{\partial x}=0, \quad \text{но} \quad \rho_- = -j_- = \psi_-^+\psi_-. \qquad (10)$$

Since any closed loop is the Fourier component of the mean, over the ground state, of a certain number of density operators:

$$\langle T\rho_+(1)\rho_+(2)\dots\rho_+(n)\rangle,$$

and analogously for left particles, the conservation laws (9) and (10) mean

$$(\omega_1 - k_1)(\omega_2 - k_2)\dots(\omega_n - k_n)\langle\rho_+(k_1)\rho_+(k_2)\dots\rho_+(k_n)\rangle = 0,$$
$$(\omega_1 + k_1)(\omega_2 + k_2)\dots(\omega_n + k_n)\langle\rho_-(k_1)\ \rho_-(k_2)\dots\rho_-(k_n)\rangle = 0. \qquad (11)$$

Our assertion about loops (it is the analog of the transversality of current in quantum electrodynamics) follows

from (11), if all the momentum intervals of the fermions coincide. It is easy to show that the situation is thus for loops consisting of three or more lines. The equality (8) can also be derived by differentiating the relation

$$G_\pm\Gamma_\pm G_\pm = \langle T\psi_\pm^+\rho_\pm\psi_\pm\rangle$$

with respect to t and using the identities (9) and (10).

We shall finally discuss loops consisting of two fermion lines. They are proportional to the integral (for + particles)

$$\int d^2p \frac{1}{\varepsilon - p + p_0 + i\delta}\frac{1}{\varepsilon - \omega - p + k + p_0 + i\delta},$$

which formally diverges. Yet according to its physical meaning as a quantity proportional to the density fluctuation, this integral is finite. This finiteness is guaranteed by the quadratic dependence of the energy of the particles on momentum at large distances from the Fermi surface $p = p_0$. Technically this means that the double integration over p and ϵ must be performed in a definite order; first over ϵ and then over p. The result of such an integration is finite and gives a quantity proportional (see below) to $k/(\omega - k)$.

We note that the physical method of cutting off that we have applied formally destroys the "relativistic invariance" of the theory. All quantities cease to be functions only of $\epsilon - p\sigma_z$ and $\omega^2 - k^2$ (σ_z is the Pauli matrix), a formal property of the relations (4), (8), (9), and (10).

We can now write a closed system of equations for G and D, by expressing the functions Γ in terms of G, by means of (8), in the formulas for Π and Σ. We have

$$\begin{aligned}\Pi_+(k) &= -\frac{i}{2\pi^2}\frac{1}{\omega - k}\int d^2p[G_+(p-k) - G_+(p)],\\ \Pi_-(k) &= -\frac{i}{2\pi^2}\frac{1}{\omega + k}\int d^2p[G_-(p-k) - G_-(p)].\end{aligned} \tag{12}$$

The equation for G_+ takes the form

$$(\varepsilon - p + p_0)G_+(p) = 1 + \frac{i}{4\pi^2}\int d^2k\frac{D_{++}(k)}{\omega - k}G_+(p-k); \tag{13}$$

as usual, we absorb all terms on the right side of (13) that have the form const · $G_+(p)$ into the renormalization of the chemical potential. To write and solve the equations for G_- is not necessary, because G_- is connected by obvious symmetry relations with G_+.

We conclude this section with a calculation of Π_+ and Π_-. On introducing a cutoff with respect to momentum $||p| - p_0| \lesssim A \ll p_0$, we have

$$\Pi_+ = -\frac{i}{2\pi^2}\frac{1}{\omega - k}\int_{p_0-A}^{p_0+A} dp\int_{-\infty}^{\infty} d\varepsilon[G_+(p-k, \varepsilon - \omega) - G_+(p, \varepsilon)].$$

The contribution from the region $||p| - p_0| \geq A$ cannot be found by formula (12), since the relations (8) are valid only near the Fermi surface. One can convince oneself that this contribution is zero for $A \gg \Lambda$ by substituting in the expression (6) the free Green function and vertices. The integral with respect to ϵ gives

$$\Pi_+ = \frac{1}{\pi(\omega - k)}\int_{p_0-A}^{p_0+A} dp[n_+(p-k) - n_+(p)],$$

where $n_+(p)$ is the distribution of the particles with respect to momentum. The integral in this formula obviously reduces to an integral over regions far from the Fermi surface:

$$\int_{p_0-A}^{p_0+A} dp\ldots = \int_{p_0-A-k}^{p_0-A} dp\, n_+(p) - \int_{p_0+A-k}^{p_0+A} dp\, n_+(p).$$

In these regions, with accuracy $\Lambda/A \to 0$, we may suppose that $n_+(p) = n_F(p)$, where n_F is the Fermi step at T = 0. Therefore

$$\Pi_+ = k/\pi(\omega - k). \tag{14}$$

Analogous calculations for Π_- give

$$\Pi_- = -k/\pi(\omega + k). \tag{15}$$

Now solving equation (5), we find

$$D_{++}(k) = (\omega - k)\frac{\lambda_1(\omega + k) + (\lambda_1^2 - \lambda_2^2)k/\pi}{\omega^2 - u^2k^2 + i0}, \tag{16}$$

$$u = \left(1 + \frac{2\lambda_1}{\pi} + \frac{\lambda_1^2 - \lambda_2^2}{\pi^2}\right)^{1/2}. \tag{17}$$

In the case $\lambda_1 = \lambda_2 = \lambda(k)$, formulas (16) and (17) give Tomonaga's result (3).

Thus in order to find G_+, it remains for us to solve the linear integral equation (13) with known D_{++}. Although this is an equation with a difference kernel and can be reduced to a differential equation by going over to ordinary coordinate space, we shall first give its solution in the momentum representation for the case $\lambda_2 = 0$ and the case of small λ_1 and λ_2.

3. THE CASES $\lambda_2 = 0$ and $\lambda_1, \lambda_2 \ll 1$

In the case $\lambda_2 = 0$, the particles on one side of the Fermi surface (+) do not interact at all with the particles on the other side of the surface (−), $D_{-+} = 0$, while

$$D_{++}(k) = \frac{\pi(w-1)(\omega - k)}{\omega - wk + i\delta}, \qquad w(k) = 1 + \lambda_1(k)/\pi. \tag{18}$$

The equation for G_+ has the form

$$(\varepsilon - p)G(p) = 1 + \frac{i}{4\pi}\int d^2k\, G(p-k)\frac{w(k) - 1}{\omega - wk + i\delta}. \tag{19}$$

Here and hereafter we omit the index + on G_+ and measure p from p_0 ($-\infty < p < \infty$). It is easily shown by direct substitution that equation (19) is satisfied by the function

$$G(p) = [(\varepsilon - p + i\delta)(\varepsilon - wp + i\delta)]^{-1/2}, \tag{20}$$

where the cut in the complex ϵ plane is chosen in the form of a straight-line segment connecting the points $p - i\delta$ and $wp - i\delta$ (sign δ = sign p). It is not difficult to show also that formula (20) for G(p) conserves the free distribution of particles with respect to momentum,

$$n(p) = n_F(p). \tag{21}$$

We proceed to the case of small λ_1 and λ_2: that is, u of (17) is close to unity. We express the integral on the right side of the equation for G in the form of a sum of two terms:

$$\begin{aligned}(\varepsilon - p)G(p) = 1 &+ \frac{i}{4\pi}\int d^2k\, G(p-k)\frac{u(k) - 1}{\omega - uk + i\delta}\\ &+ \frac{i}{4\pi^2}\int d^2k\, G(p-k)\left\{\frac{\lambda_1(\omega + k) + (\lambda_1^2 - \lambda_2^2)k/\pi}{\omega^2 - u^2k^2 + i0} - \frac{\pi(u-1)}{\omega - uk + i\delta}\right\}.\end{aligned} \tag{22}$$

We shall be concerned with the momentum region ϵ, $p \ll \Lambda$. (In the range where ϵ, $p \sim \Lambda$, as will be seen, the old answer (20) is retained with u instead of w.) By iteration of equation (22), it can be shown that the integration in the first integral extends over a narrow region $\omega \sim k \sim \epsilon \sim p$. On the contrary, in the second term

the region of integration is broad: $\epsilon, p \ll \omega, k \ll \Lambda$. Furthermore, the second region gives the logarithmic contribution to G:

$$G_{II}^{(1)} \sim \frac{1}{\varepsilon - p}\lambda_2^2 \ln\frac{\Lambda}{\varepsilon, p}.$$

The situation remains the same also in iteration of an arbitrary order in the second region:

$$G_{II}^{(n)} \sim \frac{1}{\varepsilon - p}\left[\lambda_2^2 \ln\frac{\Lambda}{\varepsilon, p}\right]^n.$$

We note further that the function (20), which satisfies equation (22) if the second integral is discarded, for small λ_1, λ_2 simply coincides in region II with $(\epsilon - p)^{-1}$. We therefore reach the conclusion that the Green function must be written in the form of the product of (20) by a slowly varying function of p and ϵ:

$$G(p) = [(\varepsilon - p)(\varepsilon - up)]^{-1/2}f(\xi), \quad \xi = \ln(\Lambda/\varepsilon, p). \tag{23}$$

In the substitution of (23) in the first integral, f may be considered constant. According to (19) the first integral gives

$$(\varepsilon - p)G(p) - f(\xi).$$

The second integral can be written in the form[4)]

$$-\frac{\lambda_2^2}{4\pi^2}\int_0^\xi f(\eta)d\eta.$$

On substituting both these epxressions in (22), we obtain the equation for f

$$f(\xi) = 1 - \frac{\lambda_2^2}{4\pi^2}\int_0^\xi f(\eta)d\eta,$$

whose solution is

$$f(\xi) = \exp(-\lambda_2^2\xi/4\pi^2).$$

Thus for small λ_1 and λ_2

$$G(p) = [(\varepsilon - p)(\varepsilon - up)]^{-1/2}\exp\left\{-\frac{\lambda_2^2}{4\pi^2}\ln\frac{\Lambda}{|p|}\right\},$$

$$u(p) \approx 1 + \lambda_1/\pi - \lambda_2^2/2\pi^2. \tag{24}$$

On calculating the particle distribution function n(p) by means of (24), it is not difficult to show that for $\lambda_2 \neq 0$ the jump at p = 0 is absent, and in its place there remains only a singularity of the form

$$n(p) = \frac{1}{2} - \frac{1}{2}\left(\frac{|p|}{\Lambda}\right)^{\lambda_2^2/4\pi^2}\operatorname{sign} p. \tag{25}$$

A formula of this type was first obtained by Lieb and Mattis[3].

4. THE CASE $\lambda_1 = \lambda_2 = \lambda$

Interactions of arbitrary intensity we shall consider only for the physically realistic case[5)] $\lambda_1 = \lambda_2 = \lambda(k)$. In this case

$$D_{++}(k) = \frac{\lambda(k)(\omega^2 - k^2)}{\omega^2 - v^2k^2 + i0}, \quad v = \left(1 + \frac{2\lambda}{\pi}\right)^{1/2}$$

and for G we have the equation

$$(\varepsilon - p)G(p) = 1 + \frac{i}{4\pi^2}\int d^2k\, G(p-k)\frac{\lambda(k)(\omega + k)}{\omega^2 - v^2k^2 + i0}. \tag{26}$$

In (26) we perform the inverse transformation to ordinary coordinate space x, t. Then (for $t \neq 0$)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)G(x,t) = K(x,t)G(x,t), \tag{27}$$

$$K(x,t) = \frac{1}{4\pi^2}\int d^2k\, e^{-i\omega t + ikx}\lambda(k)\frac{\omega + k}{\omega^2 - v^2k^2 + i0}$$

$$= \frac{\lambda}{4\pi v}\left(\frac{v+1}{x - vt - i/\Lambda} - \frac{v-1}{x + vt + i/\Lambda}\right). \tag{28}$$

The result of the integration over the momentum k depends, of course, on the specific law of approach of $\lambda(k)$ to zero for $k \gg \Lambda$. Formula (28) was obtained for the simplest form of such cutoff: we considered λ and v constant, but introduced into the integral a cutoff multiplier $e^{-|k|/\Lambda}$. Therefore the results obtained are quantitatively applicable only in the regions $x, t \gg 1/\Lambda$ and $x, t \ll 1/\Lambda$.

Equation (27) is to be solved under two boundary conditions. The first condition is that, by definition,

$$G(x, +0) - G(x, -0) = -i\delta(x). \tag{29}$$

The second condition is the well-known analyticity properties of G as a function of the frequency ϵ (see, for example,[7], Sec. 7). In the coordinate representation, this condition reads: for Re $t > 0$, the function G(t) can be continued analytically from the real axis into the right lower quadrant of the complex variable t (Re $t > 0$, Im $t < 0$); for Re $t < 0$, on the other hand, what is analytic is the continuation of G(t) from the left semiaxis into the left upper quadrant (Re $t < 0$, Im $t > 0$).

The Green function $G^{(0)}(x, t)$ of free particles, which of course satisfies both boundary conditions, has the form

$$G^{(0)}(x,t) = \frac{1}{2\pi}\frac{1}{x - t + i\delta(t)}, \tag{30}$$

where $\delta(t)$ is an infinitely small additional term (actually of order $1/p_0 \ll 1/\Lambda$), whose sign coincides with the sign of t.

Equation (27) by itself is solved trivially by introducing the new independent variables

$$r = x - t, \quad s = x + t.$$

For example, its solution for $t > 0$ satisfying condition (29) can be written in the form

$$G(t > 0) = G^{(0)}(r)\exp\left\{\tfrac{1}{2}\int^s K(r, s')ds'\right\}f_+(r), \tag{31}$$

where $f_+(r)$ is a function that must be chosen on the basis of the analyticity conditions. On calculating the integral in (31), we find

$$G(t > 0) = \frac{f_+(x-t)}{2\pi}\frac{1}{x - t + i\delta(t)}$$

$$\times\left(\frac{x - t + i/\Lambda}{x - vt + i/\Lambda}\right)^{1/2+\alpha}\left(\frac{x - t - i/\Lambda}{x + vt - i/\Lambda}\right)^{\alpha}, \quad \alpha = \frac{(v-1)^2}{8v}. \tag{32}$$

The analogous calculations for $t < 0$ give a formula that differs from (32) only by the substitution $\Lambda \to -\Lambda$ and $f_+ \to f_-$. Condition (29), however, requires that

$$f_+(x) = f_-(x) = f(x), \quad f(0) = 1. \tag{33}$$

In formula (32) and in the corresponding formula for $t < 0$, only one factor has incorrect analytic behavior, namely

$$(x - t \mp i/\Lambda)^\alpha,$$

which has a singularity in the right lower quadrant for $t > 0$ and in the left upper for $t < 0$. In order to remove the singularity without thereby violating the condition (33), it is necessary to choose f in the form

$$f(x) = (\Lambda^2x^2 + 1)^{-\alpha}.$$

By use of standard theorems on analytic continuation,

it is easy to show that this choice is unique.

The final formula for G(x, t) has the form (for both signs of t)

$$G(x,t)=\frac{1}{2\pi}\frac{1}{x-t+i\delta(t)}\left[\frac{x-t+i/\Lambda(t)}{x-vt+i/\Lambda(t)}\right]^{1/2}$$
$$\times[\Lambda^2(x-vt+i/\Lambda(t))(x+vt-i/\Lambda(t))]^{-\alpha}, \qquad (34)$$

where $\Lambda(t)$ is $\pm\Lambda$, depending on the sign of t.

Formula (34) enables us to calculate easily the distribution of particles with respect to momentum:

$$n(p)=-\frac{i}{2\pi}\int_{-\infty}^{\infty}dx\,e^{-ipx}\frac{1}{x-i0}(1+\Lambda^2x^2)^{-\alpha}$$
$$=\frac{1}{2}-\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\frac{\sin px}{x}(1+\Lambda^2x^2)^{-\alpha}. \qquad (35)$$

For weak interaction, when $\lambda \to 0$, $v \to 1$, and $\alpha \to 0$, formula (35) gives a result that coincides with (25) for $\lambda_1 = \lambda_2 = \lambda$. Formula (25) for the singularity of n(p) near the "Fermi point" p = 0 remains valid as long as $\alpha < 1/2$; that is, as long as the interaction is not too strong:

$$n(p) = 1/2 - \text{const}\cdot|p|^{2\alpha}\,\text{sign}\,p. \qquad (36)$$

With further increase of the strength of the interaction, when α becomes greater than 1/2, the leading term in the expansion of n(p) near the Fermi point becomes a linear term,

$$n(p) = 1/2 - \text{const}\cdot p.$$

We have not succeeded in calculating the Fourier components G(p, ϵ) in explicit form. It is possible, however, to find easily the behavior of G(p, ϵ) for $p \sim \epsilon \ll \Lambda$. For $\alpha < 1/2$, the value of

$$G(\varepsilon,p)=\int e^{i\varepsilon t-ipx}G(x,t)\,dx\,dt \qquad (37)$$

is determined by the region x, $t \gg 1/\Lambda$. An estimate by dimensional methods gives directly

$$G(\varepsilon\sim p)\sim\int\frac{dt\,e^{i\varepsilon t}}{t^{2\alpha}}\sim\frac{1}{\varepsilon^{1-2\alpha}}. \qquad (38)$$

For $\alpha > 1/2$, the principal contribution to the integral (37) will come from the region $x \sim t \sim 1/\Lambda$, and G will be finite on the Fermi surface $\epsilon = p = 0$:

$$G = A + B\varepsilon^{2\alpha-1}. \qquad (39)$$

For $3/2 > \alpha > 1$, on the Fermi surface

$$G = A + B\varepsilon + C\varepsilon^{2\alpha-1}$$

and so on.

Thus in the Tomonaga model, the basic assumption of the Landau theory of the Fermi liquid is not satisfied. Even for arbitrarily weak interaction, the singularity near the Fermi surface is weaker than a pole (formula (38)); and for sufficiently strong interaction, the Green function in general remains finite at the Fermi surface. The latter shows up only as a singular point of higher derivatives (Formula (39) and thereafter).

5. FINITE TEMPERATURES

The Green functions of fermions in the Tomonaga model can be found also for finite temperatures. This is so because Ward's identities (8) and all the arguments involved in their derivation remain valid also in the diagram technique for finite temperatures, which operates with the imaginary frequencies $i\epsilon_n$, $\epsilon_n = (2n+1)\pi T$, where n is an integer (see, for example, [7], Chap. III). Now Ward's identities connect the temperature-dependent Green functions of the fermions $\mathfrak{G}_\pm(p)$, p = (p, $i\epsilon_n$), with the corresponding vertex parts $\mathcal{T}(p, k)$, k = (k, $i\omega_n$), $\omega_n = 2\pi nT$. For the temperature functions $\mathfrak{G}$, T and effective potential $\mathfrak{D}$, all the formulas (5)–(8) and (13)–(17) remain valid, with the substitutions $G \to \mathfrak{G}$, $\Gamma \to \mathcal{T}$, $D \to \mathfrak{D}$, and also $\epsilon, \omega \to i\epsilon_n, i\omega_n$,

$$\frac{1}{2\pi}\int d\varepsilon\ldots \to iT\sum_n\ldots$$

For simplicity we restrict ourselves to the case $\lambda_2 = 0$. Then

$$\mathfrak{D}(k)=\frac{\pi(w-1)(i\omega_n-k)}{i\omega_n-wk},\qquad w=1+\frac{\lambda_1}{\pi}, \qquad (40)$$

$$(i\varepsilon_n-p)\mathfrak{G}(p)=1-\frac{T}{2}\sum_n\int dk\frac{w-1}{i\omega_n-wk}\mathfrak{G}(p-k). \qquad (41)$$

We go over to the coordinate representation for the Green functions: space coordinate x and imaginary time $i\tau$. As is well known (see [7], Sec. 11), the Green functions of fermions have the periodicity property

$$\mathfrak{G}(\tau) = -\mathfrak{G}(\tau + 1/T),$$

therefore it is sufficient to find $\mathfrak{G}$ in the interval $0 < \tau < \beta = 1/T$. Equation (41) transforms to

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial\tau}\right)\mathfrak{G}(x,\tau)=\mathfrak{K}(x,\tau)\mathfrak{G}(x,\tau), \qquad (42)$$

$$\mathfrak{K}(x,\tau)=-\frac{w-1}{2}i\int_{-\Lambda}^{\Lambda}dk\frac{e^{ikx-wk\tau}}{1-e^{-wk\beta}}, \qquad (43)$$

where we have introduced explicitly, in the formula for $\mathfrak{K}$, the cutoff with respect to Λ. As for T = 0, an additional condition is the condition for $\tau = 0$, which by virtue of the periodicity condition is equivalent to

$$\mathfrak{G}(x,+0)+\mathfrak{G}(x,\beta-0)=-\delta(x). \qquad (44)$$

On introducing, as before, the new variables

$$\tau = x + i\tau, \quad s = x - i\tau,$$

we can put the solution into the form

$$\mathfrak{G}=\mathfrak{G}^{(0)}(\mathfrak{r})\exp\left\{\frac{1}{2}\int^{\mathfrak{s}}\mathfrak{K}(\mathfrak{r},\mathfrak{s}')\,d\mathfrak{s}'+f(\mathfrak{r})\right\},$$

where $\mathfrak{G}^{(0)}$ is the Green function of free particles:

$$\mathfrak{G}^{(0)}(\mathfrak{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\frac{e^{ip\mathfrak{r}}}{1+e^{-p\beta}},\quad \tau>0. \qquad (45)$$

The function $f(\mathfrak{r})$ must be so chosen that the whole function in the exponent of the exponential will be periodic in τ with period β and, furthermore, will vanish at $x = \tau = 0$. It is not difficult to show that the stated conditions are satisfied by the following formula for $\mathfrak{G}(x, t)$:

$$\mathfrak{G}(x,\tau)=\mathfrak{G}^{(0)}(x,\tau)\exp\left\{\frac{1}{2}\int_{-\Lambda}^{\Lambda}\frac{dk}{k}\left(\frac{e^{ikx-wk\tau}-1}{1-e^{-wk\beta}}-\frac{e^{ikx-k\tau}-1}{1-e^{-k\beta}}\right)\right\}. \qquad (46)$$

We shall not make a detailed investigation of this formula.

6. ONE-DIMENSIONAL LIQUID

From Tomonaga's formula for the speed of sound, which was given in the Introduction,

$$v^2=\frac{\pi^2n^2}{4m^2}+\frac{\lambda n}{m} \qquad (47)$$

(we write it now in ordinary units), it is seen that for sufficiently strong attraction the speed becomes a pure imaginary (for given n). This fact is considered by many to indicate that at sufficiently strong attraction, the system cannot exist at all. It is easy to realize, however, that this is not so. Formula (47) shows only that in the attraction case the system is a liquid and not a gas; that is, it does not require external walls for containment in a finite region, and it itself determines its own equilibrium density at zero pressure.

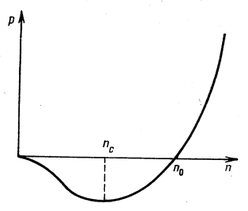


FIG. 5

We shall calculate, for example, the equation of state of such a liquid by using the usual formula

$$v^2=\frac{1}{m}\frac{dP}{dn}.$$

Hence

$$P = \pi^2n^3/12\,m + \lambda n^2/2.$$

Figure 5 shows the curve P(n) in the attraction case. The equilibrium density for which P = 0 is $n_0 = -6\lambda m/\pi^2$. The density range below n_0, where the pressure is negative, is metastable down to the value $n_c = -4\lambda m/\pi^2$; to the left of this, the system is absolutely unstable (dP/dn < 0).

It is instructive to quote the values of the speed of sound v, the Fermi velocity v_F, and the parameter $\alpha = (v - v_F)^2/8vv_F$, which determines the intensity of the interaction (see Section 4), at the point where P = 0:

$$v_0 = -\sqrt{3}\lambda/\pi,\quad v_{F0} = -3\lambda/\pi,\quad \alpha_0 = (\sqrt{3}-1)^2/8\sqrt{3} \approx 0{,}05.$$

It is seen that numerically (see formulas (36) and (38)) the deviations of the particle distribution and of the Green function from their values according to the Fermi-liquid theory, being determined by the value of $2\alpha_0 \approx 0.1$, are quite small.

7. INFRARED CATASTROPHE

It has already been pointed out [5] that in a one-dimensional Fermi system an infrared catastrophe occurs. Any particle lying near one of the Fermi points can emit, consistently with both conservation laws—energy and momentum—, an arbitrary number of real particle-hole pairs lying in the neighborhood of the same Fermi point. Mathematically, the infrared catastrophe shows up in the fact that in the series for the characteristic energy $\Sigma(p)$, there occur pole terms of high order, of the type (see [5])

$$\lambda^n p^n/(\varepsilon - p)^{n-1}. \qquad (48)$$

It is not difficult to see that this series for Σ practically coincides with the series for G(p), which we summed in Sec. 3, for the case $\lambda_2 = 0$. The single inconsequential difference consists in the fact that in [5], where short-range forces were being considered, the diagram of Fig. 6 gave only a correction to the chemical potential, whereas in the case of long-range interaction that we have considered, this diagram gives also a correction to the velocity on the Fermi surface. Therefore in order to obtain an expression for Σ in the case of short-range action, it is necessary to subtract from formula (20) a correction derived from the diagram of Fig. 6.

FIG. 6

For this purpose one can proceed as follows. The expression for Σ corresponding to (20) has the form

$$\Sigma(p) = \varepsilon - p - [(\varepsilon - p)(\varepsilon - p - \lambda p/\pi)]^{1/2}. \qquad (49)$$

On expanding it as a series in λ, we get

$$\Sigma(p)=\frac{\lambda p}{2\pi}+\frac{\lambda^2p^2}{8\pi^2(\varepsilon-p)}+\ldots$$

All the terms of this series except the first have the same structure as the series (48). One can show that the specific properties of long-range interaction manifest themselves only in the presence of the first term; the contribution of the essentially long-range forces to the remaining terms of the series is zero.

Thus the effect of the infrared catastrophe [5] on the characteristic energy Σ is given by the expression (49) after subtraction of $\lambda p/2\pi$. The corresponding formula for the Green function has the form

$$G(p)=\frac{1}{\varepsilon-p}f\left(\frac{\lambda p}{\varepsilon-p}\right),\quad \frac{1}{f(x)}=(1-x)^{1/2}+\frac{x}{2}.$$

8. QUASIUNIDIMENSIONAL METAL WITH COULOMB INTERACTION

The results obtained above can be generalized to the case of a quasiunidimensional metal in which the distances between conducting threads are sufficiently large so that the electrons cannot jump from thread to thread. It is not permissible, however, to neglect the interaction between electrons on different threads. It is important that all the relations that follow from Ward's identity are still applicable in this case. All that changes is the form of the function D(k) that describes the interaction between electrons. If the quasiunidimensional metal is a system of parallel threads elongated along the z axis, then the equation for the function $D_{l\,m}$ has the form

$$D_{lm}(k)=\sum_n V_{ln}[\delta_{nm}+\Pi(k)D_{nm}(k)]. \qquad (50)$$

Here l, m, n enumerate the conducting threads, and the potential V describes the priming interaction. For Coulomb interaction, the solution of equation (50) has the form

$$D(k)\equiv D_{ll}(k)=S\int\frac{d^2q}{(2\pi)^2}\left[S\left(\sum_\varkappa\frac{4\pi e^2}{(q+\varkappa)^2+k^2}\right)^{-1}-\Pi(k)\right]^{-1}, \qquad (51)$$

where S is the area of the elementary cell, the integration over d^2q extends over the area of the reciprocal cell, and the summation over κ extends over all vectors of the reciprocal lattice. The polarization operator $\Pi(k)$ is unchanged in form and equal to the sum of the expressions (14) and (15).

We shall suppose that the Coulomb interaction is weak ($e^2 \ll \hbar v_F = 1$). Then the Debye radius $e^{-1}S^{1/2}$ is large in comparison with the distance $S^{1/2}$ between threads. The important q's in the integral (51) are of the order of the reciprocal of the Debye radius; therefore in the sum over κ, it is necessary to keep only the term with $\kappa = 0$.

We shall be interested in regions distant from the

Fermi surface by less than the inverse Debye radius. Therefore the important k's satisfy the condition $k \ll q$, and the expression (51) takes the form

$$D(k)=\frac{\pi S}{2}\int\frac{d^2q}{(2\pi)^2}(v_q^2-1)\frac{\omega^2-k^2}{\omega^2-v_q^2k^2}, \tag{52}$$

$$v_q^2=1+8e^2/q^2S. \tag{53}$$

On substituting the expression (52) into formulas (13) and (31), we get for the Green function the expression

$$G(x,t)=\frac{1}{x-t+i\delta(t)}$$
$$\times\exp\left\{\int\frac{S\,d^2q}{(2\pi)^2}\left[\frac{1}{2}\ln\frac{x-v_qt}{x-t}-\frac{(v_q-1)^2}{8v_q}\ln(\Lambda^2(x^2-v_q^2t^2))\right]\right\}, \tag{54}$$

where the cutoff parameter Λ is of the order of the reciprocal of the Debye radius $eS^{-1/2}$. For clarification of the rules for going around singularities, it is necessary to replace the time t by $t - i/\Lambda(t)$.

In the case of small charge e^2, it is possible, by using the method of Section 3, to obtain an expression for the Green function directly in the momentum representation. It is easily shown that in the integral over the momentum range close to p (the analog of the first integral in (22)), the large momenta $q \sim S^{-1/2}$ will be important; and that in the integral over the large logarithmic range, momenta $q \sim S^{-1/2}e$ will be important for $p \ll S^{-1/2}$, and momenta $q \sim S^{-1/2}$ for $p \gtrsim S^{-1/2}$. Therefore the formula for G(p) retains the form (23):

$$G(p)=[(\varepsilon-p)(\varepsilon-up)]^{-1/2}f(\ln|p|);$$
$$u=\begin{cases}1+2e^2\pi^{-1}\ln(\Lambda/|p|), & p\gtrsim S^{-1/2}\\ 1+2e^2\pi^{-1}\ln(\Lambda S^{1/2}), & p\ll S^{-1/2}\end{cases}$$
$$f=\begin{cases}\exp\left\{-\dfrac{e^2}{2\pi}\ln\dfrac{e}{|p|S^{1/2}}\right\}, & p\ll S^{-1/2}e\\ \exp\left\{-\dfrac{e^4}{3\pi^2}\ln^3\dfrac{\Lambda}{|p|}\right\}, & p\gtrsim S^{-1/2}\end{cases} \tag{55}$$

Formula (55) corresponds to a momentum distribution of particles near the Fermi surface of the form

$$n(p)=\frac{1}{2}-\frac{1}{2}\left(\frac{|p|S^{1/2}}{e}\right)^{e^2/2\pi}\operatorname{sign}p. \tag{56}$$

We shall consider the case in which the conducting threads form a three-dimensional lattice and are directed along three mutually perpendicular directions. The interaction between electrons located on a thread directed along the x axis has the form

$$D(k_x)=\int\frac{S\,dk_y\,dk_z}{(2\pi)^2}\left[S\frac{k_x^2+k_y^2+k_z^2}{4\pi e^2}-\right.$$
$$\left.-\frac{2}{\pi}\left(\frac{k_x^2}{\omega^2-k_x^2}+\frac{k_y^2}{\omega^2-k_y^2}+\frac{k_z^2}{\omega^2-k_z^2}\right)\right]^{-1}. \tag{57}$$

As above, we use the fact that $\omega, k_x \ll k_y, k_z$. Therefore formula (57) takes the form of the expression (52). All that changes is the expression (53) for v_q:

$$v_q^2=1+(2+Sq^2/8e^2)^{-1}. \tag{58}$$

With allowance for this change, formulas (54) and (55) retain their form. The momentum distribution of particles near the Fermi surface has in this case the form

$$n(p)=\frac{1}{2}\left[1-\left(\frac{|p|S^{1/2}}{e}\right)^{\gamma}\operatorname{sign}p\right],\quad \gamma=\frac{e^2(5-\sqrt{24})}{2\pi}. \tag{59}$$

[1)] In fact, the Green functions and the Feynman diagrams have been calculated by this technique (see, for example, [7], Chap. II).

[2)] Everywhere hereafter, we shall designate by p both the one-dimensional momentum and (in arguments of functions) the "momentum-frequency" pair (p, ϵ). Furthermore, $\hbar = 1$ everywhere.

[3)] For brevity we shall use a system of units in which the velocity of free particles $v_F = 1$.

[4)] We have gone over here, as always, to the logarithmic variable of integration $\eta = \ln(\Lambda/p)$ (see, for example, [5,6]).

[5)] Recall that we introduced different λ_1 and λ_2 purely formally. For an arbitrary interaction, described by a potential $\lambda(|x|)$, always $\lambda_1 = \lambda_2$.

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Frequency dependence of the conductivity of one-dimensional systems

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The dependence of the conductivity of one-dimensional systems on the frequency of an external electric field, ω, is investigated. It is shown that the static conductivity ($\omega = 0$) is zero. For sufficiently high electron Fermi energies the conductivity tends to zero not slower than $\sim|\omega|$.

1. The conductivity of one-dimensional systems has been the subject of many investigations. Nonetheless, the dependence of the conductivity on the frequency of the external electric field has not been determined so far. All the arguments advanced to date concerning this dependence were exclusively indirect and based on investigations of the energy spectra of one-dimensional systems. These include the researches by Mott and Twoose[1] and Borland[2]. Using purely intuitive considerations (Mott) or investigating the ergodicity properties of the equation for the distribution function of the phase shifts of the particle wave function (Borland), these authors reached the conclusion that the wave function is localized, and indicated by the same token the possibility that one-dimensional systems have no static conductivity. No direct calculation of the conductivity was made, however, and the question remained open.

We show here on the basis of exact equations for the conductivity that it is equal to zero at $\omega = 0$ and that it decreases with frequency at a rate not lower than $\sim|\omega|$ at sufficiently high energies.

The first exact equations for the conductivity were derived by Halperin[3] for the particular case of white noise. Dykhne and I[4] investigated the problem of averaging the product of two Green functions of a particle. The present paper deals with the case when the impurity potential is of the form $U(x) = U_0\sum_i\delta(x - x_i)$ and the distances between impurities have a Poisson distribution.

The plan of the paper is the following. In Sec. 2 are introduced the characteristic functions $f^{(0,1)}(z_1, z_2)$ and the equations they satisfy [Eqs. (12)] are derived. The conductivity σ is expressed in terms of these functions with the aid of formulas (1) and (10). We note here that the quantity $[f(\epsilon+\omega) - f(\epsilon)]/\omega$ is replaced in (1) by the derivative $\partial f/\partial\epsilon$. Equations (12) are investigated in Sec. (12) in the case of high energies, when the collision term in (12) takes the form (13). It is shown that in this approximation the conductivity decreases with frequency no slower than $\sim|\omega|$, and the limiting expression for the function $f^{(0)}(z_1, z_2)$ takes the form of a δ-function, i.e., $f^{(0)}(z_1, z_2) \to \delta(z_1 - z_2)$ as $\omega \to 0$. It is shown in Sec. 4 that $f^{(0)}(z_1, z_2) \propto \delta(z_1 - z_2)$ is an exact solution of (12) at all values of the energy if $\omega = 0$. From this and from formulas (10) and (1) it follows that there is no static conductivity at any energy.

2. We start with the well known expression for the conductivity in an external electric field of frequency ω (we put $\hbar = 1$ throughout):

$$\sigma(\omega)=-\pi e^2\int d\varepsilon\frac{\partial f(\varepsilon)}{\partial\varepsilon}\Phi(\varepsilon,\omega), \tag{1}$$

where ϵ is the particle energy, $f(\epsilon)$ is the particle energy distribution function, and

$$\Phi(\varepsilon,\omega)=\frac{1}{L}\sum_{n,m}|v_{nm}|^2\delta(\varepsilon_n-\varepsilon)\delta(\varepsilon_m-\varepsilon'),$$

where v_{nm} is the matrix element of the velocity operator, $\epsilon' = \epsilon + \omega$, and L is the dimension of the system (we are, of course, interested in the limit as $L \to \infty$). In formula (2), the matrix elements are calculated between exact wave functions of a system having a potential energy (x_i are the impurity coordinates)

$$U(x)=U_0\sum_i\delta(x-x_i).$$

In the one-dimensional case, formula (2) can be represented in a different form by introducing the logarithmic derivative of the wave function of the electron, $z(x) = \psi^{-1}(x)d\psi(x)/dx$, which satisfies the obvious equation ($k^2 = 2m\epsilon$):

$$dz(x)/dx+z^2(x)+k^2=2mU_0\sum_i\delta(x-x_i).$$

The energy levels of a system with dimensions L are determined from the boundary conditions (the energy is a parameter in Eq. (3))

$$z(x=0,\varepsilon)=z_0,\quad z(x=L,\varepsilon)=z_L.$$

It is now easy to rewrite the expression for $\Phi(\epsilon, \omega)$ in the form

$$L\Phi(\varepsilon,\omega)=\frac{1}{(2m)^2}\left\langle\left[\int_0^L\psi^2(x,\varepsilon)\,dx\int_0^L\psi^2(x',\varepsilon')\,dx'\right]^{-1}\right.$$
$$\times\left\{\int_0^L\left[\psi(x,\varepsilon)\frac{d\psi(x,\varepsilon')}{dx}-\psi(x,\varepsilon')\frac{d\psi(x,\varepsilon)}{dx}\right]dx\right\}^2$$
$$\left.\times\left|\frac{\partial z(L,\varepsilon)}{\partial\varepsilon}\frac{\partial z(L,\varepsilon')}{\partial\varepsilon'}\right|\delta[z_L-z(L,\varepsilon)]\delta[z_L-z(L,\varepsilon')]\right\rangle.$$

The angle brackets denote averaging over the impurity coordinates x_i. From (3) we find that

$$\frac{\partial z(x,\varepsilon)}{\partial\varepsilon}=-2m\psi^{-2}(x,\varepsilon)\int_0^x\psi^2(x',\varepsilon)\,dx',$$

and finally, we have for $\Phi(\epsilon, \omega)$

$$L\Phi[\varepsilon,\omega]=\left\langle\psi^{-2}(L,\varepsilon)\psi^{-2}(L,\varepsilon')\delta[z_L-z(L,\varepsilon)]\delta[z_L-z(L,\varepsilon')]\right.$$
$$\left.\times\left\{\int_0^L\left[\psi(x,\varepsilon)\frac{d\psi(x,\varepsilon')}{dx}-\psi(x,\varepsilon')\frac{d\psi(x,\varepsilon)}{dx}\right]dx\right\}^2\right\rangle.$$

We note here that in the derivation of the expression for $\Phi(\epsilon, \omega)$ in (4) we have used the definition (2) of Φ, and then have written down in explicit form the square of the matrix element of the current and have taken into account the fact that