Some Extensions of a Lemma of Kotlarski

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Abstract

This note demonstrates that the conditions of Kotlarski's (1967) lemma can be substantially relaxed. In particular, the condition that the characteristic functions of M, U_1, and U_2 are non-vanishing can be replaced with much weaker conditions: the characteristic function of U_1 can be allowed to have real zeros, as long as the derivative of its characteristic function at those points is not also zero; that of U_2 can have a countable number of zeros; and that of M need satisfy no restrictions on its zeros.

We also show that Kotlarski's (1967) lemma holds when the tails of U_1 are no thicker than exponential, regardless of the zeros of the characteristic functions of U_1, U_2, or M.

1 Introduction

This note provides new regularity conditions ensuring that the conclusion of Kotlarski's (1967) lemma holds. Kotlarski's result may be explained as follows. Suppose one observes the joint distribution of two noisy measurements $(Y_1, Y_2) = (M + U_1, M + U_2)$ of a random variable M, where random variables U_1 and U_2 are measurement errors. Kotlarski showed that when (M, U_1, U_2) are mutually independent, $E[U_1] = 0$, and the characteristic functions of M, U_1, and U_2 are non-vanishing, it is possible to recover the unknown distributions of M, U_1, and U_2 from the joint distribution of (Y_1, Y_2).

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Kotlarski's lemma has been applied to identify and estimate a wide variety of models in economics, such as measurement error models (e.g., Li and Vuong 1998, Schennach 2004), auction models (e.g., Li et al. 2000, Krasnokutskaya 2011), panel data models (e.g., Arellano and Bonhomme 2009, Evdokimov 2008, 2010), and in various labor economics applications (e.g., Bonhomme and Robin 2010, Kennan and Walker 2011).

Kotlarski's lemma requires that the characteristic functions of the random variables M, U_1, and U_2 do not have real zeros. This is restrictive; the characteristic functions of many standard distributions have zeros (e.g. the uniform, the truncated normal, and many discrete distributions). Thus, it is important to consider identification when the characteristic functions may have real zeros.[1] Our aim here is to provide less restrictive alternative conditions for Kotlarski's conclusions to still hold.

Instead of requiring that the characteristic functions of M, U_1, and U_2 are non-vanishing, we require that the sets of zeros of the characteristic function of U_1 and its derivatives have empty intersection and that the real zeros of the characteristic function of U_2 are isolated. We impose no restrictions on the zeros of the characteristic function of M.

We also show that the conclusion of Kotlarski's lemma holds when U_1 has tails that are no thicker than exponential. This alternative result imposes strong restrictions on the tails of one of the measurement errors, but does not require any assumptions on its characteristic function, aiding economic interpretability. Further, the distributions of M and U_2 are completely unrestricted, apart from a first moment restriction.

Thus, we not only relax the assumption of nonvanishing characteristic functions of the errors U_1 and U_2, but we also provide conditions that may have a direct economic interpretation and that may thus be more appealing to researchers than those previously imposed.

2 Main Results

Let $\phi_{\mathcal{X}}$ denote the characteristic function of $\mathcal{X}$, $\phi_{\mathcal{X}}(s) \equiv E\left[\exp(is\mathcal{X})\right]$, $s \in \mathbb{R}$, where $i \equiv \sqrt{-1}$. We write $\phi'_{\mathcal{X}} \equiv (\partial/\partial s)\phi_{\mathcal{X}}$, and let λ denote Lebesgue measure. We impose the following assumption:

Assumption A: (i) *M, U_1, and U_2 are mutually independent*; *and $Y_1 \equiv M + U_1$ and*

$Y_2 \equiv M + U_2$; *(ii)* $E\left[|Y_1| + |Y_2|\right] < \infty$ *and* $E(U_1) = 0$; *(iii) the real zeros of* ϕ_{U_1} *and* ϕ'_{U_1} *are disjoint; and (iv)* ϕ_{U_2} *has only isolated real zeros.*

Given $A(i)$, the moment condition $A(ii)$ implies $E|M| < \infty$ and $E|U_2| < \infty$. Given $A(i)$ and $E(U_1) = 0$, it suffices for $E\left[|Y_1| + |Y_2|\right] < \infty$ that $E\left|Y_2\right| < \infty$, but we write the condition as we do to avoid obscuring the moment requirements on Y_1 and Y_2.

Let $\mathcal{Z}_0$ denote the set of real zeros of $\phi_{Y_1 - Y_2}$. Also, define the characteristic function $\phi_{Y_1,Y_2}\left(s_1, s_2\right) = E\left[\exp(is_1 Y_1 + is_2 Y_2)\right]$, the set of singular points $S_0 \equiv \{s \in \mathcal{Z}_0 : \limsup_{\xi \to s} \left|\frac{(\partial/\partial s_1)\phi_{Y_1,Y_2}(\xi,-\xi)}{\phi_{Y_1,Y_2}(\xi,-\xi)}\right| = \infty\}$, and the function

$$\psi\left(s\right) \equiv \begin{cases} \frac{(\partial/\partial s_1)\phi_{Y_1,Y_2}(s,-s)}{\phi_{Y_1,Y_2}(s,-s)}, \text{ if } s \notin \mathcal{Z}_0; \\ 0, \text{ if } s \in \mathcal{Z}_0. \end{cases}$$

Below we show that A implies that all elements of $\mathcal{Z}_0$ are isolated (and hence are countable). Since S_0 is a subset of $\mathcal{Z}_0$, we can enumerate all positive elements of S_0. Placing these in increasing order, for $k > 0$ we let $s_0\left(k\right)$ be the kth smallest positive element of S_0. Similarly, for $k < 0$, we let $s_0\left(k\right)$ be the $-k$th largest negative element of S_0. Thus, $S_0 = \left\{\ldots, s_0\left(-2\right), s_0\left(-1\right), s_0\left(1\right), s_0\left(2\right), \ldots\right\}$, and $s_0\left(k\right) < s_0\left(l\right)$ for all $k < l$. In addition, for notational convenience, denote $s_0\left(0\right) = 0$. For all $s \geq 0$ let $\overline{k}_0\left(s\right)$ be the largest k such that $s_0\left(k\right) \leq s$. Thus $\overline{k}_0\left(s\right) = 0$ for all $s \in \left[0, s_0\left(1\right)\right)$, $\overline{k}_0\left(s\right) = 1$ for all $s \in \left[s_0\left(1\right), s_0\left(2\right)\right)$ and so on. We extend Kotlarski's (1967) lemma as follows.

Lemma 1

(a) Let (L, V_1, V_2) *be random, and let* $(Z_1, Z_2) \equiv (L + V_1, L + V_2)$, *with* (V_1, V_2) *distributed identically to* (U_1, U_2). *If* $A(i)$ *holds for both* (M, U_1, U_2) *and* (L, V_1, V_2), *then* $\mathcal{Z}_0$ *is also the zero set of* $\phi_{Z_1 - Z_2}$. *If* $A(i, ii)$ *hold for* (M, U_1, U_2) *and* $\lambda(\mathcal{Z}_0) > 0$, *then* $A\,(iii)$ *or* $A\,(iv)$ *fail for* (M, U_1, U_2) *and there exist* (L, V_1, V_2) *such that* $\phi_{Z_1,Z_2} = \phi_{Y_1,Y_2}$ *but* $\phi_L \neq \phi_M$.

(b) if $A(i) - (iv)$ *hold, then, with* $\mu_1 \equiv E(Y_1)$, *for all* $s \in \mathbb{R}^+ \backslash S_0$

$$\phi_{U_1}(s) = \exp[-is\mu_1] \lim_{\varepsilon \searrow 0} \left[(-1)^{\overline{k}_0(s)} \prod_{0 < k \leq \overline{k}_0(s)} \exp\left\{ \int_{s_0(k-1)+\varepsilon}^{s_0(k)-\varepsilon} \psi\left(\xi\right) d\xi \right\} \times \exp\left\{ \int_{s_0\left(\overline{k}_0(s)\right)+\varepsilon}^{s} \psi\left(\xi\right) d\xi \right\} \right]. \quad (1)$$

A similar formula holds[2] *for all* $s \in \mathbb{R}^- \backslash S_0$. *Then* $\phi_M(s) = \phi_{Y_1}(s)/\phi_{U_1}(s)$ *and* $\phi_{U_2}(-s) = \phi_{Y_1,Y_2}(s,-s)/\phi_{U_1}(s)$ *for all* $s \notin \mathcal{Z}_0$. *Moreover, the functions* $\phi_M(\cdot)$, $\phi_{U_1}(\cdot)$, *and* $\phi_{U_2}(\cdot)$ *are continuous on* $\mathbb{R}$ *and hence can be uniquely extended from* $\mathbb{R}\backslash\mathcal{Z}_0$ *to* $\mathbb{R}$.

Proof: We begin with some simple but useful Facts:

(1) Given $A(i)$, we have

$$\begin{aligned}\phi_{Y_1,Y_2}(s_1,s_2) &= E[\exp(i(s_1+s_2)M + is_1U_1 + is_2U_2)] \\ &= \phi_M(s_1+s_2)\phi_{U_1}(s_1)\phi_{U_2}(s_2).\end{aligned}$$

Letting $s_1 = s$ and $s_2 = -s$ gives $\phi_{Y_1,Y_2}(s,-s) = \phi_{Y_1-Y_2}(s) = \phi_M(0)\phi_{U_1}(s)\phi_{U_2}(-s) = \phi_{U_1}(s)\bar{\phi}_{U_2}(s)$, as $\phi_M(0) = 1$ and $\phi_{U_2}(-s) = \bar{\phi}_{U_2}(s)$. Thus, the zero set $\mathcal{Z}_0$ of $\phi_{Y_1-Y_2}$ is the union of the zero sets $\mathcal{Z}_{01}$ of ϕ_{U_1} and $\bar{\mathcal{Z}}_{02}$ of $\bar{\phi}_{U_2}$. As the zeros of $\bar{\phi}_{U_2}$ are identical to the zeros of ϕ_{U_2}, say $\mathcal{Z}_{02}$, we have $\bar{\mathcal{Z}}_{02} = \mathcal{Z}_{02}$. Thus, $A(i)$ implies $\mathcal{Z}_0 = \mathcal{Z}_{01} \cup \mathcal{Z}_{02}$.

(2) $A(i,ii)$ imply $E[|M| + |U_1| + |U_2|] < \infty$, which in turn implies that the functions ϕ_{Y_1,Y_2}, ϕ_{U_1}, ϕ_{U_2}, and ϕ_M are continuously differentiable.

(3) $A(i) - (iii)$ imply that $\mathcal{Z}_{01}$ has no limiting points; hence, all elements of $\mathcal{Z}_{01}$ are isolated (in $\mathbb{R}$) and $\mathcal{Z}_{01}$ is a countable set. To prove this, suppose there exists a sequence of points $\{\xi_k\}_{k=1}^{\infty}$, such that $\xi_k \neq \xi_0$ for all k, $\xi_0 = \lim_{k\to\infty}\xi_k$, and $\phi_{U_1}(\xi_k) = 0$ for all k. By Fact (2), the function ϕ_{U_1} is continuously differentiable. Then $\phi_{U_1}(\xi_0) = \lim_{k\to\infty}\phi_{U_1}(\xi_k) = 0$, and $\phi'_{U_1}(\xi_0) = \lim_{k\to\infty}\left(\phi_{U_1}(\xi_k) - \phi_{U_1}(\xi_0)\right)/(\xi_k - \xi_0) = 0$, which contradicts $A(iii)$.

(4) By Fact (3) and $A(iv)$, $\mathcal{Z}_0 = \mathcal{Z}_{01} \cup \mathcal{Z}_{02}$ is countable. The Lebesgue measure of a countable set is zero, so Assumption A implies $\lambda(\mathcal{Z}_0) = \lambda(\mathcal{Z}_{01}) = \lambda(\mathcal{Z}_{02}) = 0$.

We are now ready to prove the lemma:

(a) Because (U_1,U_2) and (V_1,V_2) are identically distributed, $\phi_{U_1} = \phi_{V_1}$ and $\phi_{U_2} = \phi_{V_2}$, so the zero sets of ϕ_{V_1} and ϕ_{V_2} are $\mathcal{Z}_{01}$ and $\mathcal{Z}_{02}$, respectively. Given this and $A(i)$, Fact (1) ensures that $\mathcal{Z}_0$ is the zero set of both $\phi_{Y_1-Y_2}$ and $\phi_{Z_1-Z_2}$. If $A(i,ii)$ hold and $\lambda(\mathcal{Z}_0) > 0$ then $A(iii)$ or $A(iv)$ must fail due to Fact (4). The proof is completed by the example of Kotlarski (1967, p.72), which specifies two random triplets having the given properties with $\lambda(\mathcal{Z}_0) > 0$ and with $\phi_{Z_1,Z_2} = \phi_{Y_1,Y_2}$ but $\phi_L \neq \phi_M$.

(b) (1) By Fact (2), $(\partial/\partial s_1)\phi_{Y_1,Y_2}$ exists and

$$\begin{aligned}
(\partial/\partial s_1)\phi_{Y_1,Y_2}(s_1,s_2) &= \phi'_M(s_1+s_2)\phi_{U_1}(s_1)\phi_{U_2}(s_2)+\phi_M(s_1+s_2)\phi'_{U_1}(s_1)\phi_{U_2}(s_2); \qquad \text{so} \\
(\partial/\partial s_1)\phi_{Y_1,Y_2}(s,-s) &= \phi'_M(0)\phi_{U_1}(s)\phi_{U_2}(-s)+\phi'_{U_1}(s)\phi_{U_2}(-s).
\end{aligned}$$

Suppose $s \notin \mathcal{Z}_0$. Then

$$\psi(s)=\frac{(\partial/\partial s_1)\phi_{Y_1,Y_2}(s,-s)}{\phi_{Y_1,Y_2}(s,-s)}=\frac{\phi'_M(0)\phi_{U_1}(s)\phi_{U_2}(-s)+\phi'_{U_1}(s)\phi_{U_2}(-s)}{\phi_{U_1}(s)\phi_{U_2}(-s)}=i\mu_1+\frac{\phi'_{U_1}(s)}{\phi_{U_1}(s)}. \tag{2}$$

Note that for $s \notin \mathcal{Z}_0$ we can write $\psi(s)=(\partial/\partial s_1)\ln\phi_{Y_1,Y_2}(s,-s)$. Also note that for $s \in S_0$, $\limsup_{\xi\to s}|\psi(\xi)|$ is infinite, which implies that $\phi_{U_1}(s)=0$, because the function ϕ_{U_2} is bounded and the function ϕ'_{U_1} is locally bounded away from both zero and infinity. Thus $\phi_{U_1}(s)=0$ for all $s \in S_0$.

The proof now proceeds by induction. First, for all $s \in [0, s_0(1))$, i.e., all s such that $\overline{k}_0(s)=0$, the right hand side of expression (1) simplifies to

$$\begin{aligned}
& \exp[-is\mu_1] \lim_{\varepsilon\searrow 0}\exp\left\{\int_\varepsilon^s \psi(\xi)\,d\xi\right\} \\
= \; & \exp[-is\mu_1] \lim_{\varepsilon\searrow 0}\exp\left\{\int_\varepsilon^s \left(i\mu_1+\frac{\phi'_{U_1}(\xi)}{\phi_{U_1}(\xi)}\right)d\xi\right\} \\
= \; & \lim_{\varepsilon\searrow 0}\exp[-i\varepsilon\mu_1]\exp\left\{\int_\varepsilon^s \frac{\partial}{\partial s}\ln\left(\phi_{U_1}(\xi)\right)d\xi\right\} \\
= \; & \phi_{U_1}(s),
\end{aligned}$$

where $\ln\left(\phi_{U_1}(\xi)\right)$ is the principal value of the logarithm and is well defined since $\phi_{U_1}(\xi)\neq 0$ for all $\xi \in [0, s_0(1))$ and $\phi_{U_1}(0)=1$. Hence, formula (1) is shown to hold for all $s \in [0, s_0(1))$.

Now suppose (1) holds for all s such that $\overline{k}_0(s) \leq K$, i.e., it holds for all $s \in \cup_{1\leq k\leq K}(s_0(k-1), s_0(k))$. We now show that this implies that (1) also holds for all $s \in (s_0(K), s_0(K+1))$. Write $\hat{s} \equiv (s_0(K)+s_0(K-1))/2$. Since $\phi_{U_1}(\hat{s}) \neq 0$ and (1) holds for $\hat{s}$, for any $s \in (s_0(K), s_0(K+1))$ we can write the right hand side of (1) as

$$
\begin{aligned}
& \phi_{U_1}(\hat{s}) \exp[-i(s-\hat{s})\mu_1] \lim_{\varepsilon \searrow 0} \left[(-1) \exp\left\{ \int_{\hat{s}}^{s_0(K)-\varepsilon} \psi(\xi)\, d\xi \right\} \times \exp\left\{ \int_{s_0(K)+\varepsilon}^{s} \psi(\xi)\, d\xi \right\} \right] \\
= \; & \phi_{U_1}(\hat{s}) \lim_{\varepsilon \searrow 0} \left[\exp[-2i\varepsilon\mu_1](-1) \frac{\phi_{U_1}(s_0(K)-\varepsilon)}{\phi_{U_1}(\hat{s})} \frac{\phi_{U_1}(s)}{\phi_{U_1}(s_0(K)+\varepsilon)} \right] \\
= \; & \phi_{U_1}(s) \lim_{\varepsilon \searrow 0} \left[\exp[-2i\varepsilon\mu_1] \frac{0-\phi_{U_1}(s_0(K)-\varepsilon)}{\phi_{U_1}(s_0(K)+\varepsilon)-0} \right] \\
= \; & \phi_{U_1}(s) \lim_{\varepsilon \searrow 0} \left[\exp[-2i\varepsilon\mu_1] \frac{\phi'_{U_1}(\xi_1)\,\varepsilon}{\phi'_{U_1}(\xi_2)\,\varepsilon} \right] \\
= \; & \phi_{U_1}(s),
\end{aligned}
$$

where $\xi_1 \in (s_0(K)-\varepsilon, s_0(K))$, $\xi_2 \in (s_0(K), s_0(K)+\varepsilon)$. The second equality holds because $\phi_{U_1}(s_0(K)) = 0$ and because $\phi_{U_1}(\hat{s})$ can be cancelled out, since $\phi_{U_1}(\hat{s}) \neq 0$ by construction of $\hat{s}$. The third equality follows from the mean value theorem applied to $\phi_{U_1}(s_0(K)) = 0$, and the last equality holds by $\phi'_{U_1}(s_0(K)) \neq 0$ and the continuity of ϕ'_{U_1} from Fact (2). Thus, we have shown that (1) holds for $\overline{k}_0(s) = K+1$, i.e. for all $s \in \cup_{1 \leq k \leq K+1} (s_0(k-1), s_0(k))$. The proof by induction is therefore complete.

The above establishes identification of $\phi_{U_1}(s)$ for all $s \in \mathbb{R} \backslash \mathcal{Z}_0$. Then we also identify $\phi_M(s) = \phi_{Y_1}(s)/\phi_{U_1}(s)$ and $\phi_{U_2}(s) = \phi_{Y_2-Y_1}(s)/\phi_{U_1}(-s)$ for all $s \in \mathbb{R}\backslash\mathcal{Z}_0$. Finally, the continuity of ϕ_M, ϕ_{U_1}, and ϕ_{U_2} implies the uniqueness of their continuous extension from $\mathbb{R}\backslash\mathcal{Z}_0$ to $\mathbb{R}$. ■

Remark 1: When the characteristic function $\phi_{Y_1-Y_2}$ has no zeros, eq. (1) becomes

$$
\phi_{U_1}(s) = \exp[-is\mu_1] \exp\left\{ \int_0^s \psi(\xi)\, d\xi \right\} = \exp[-is\mu_1] \exp\left\{ \int_0^s \frac{(\partial/\partial s_1)\phi_{Y_1,Y_2}(\xi,-\xi)}{\phi_{Y_1,Y_2}(\xi,-\xi)} d\xi \right\}, \tag{3}
$$

which is exactly the expression obtained in Evdokimov (2008), who assumes that the characteristic functions ϕ_{U_1} and ϕ_{U_2} are nonvanishing. Similar to Evdokimov (2008), Lemma 1 relaxes Kotlarski's condition that the characteristic function ϕ_M is nonvanishing.

Remark 2: In Lemma 1, we essentially recover $\phi_{U_1}(s)$ by observing $\psi(s) - i\mu_1$, which equals the ratio $\phi'_{U_1}(s)/\phi_{U_1}(s) = (\partial/\partial s)\ln\left(\phi_{U_1}(s)\right)$, and by imposing the initial condition $\phi_{U_1}(0) = 1$. When $\phi_{U_1}(s)$ is nonzero, solving the differential equation (2) immediately yields eq. (3). Nevertheless, we run into obvious problems when $\phi_{U_1}(s_0) = 0$ for some s_0. Here, $A(iii)$ is very important; for a small $\varepsilon > 0$ we can write $\phi_{U_1}(s_0+\varepsilon) = \phi_{U_1}(s_0-\varepsilon) + 2\phi'_{U_1}(s_0)\varepsilon + o(\varepsilon)$ and hence "jump" through the singular point s_0. This expression is uninformative unless $\phi'_{U_1}(s_0) \neq 0$. For example, the functions $\phi_A(s) = (1-s)^2$ and $\phi_B(s) = (1-s)|1-s|$ for $s \geq 0$ (although not proper characteristic functions) have $\phi'_A(1) = \phi_A(1) = \phi'_B(1) = \phi_B(1) = 0$ and thus violate $A(iii)$. Indeed, one cannot distinguish between these two functions based on $\phi'(s)/\phi(s)$ because for both functions $\psi(s) - \mu_1 = \phi'_A(s)/\phi_A(s) = \phi'_B(s)/\phi_B(s) = 2/(s-1)$ for $s \geq 0$.

Remark 3: If the zeros of ϕ_{U_1} and ϕ'_{U_1} are not disjoint, identification may be obtained by considering higher-order derivatives, say $\phi^{(n)}_{U_1}$, $n > 1$. For example, suppose that $\phi_{U_1}(\xi_0) = \phi'_{U_1}(\xi_0) = 0$, but ϕ''_{U_1} $(= \phi^{(2)}_{U_1})$ exists and is continuous (so that U_1 has finite second moment), and that $\phi''_{U_1}(\xi_0) \neq 0$, so that the zeros of ϕ'_{U_1} and ϕ''_{U_1} are disjoint at ξ_0. If so, a similar argument delivers identification. If $\phi''_{U_1}(\xi_0) = 0$, one can consider the next higher derivative, and so on. That is, identification continues to hold, given that the characteristic function ϕ_{U_1} is sufficiently continuously differentiable and its higher-order derivatives have suitably disjoint zeros. The (-1) factor in equation (1) appears only when n is even, with $\phi^{(n+1)}_{U_1}(\xi_0) \neq \phi^{(n)}_{U_1}(\xi_0) = 0$. A sufficient (but not necessary) condition for $A(iv)$ is the disjointness of the zeros of ϕ_{U_2} and ϕ'_{U_2}, since ϕ'_{U_2} exists by Fact (2). This holds by the argument of Fact (3). Just as for U_1, the properties of higher-order derivatives of ϕ_{U_2} can also ensure $A(iv)$.

Remark 4: When $A(i)$ holds, the assumptions $E[|Y_1| + |Y_2|] < \infty$, $A(iii)$, and $A(iv)$ can be checked for any given ϕ_{Y_1,Y_2}.

Remark 5: Although the uniform distribution is not a common measurement error distribution, it nicely illustrates the power of $A(iii)$. If $U_1 \sim U[-a,a]$ then for any value of $a > 0$ the functions ϕ_{U_1} and ϕ'_{U_1} have real zeros, but these zeros never coincide. Thus, the original result of Kotlarski (1967) as well as the lemmas of Li and Vuong (1998), Schennach (2004), and Evdokimov (2008) do not apply, yet our Lemma 1 does guarantee identification.

The assumptions of Lemma 1 are weak and hold for all standard probability distributions. However, they are stated in terms of characteristic functions. Economic models rarely impose restrictions on characteristic functions; hence any assumptions stated in terms of characteristic functions might lack an economic interpretation. To address this issue, we introduce an alternative assumption and identification lemma.

Assumption B: *$A(i)$ and $A(ii)$ hold, and (iii) there exist positive constants c_1 and c_2 such that the density of U_1 satisfies $f_{U_1}(u) < c_1 \exp(-c_2 |u|)$ for large u.*

Lemma 2 *Let Assumption B hold. Then the distributions of M, U_1, and U_2 are identified.*

Proof: The characteristic functions ϕ_{U_1} and ϕ_{U_2} are continuous, and $\phi_{U_1}(0) = \phi_{U_2}(0) = 1$. Thus, there is an $\overline{s} > 0$ such that $\left|\phi_{U_j}(s)\right| > 1/2$ for all $s \in [-\overline{s}, \overline{s}]$ and $j = 1, 2$. Then $\left|\phi_{Y_1,Y_2}(s, -s)\right| = \left|\phi_{U_1}(s)\, \phi_{U_2}(-s)\right| > 1/4$ for all $s \in [-\overline{s}, \overline{s}]$. Thus, B ensures that equation (3) applies for all $s \in [-\overline{s}, \overline{s}]$, identifying $\phi_{U_1}(s)$ on this interval.

$B(iii)$ implies that ϕ_{U_1} is analytic on $\mathbb{R}$; see page 3 of Paley and Wiener (1934). Then, by the properties of analytic functions, ϕ_{U_1} is identified not only on the interval $[-\overline{s}, \overline{s}]$ but also on the whole real line. Moreover, functions analytic on $\mathbb{R}$ may only have isolated real zeros, and hence $\phi_M(s) = \phi_{Y_1}(s) / \phi_{U_1}(s)$ for all points $s \in \mathbb{R}$, except for at most a countable number of the isolated zeros of ϕ_{U_1}. Then, by continuity, ϕ_M is identified on the whole real line. We identify ϕ_{U_2} in a similar way as $\phi_{U_2}(s) = \phi_{Y_2 - Y_1}(s) / \phi_{U_1}(-s)$. ■

Remark 6: Here, Assumption $B(iii)$ replaces $A(iii)$, and it makes $A(iv)$ unnecessary. Clearly, $B(iii)$ is strong for U_1. The advantage of this assumption is its potential economic interpretability; in a variety of economic applications, researchers may have some intuition or economic model that implies that one of the measurement errors, U_1, has thin tails (or even bounded support). Apart from the requirement that U_2 has finite first moment (implied by $B(ii)$), its distribution is completely unrestricted.

Remark 7: The key property of ϕ_{U_1} ensured by $B(iii)$ is its analyticity on $\mathbb{R}$. Although economically interpretable conditions are more compelling, any other condition ensuring this analyticity can replace $B(iii)$ to deliver the same conclusion.

Remark 8: Lemma 2 is not a corollary of Lemma 1, as $B(iii)$ (or analyticity of ϕ_{U_1}) does not imply $A(iii)$, and it says nothing about ϕ_{U_2}.

Remark 9: An interesting topic for future research is whether our approach can be applied or adapted to models identified using generalized functions and their Fourier transforms, as in Schennach (2007) and Zinde-Walsh (2010).

Notes

[1]Note that when the distribution of the error term is *known*, deconvolution can be performed even when the characteristic function of the distribution of the error has real zeros; see Devroye (1989) and Carrasco and Florens (forthcoming).

[2]For all $s \leq 0$ let $\underline{k}_0(s)$ be the smallest k such that $s_0(k) \geq s$. Then for all $s \in \mathbb{R}^- \backslash S_0$,

$$\phi_{U_1}(s) = \exp[-is\mu_1] \lim_{\varepsilon \searrow 0} \left[(-1)^{\underline{k}_0(s)} \prod_{\underline{k}_0(s) \leq k < 0} \exp\left\{ \int_{s_0(k+1)-\varepsilon}^{s_0(k)+\varepsilon} \psi(\xi)\, d\xi \right\} \times \exp\left\{ \int_{s_0(\underline{k}_0(s))-\varepsilon}^{s} \psi(\xi)\, d\xi \right\} \right].$$

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