Phase Space

The phase space of a monatomic ideal gas is defined as the set of all equilibrium configurations of the gas. Because every such state may be represented by its pressure and volume, we can represent phase space on a $p-V$ diagram, in which the $x$-axis is labelled with volume and the $y$ axis is labelled with pressure. Parametric curves on such a diagram correspond to processes that proceed quasi-statically, meaning slowly enough that the intermediates of the process are also in equilibrium, and processes that form a closed loop are called cycles. Scalar fields on phase space are called state functions. You are given the following useful thermodynamic equations, where $p$ is pressure, $V$ is volume, $N$ is particle number, $k$ is Boltzmann’s constant, $T$ is temperature, $Q$ is heat, $h$ is Plank’s constant, $m$ is particle mass, and $S$ is entropy:

- **ideal gas law**: $T = pV/(kN)$
- definition of differential of work done by the gas: $\delta W = p\,dV$ (why?)
- definition of internal energy: $U = \frac{3}{2}pV$
- first law of thermodynamics: $dU = \delta Q - \delta W$
- definition of thermal wavelength: $\Lambda = h(2\pi mpV/N)^{-1/2}$
- the Sackur-Tetrode equation for entropy: $S/(kN) = \log(V/(N\Lambda^3)) + \frac{5}{2}$

Do the following:

a) Write an expression for $\delta Q$ in terms of $p$ and $V$ and their differentials.

b) Along a given path, heat capacity is defined as $C = dQ/dT$. Calculate $C$ assuming the process proceeds at constant pressure.

c) Calculate $dS$ and show that $dS = \delta Q/T$.

d) A process is adiabatic if $dS$ along the path is always zero. Characterize adiabatic paths through phase space.

e) Calculate the total heat added to the gas in one stroke of a rectangular cycle with lower corner $(V_l,p_l)$ and upper corner $(V_u,p_u)$.

f) Write the equations of a coordinate transformation from $(V,p)$ coordinates to $(S,T)$ coordinates, and express the area element $dV\,dp$ in those coordinates. What is the physical interpretation of $dV\,dp$, and where are $(S,T)$ coordinates bad?

g) Write $\delta W$ in $(S,T)$ coordinates.

Phase space has a different, but related, meaning in classical physics. If a particle free to move in one dimension has position $q$ and momentum $p$, then the phase space of that particle’s motion is the set of all $(q,p)$ pairs. We can define on this space a Hamiltonian function, which in many cases corresponds to the total energy of the system. For a one-particle system with a given Hamiltonian $H$, Hamilton’s equations for the time-evolution of an initial state $(q,p)$ are:

$$\frac{\partial H}{\partial q} = -\frac{dp}{dt},$$
$$\frac{\partial H}{\partial p} = \frac{dq}{dt}.$$  

Thus it is natural to define a vector field on phase space called the Hamiltonian flow by $v = (\partial H/\partial p, -\partial H/\partial q)$. Note that solutions to Hamilton’s equations are integral curves of the flow.

h) Show that the divergence of the flow is zero

i) Suppose at $t = 0$ each point in phase space $q,p$ is assigned a number $\rho(q,p)$. Assuming those points evolve in time according to Hamilton’s equations, what is $\partial \rho/\partial t$?

Classical Newtonian gravity is done in terms of the gravitational potential \( \phi \). For any particle, the gravitational acceleration it feels is given by \( g = -\nabla\phi \), and for a mass density \( \rho \), Gauss’s law for gravity states that \( \nabla^2 \phi = 4\pi G \rho \).

a) Find the most general spherically symmetric solution to Gauss’s law given that \( \rho \) is zero everywhere except \( r = 0 \).

Consider now a cloud of test particles in a gravitational potential. Single out one particle as the fiducial (reference) particle and denote the position of the \( i \)’th other particle relative to this test particle with the vector \( \delta x_i \). Define the Hessian of the potential as the matrix whose \( (i, j) \)'th entry is given by:

\[
J_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j},
\]

where \( x^1 \) corresponds to \( x \), \( x^2 \) corresponds to \( y \), and \( x^3 \) corresponds to \( z \) (i.e. the superscripts are not exponents but labels on the coordinates).

b) Show that \( \frac{d^2}{dt^2} \delta x_i = J \delta x_i \). This means that the relative (tidal) accelerations of the particles are encapsulated by the matrix \( J \).

c) Show that \( \text{Tr} J \) is the rate of change of an infinitesimal volume element. What is the rate of change of the volume of a cloud of negligible-mass test particles in a region with no other matter?

d) Calculate \( J \) for the potential you derived in (a).

e) Suppose we make a coordinate transformation to new coordinates \( y^j(x^i) \). In terms of derivatives of the coordinates with respect to each other, what are the coordinates of \( J \) in the new coordinate system?

Suppose we have a fluid in which pressure and density are related by \( p = K \rho^{1+1/n} \), where \( n \) is a constant called the polytropic index. To describe the motion of this fluid, we use one Euler’s equation for momentum:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla p = \mathbf{a},
\]

where \( \mathbf{a} \) is the net external acceleration imposed on the fluid.

f) Find an ordinary differential equation for static, spherically symmetric solutions \( \rho(r) \) to this equation.

g) Find a the cylindrically symmetric dynamic solutions assuming \( n >> 1 \).

Gravitoelectromagnetism is an outdated physical theory, proposed near the turn of the century, that encapsulates some relativistic gravitational effects. It is named for its close similarity to Maxwell’s formulation of electromagnetism. The equations of GEM are:

\[
\begin{align*}
\nabla \cdot \mathbf{g} &= -4\pi G \rho \\
\nabla \cdot \mathbf{B}_g &= 0 \\
\nabla \times \mathbf{g} &= -\frac{\partial \mathbf{B}_g}{\partial t} \\
\n\nabla \times \mathbf{B}_g &= \left( -\frac{16\pi G}{c^2} \rho \mathbf{v} + \frac{4}{c^2} \frac{\partial \mathbf{E}_g}{\partial t} \right)
\end{align*}
\]

where \( \mathbf{B}_g \) is the “gravitomagnetic field” (it exerts forces on particles in a velocity-dependent way analogously to the electromagnetic \( \mathbf{F} = q \mathbf{v} \times \mathbf{B} \)).

h) Show that these equations permit gravitational waves that propagate at light speed.

i) Can spherically symmetric configurations of matter (\( \rho \) and \( \mathbf{v} \)) produce gravitational waves?