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Efficiency loss in a Cournot oligopoly with convex market demand

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ABSTRACT

We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities. We compare the social welfare achieved at a Cournot equilibrium to the maximum possible, for the case where the inverse market demand function is convex. We establish a lower bound on the efficiency of Cournot equilibria in terms of a scalar parameter derived from the inverse demand function, namely, the ratio of the slope of the inverse demand function at the Cournot equilibrium to the average slope of the inverse demand function between the Cournot equilibrium and a social optimum. Also, for the case of a single, monopolistic, profit maximizing supplier, or of multiple suppliers who collude to maximize their total profit, we establish a similar but tighter lower bound on the efficiency of the resulting output. Our results provide nontrivial quantitative bounds on the loss of social welfare for several convex inverse demand functions that appear in the economics literature.

1. Introduction

We consider a Cournot oligopoly model where multiple suppliers compete by choosing quantities, with a focus on the case where the inverse market demand function is convex. Our objectives are to compare the optimal social welfare to: (i) the social welfare at a Cournot equilibrium and (ii) the social welfare achieved when the suppliers collude to maximize the total profit, or, equivalently, when there is a single supplier.

1.1. Background

In a book on oligopoly theory (see Chapter 2.4 of Friedman (1983)), Friedman raises two questions on the relation between Cournot equilibria and competitive equilibria. First, "is the Cournot equilibrium close, in some reasonable sense, to the competitive equilibrium?" Furthermore, "will the two equilibria coincide as the number of firms goes to infinity?" The second question has been much explored, and the answer is generally positive; see, e.g., Gabszewicz and Vial (1972), Novshek and Sonnenschein (1978) and Novshek (1980).

In more recent work, attention has turned to the efficiency of Cournot equilibria in settings that involve an arbitrary (possibly small) number of suppliers or consumers. Anderson and Renault (2003) quantify the efficiency loss in Cournot oligopoly models with concave demand functions. However, most of their results focus on the relation between consumer surplus, producer surplus, and the aggregate social welfare achieved at a Cournot equilibrium, rather than on the relation between the social welfare achieved at a Cournot equilibrium and the optimal social welfare.

We define the efficiency of a Cournot equilibrium as the ratio of its aggregate social welfare to the optimal social welfare, which provides a natural measure of the difference between a Cournot equilibrium and a socially optimal competitive equilibrium. The related concept of the percentage of welfare loss dates back to at least the 1950s in the economics literature (Harberger, 1954) and is also intimately related to the concept of "price of anarchy" of Koutsoupias and Papadimitriou (1999).

Recent works have reported various efficiency bounds for Cournot oligopoly with affine demand functions. Kluberg and Perakis (2012) compare the social welfare and the aggregate profit earned by the suppliers under Cournot competition to the corresponding maximum possible, for the case where suppliers produce multiple differentiated products and demand is an affine function of the price. Closer to the present paper, Johari and Tsitsiklis (2005) establish a 2/3 lower bound on the efficiency of a Cournot equilibrium, when the inverse demand function is affine. They also show that the 2/3 lower bound applies to a monopoly model with general concave demand. Furthermore, there are some recent efficiency loss results, by Corchon (2008), for the special case of convex inverse demand functions $p(q) = \alpha - \beta q^{\gamma}$, with $\gamma > 0$, and by Guo

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and Yang (2005), for a class of inverse demand functions that solve a certain differential equation (e.g., with constant elasticity).

In this paper, we study the efficiency loss in a Cournot oligopoly model with general convex demand functions.[1] Convex demand functions, such as the negative exponential and the constant elasticity demand curves, have been widely used in oligopoly analysis and marketing research (Bulow and Pfleiderer, 1983; Fabinger and Weyl, 2009; Tyagi, 1999). We do not address the case of concave inverse demand functions, which appears to be qualitatively different, as will be illustrated by an example in Section 2.5.

The general methodology used in this paper is similar in spirit to that used in a companion paper Tsitsiklis and Xu (2013) to derive upper bounds on the (aggregate) profit loss of Cournot equilibria. However, the assumptions, the details, and the expressions in the various results are different. For instance, the assumption in Tsitsiklis and Xu (2013) that a monopolist's revenue is a concave function is replaced here by an assumption that the inverse demand function is convex.

1.2. Summary of contributions

Before continuing, we provide here a roadmap of the paper together with a summary of our main contributions. In the next section, we provide some mathematical preliminaries on Cournot equilibria that will be useful later, including the fact that efficiency lower bounds can be obtained by restricting to linear cost functions. In Section 3, we consider affine inverse demand functions and derive a refined lower bound on the efficiency of Cournot equilibria that depends on a small amount of ex post information. We also show this bound to be tight.

In Section 4, we show that for convex inverse demand functions, and for the purpose of studying the worst case efficiency loss, it suffices to restrict to a special class of piecewise linear inverse demand functions. We then obtain an efficiency lower bound in terms of the ratio of two parameters, which is in some sense, a measure of nonlinearity of the inverse demand function. For affine inverse demand functions, our efficiency bound becomes 2/3, consistent with the bound in Johari and Tsitsiklis (2005). We then derive a number of corollaries that provide efficiency lower bounds that can be calculated without detailed information on these equilibria, and apply these results to various commonly encountered convex inverse demand functions.

In Section 6, we consider the important special case where $N = 1$, i.e., of a single monopolistic supplier; we note that this case is also mathematically equivalent to a setting where multiple suppliers choose to collude and coordinate production so as to maximize their total profit. Our earlier lower bounds continue to hold. However, by using the additional assumption that $N = 1$, we can obtain sharper (i.e., larger) lower bounds. Finally, in Section 7, we make some brief concluding remarks.

2. Preliminaries on Cournot equilibria

In this section, we define the Cournot competition model, and then introduce some definitions and assumptions. The proofs of all results in this section are given in Appendix A.

We consider a market for a single homogeneous good with inverse demand function $p : [0, \infty) \to [0, \infty)$ and N suppliers. Supplier $n \in \{1, 2, \ldots, N\}$ has a cost function $C_n : [0, \infty) \to [0, \infty)$. Each supplier n chooses a nonnegative real number x_n, which is the amount of the good to be supplied by her. The *strategy profile* $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ results in a total supply denoted by $X = \sum_{n=1}^{N} x_n$, and a corresponding market price $p(X)$. The payoff to supplier n is

$$\pi_n(x_n, \mathbf{x}_{-n}) = x_n p(X) - C_n(x_n),$$

where we have used the standard notation $\mathbf{x}_{-n}$ to indicate the vector $\mathbf{x}$ with the component x_n omitted. A strategy profile $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is a Cournot (or Nash) equilibrium if

$$\pi_n(x_n, \mathbf{x}_{-n}) \geq \pi_n(x, \mathbf{x}_{-n}), \quad \forall\, x \geq 0,\ \forall\, n \in \{1, 2, \ldots, N\}.$$

In the sequel, we denote by f' and f'' the first and second, respectively, derivatives of a scalar function f, if they exist. For a function defined on a domain $[0, Q]$, the derivatives at the endpoints 0 and Q are defined as right and left derivatives, respectively. For points in the interior of the domain, and if the derivative is not guaranteed to exist, we use the notation $\partial_+ f$ and $\partial_- f$ to denote the right and left, respectively, derivatives of f; these are guaranteed to exist for convex or concave functions f.

Assumption 1. For any n, the cost function $C_n : [0, \infty) \to [0, \infty)$ is convex, nondecreasing, and continuously differentiable. Furthermore, $C_n(0) = 0$.

Assumption 2. The inverse demand function $p : [0, \infty) \to [0, \infty)$ is convex, continuous, nonnegative, and nonincreasing, with $p(0) > 0$.

Note that some parts of our assumptions are redundant, but are included for easy reference. For example, if C_n is convex and nonnegative, with $C_n(0) = 0$, then it is automatically continuous and nondecreasing.

Definition 1. The *optimal social welfare* is the optimal objective value in the following optimization problem,

$$\begin{aligned} &\text{maximize} \quad \int_0^X p(q)\, dq - \sum_{n=1}^{N} C_n(x_n) \\ &\text{subject to} \quad x_n \geq 0,\ n = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $X = \sum_{n=1}^{N} x_n$.

In the above definition, $\int_0^X p(q)\, dq$ is the aggregate consumer surplus and $\sum_{n=1}^{N} C_n(x_n)$ is the total cost of the suppliers. The objective function in (1) is a measure of the social welfare across the entire economy of consumers and suppliers, the same measure as the one used in Anderson and Renault (2003).

For a model with a nonincreasing continuous inverse demand function and continuous convex cost functions, the following assumption guarantees the existence of an optimal solution to (1), because it essentially restricts the optimization to the compact set of vectors $\mathbf{x}$ for which $x_n \leq R$, for all n.

Assumption 3. There exists some $R > 0$ such that $p(R) \leq \min_n\{C_n'(0)\}$.

2.1. Optimality and equilibrium conditions

We observe that under Assumptions 1 and 2, the objective function in (1) is concave. Hence, we have the following *necessary and sufficient* conditions for a vector $\mathbf{x}^S$ to achieve the optimal social welfare:

$$\begin{cases} C_n'(x_n^S) = p\left(X^S\right), & \text{if } x_n^S > 0, \\ C_n'(0) \geq p\left(X^S\right), & \text{if } x_n^S = 0, \end{cases} \tag{2}$$

where $X^S = \sum_{n=1}^{N} x_n^S$.

[1] Since a demand function is generally nonincreasing, the convexity of a demand function implies that the corresponding inverse demand function is also convex.

The social optimization problem (1) may admit multiple optimal solutions. However, as we now show, they must all result in the same price. We note that the differentiability of the cost functions is crucial for this result to hold. In this, and in all subsequent results, all of the assumptions that we have introduced will be assumed to be in effect.

Proposition 1. *All optimal solutions to (1) result in the same price.*

There are similar equilibrium conditions for a strategy profile $\mathbf{x}$. In particular, under Assumptions 1 and 2, if $\mathbf{x}$ is a Cournot equilibrium, then

$$C_n'(x_n) \le p(X) + x_n \cdot \partial_- p(X), \quad \text{if } x_n > 0, \tag{3}$$

$$C_n'(x_n) \ge p(X) + x_n \cdot \partial_+ p(X), \tag{4}$$

where again $X = \sum_{n=1}^{N} x_n$. Note, however, that in the absence of further assumptions, the payoff of supplier n need not be a concave function of x_n and these conditions are, in general, not sufficient.

We will say that a nonnegative vector $\mathbf{x}$ is a *Cournot candidate* if it satisfies the necessary conditions (3)–(4). Note that for a given model, the set of Cournot equilibria is a subset of the set of Cournot candidates. Most of the results obtained in this section, including the efficiency lower bound in Proposition 3, apply to all Cournot candidates.

For convex inverse demand functions, the necessary conditions (3)–(4) can be further refined.

Proposition 2. *If $\mathbf{x}$ is a Cournot candidate with $X = \sum_{n=1}^{N} x_n > 0$, then the function p must be differentiable at X, i.e.,*

$$\partial_- p(X) = \partial_+ p(X).$$

Because of the above proposition, when Assumptions 1 and 2 hold, we have the following necessary (and, by definition, sufficient) conditions for a nonzero vector $\mathbf{x}$ to be a Cournot candidate:

$$\begin{cases} C_n'(x_n) = p(X) + x_n p'(X), & \text{if } x_n > 0, \\ C_n'(0) \ge p(X) + x_n p'(X), & \text{if } x_n = 0. \end{cases} \tag{5}$$

2.2. Efficiency of Cournot equilibria

If $p(0) \le \min_n\{C_n'(0)\}$, then the model is uninteresting, because no supplier has an incentive to produce and the optimal social welfare is zero (Friedman, 1977). This motivates the assumption that follows.

Assumption 4. The price at zero supply is larger than the minimum marginal cost of the suppliers, i.e.,

$$p(0) > \min_n\{C_n'(0)\}.$$

We now define the efficiency of a Cournot equilibrium as the ratio of the social welfare that it achieves to the optimal social welfare. It is actually convenient to define the efficiency of a general vector $\mathbf{x}$, not necessarily a Cournot equilibrium.

Definition 2. The *efficiency* of a nonnegative vector $\mathbf{x} = (x_1, \ldots, x_N)$ is defined as

$$\gamma(\mathbf{x}) = \frac{\int_0^X p(q)\,dq - \sum_{n=1}^{N} C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^{N} C_n(x_n^S)}, \tag{6}$$

where $\mathbf{x}^S = (x_1^S, \ldots, x_N^S)$ is an optimal solution of the optimization problem in (1) and $X^S = \sum_{n=1}^{N} x_n^S$.

We note that $\gamma(\mathbf{x})$ is well defined: because of Assumption 4, the denominator on the right-hand side of (6) is guaranteed to be positive. Note that $\gamma(\mathbf{x}) \le 1$ for every nonnegative vector $\mathbf{x}$.

2.3. Restricting to linear cost functions

We next show that in order to study the worst-case efficiency of Cournot equilibria, it suffices to consider linear cost functions.

Proposition 3. *Let $\mathbf{x}$ be a Cournot candidate which is not socially optimal, and let $\alpha_n = C_n'(x_n)$. Consider a modified model in which we replace the cost function of each supplier n by a new function $\overline{C}_n$, defined by*

$$\overline{C}_n(x) = \alpha_n x, \quad \forall\, x \ge 0.$$

Then, for the modified model, Assumptions 1–4 still hold, the vector $\mathbf{x}$ is a Cournot candidate, and its efficiency, denoted by $\overline{\gamma}(\mathbf{x})$, satisfies $0 < \overline{\gamma}(\mathbf{x}) \le \gamma(\mathbf{x})$.

If $\mathbf{x}$ is a Cournot equilibrium, then it satisfies Eqs. (3)–(4), and therefore is a Cournot candidate. Hence, Proposition 3 applies to all Cournot equilibria that are not socially optimal.

We note that even if $\mathbf{x}$ is a Cournot equilibrium in the original model, it need not be a Cournot equilibrium in the modified model with linear cost functions. On the other hand, Proposition 3 asserts that a Cournot candidate in the original model remains a Cournot candidate in the modified model. Hence, to lower bound the efficiency of a Cournot equilibrium in the original model, it suffices to lower bound the efficiency achieved at a worst Cournot candidate for the modified model. Accordingly, and for the purpose of deriving lower bounds, we can (and will) restrict to the case of linear cost functions, and study the worst case efficiency over all Cournot candidates.

2.4. Other properties of Cournot candidates

Proposition 1 shows that all social optima lead to a unique "socially optimal" price. Combining with Proposition 4, below, we may conclude that a Cournot candidate is socially optimal if and only if it results in the socially optimal price.

Proposition 4. *Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and an optimal solution to (1), respectively.*

(a) *If $p(X) \ne p(X^S)$, then $p(X) > p(X^S)$ and $X < X^S$.*
(b) *If $p(X) = p(X^S)$, then $p'(X) = 0$ and $\gamma(\mathbf{x}) = 1$.*

2.5. Concave inverse demand functions

The example that follows shows that if the inverse demand function is concave, then arbitrarily high efficiency losses are possible, even if $X = X^S$ and $p(X) = p(X^S)$; this is to be contrasted to the full efficiency result for the convex case (cf. Proposition 4(b)). For this reason, the study of the concave case would require a very different line of analysis, and will not be pursued further in this paper.

Example 1. Consider a model involving two suppliers ($N = 2$), with $C_1(x) = x$ and $C_2(x) = x^2$. The inverse demand function is concave on the interval where it is positive, of the form

$$p(q) = \begin{cases} 1, & \text{if } 0 \le q \le 1, \\ \max\{0, -M(q-1)+1\}, & \text{if } 1 < q, \end{cases}$$

where $M > 2$. It is not hard to see that the vector $(0.5, 0.5)$ satisfies the optimality conditions in (2), and is therefore socially optimal. We now argue that $(1/M, 1-1/M)$ is a Cournot equilibrium. Given

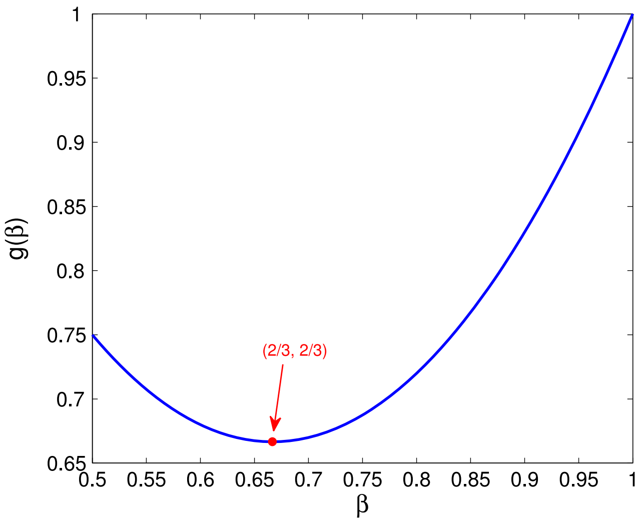

Fig. 1. A tight lower bound on the efficiency of Cournot equilibria for the case of affine inverse demand functions.

the action $x_2 = 1/M$ of supplier 2, any action on the interval $[0, 1 - 1/M]$ is a best response for supplier 1. Given the action $x_1 = 1 - (1/M)$ of supplier 1, a simple calculation shows that

$$\arg\max_{x\in[0,\infty)} \left\{x \cdot p(x + 1 - 1/M) - x^2\right\} = 1/M.$$

Hence, $(1/M, 1 - 1/M)$ is a Cournot equilibrium. Note that $X = X^S = 1$. However, the optimal social welfare is 0.25, while the social welfare achieved at the Cournot equilibrium is $1/M - 1/M^2$. By considering arbitrarily large M, the corresponding efficiency can be made arbitrarily small. □

3. Affine inverse demand functions

We now turn our attention to the special case of affine inverse demand functions. It is already known from Johari and Tsitsiklis (2005) that 2/3 is a tight lower bound on the efficiency of Cournot equilibria. In this section, we refine this result by providing a tighter lower bound, based on a small amount of ex post information about a Cournot equilibrium.

Throughout this section, we assume an inverse demand function of the form

$$p(q) = \begin{cases} b - aq, & \text{if } 0 \le q \le b/a, \\ 0, & \text{if } b/a < q, \end{cases} \tag{7}$$

where a and b are positive constants. Under the assumption of convex costs (Assumption 1), a Cournot equilibrium is guaranteed to exist (Novshek, 1985).

The main result of this section follows.

Theorem 1. *Suppose that the inverse demand function is affine, of the form (7) with $b > \min_n\{C_n'(0)\}$. Let $\mathbf{x}$ be a Cournot equilibrium, and let $\alpha_n = C_n'(x_n)$. Let also*

$$\beta = \frac{aX}{b - \min_n\{\alpha_n\}}.$$

If $X > b/a$, then $\mathbf{x}$ is socially optimal. Otherwise:

(a) *We have $1/2 \le \beta < 1$.*
(b) *The efficiency of $\mathbf{x}$ satisfies,*

$$\gamma(\mathbf{x}) \ge g(\beta) = 3\beta^2 - 4\beta + 2.$$

(c) *The bound in part (b) is tight. That is, for every $\beta \in [1/2, 1)$ and every $\epsilon > 0$, there exists a model with a Cournot equilibrium whose efficiency is no more than $g(\beta) + \epsilon$.*
(d) *The function $g(\beta)$ is minimized at $\beta = 2/3$ and the worst case efficiency is 2/3.*

Theorem 1 is proved in Appendix B.1. The lower bound $g(\beta)$ is illustrated in Fig. 1. Consider a Cournot equilibrium such that $X \le b/a$. For the special case where all the cost functions are linear, of the form $C_n(x_n) = \alpha_n$, Theorem 1 has an interesting interpretation. We first note that a socially optimal solution is obtained when the price $b - aq$ equals the marginal cost of a "best" supplier, namely $\min_n \alpha_n$. In particular, $X^S = (b - \min_n\{\alpha_n\})/a$, and $\beta = X/X^S$. Since $p'(X) = -a < 0$, Proposition 4 implies that $p(X) \ne p(X^S)$ and that $\beta < 1$. Theorem 1 further states that $\beta \ge 1/2$, i.e., the total supply at a Cournot equilibrium is at least half of the socially optimal supply. Clearly, if β is close to 1 we expect the efficiency loss due to the difference $X^S - X$ to be small. However, efficiency losses may also arise if the total supply at a Cournot equilibrium is provided by less efficient suppliers. (As shown in Example 1, in the nonconvex case this effect can be substantial.) Our result shows that, for the convex case, β can be used to lower bound the total efficiency loss due to this second factor as well; when β is close to 1, the efficiency indeed remains close to 1. (This is in sharp contrast to the nonconvex case where we can have $X = X^S$ but large efficiency losses.) Somewhat surprisingly, the worst case efficiency also tends to be somewhat better for low β, that is, when β approaches 1/2, as compared to intermediate values ($\beta \approx 2/3$).

4. Convex inverse demand functions

In this section we study the efficiency of Cournot equilibria under more general assumptions. Instead of restricting the inverse demand function to be affine, we will only assume that it is convex. A Cournot equilibrium need not exist in general, but it does exist under some conditions; see, e.g., Novshek (1985), Gaudet and Salant (1991) and Amir (1996). Our results apply whenever a Cournot equilibrium happens to exist.

We first show that a lower bound on the efficiency of a Cournot equilibrium can be established by calculating its efficiency in another model with a carefully chosen piecewise linear inverse demand function of a special kind, which preserves the first order necessary conditions at the Cournot equilibrium $\mathbf{x}$ (of the original model), as well as the optimality of the original social optimum. Then, in Theorem 2, we establish a lower bound on the efficiency of Cournot equilibria, as a function of the ratio of the slope of the inverse demand function at the Cournot equilibrium to the average slope of the inverse demand function between the Cournot equilibrium and a socially optimal point. Subsequently, in Section 5, we will apply Theorem 2 to specific convex inverse demand functions. Recall our definition of a Cournot candidate as a vector $\mathbf{x}$ that satisfies the necessary conditions (3)–(4).

Proposition 5. *Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and an optimal solution to (1), respectively. Assume that $p(X) \ne p(X^S)$ and let*[2] *$c = |p'(X)|$. Consider a modified model in which we replace the inverse demand function by a new function p^0, defined by*

$$p^0(q) = \begin{cases} -c(q - X) + p(X), & \text{if } 0 \le q \le X, \\ \max\left\{0, \dfrac{p(X^S) - p(X)}{X^S - X} \right. & \\ \quad \left. \times\ (q - X) + p(X)\right\}, & \text{if } X < q. \end{cases} \tag{8}$$

[2] According to Proposition 2, $p'(X)$ must exist.

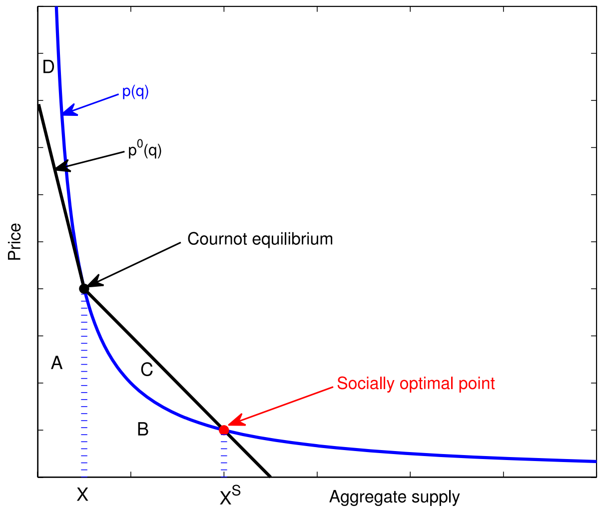

Fig. 2. The efficiency of a Cournot equilibrium cannot increase if we replace the inverse demand function by the piecewise linear function p^0. The function p^0 is tangent to the inverse demand function p at the equilibrium point, and connects the Cournot equilibrium point with the socially optimal point.

Then, for the modified model, with inverse demand function p^0, the vector $\mathbf{x}^S$ remains socially optimal, and the efficiency of $\mathbf{x}$, denoted by $\gamma^0(\mathbf{x})$, satisfies

$$\gamma^0(\mathbf{x}) \le \gamma(\mathbf{x}).$$

Proof. Since $p(X) \neq p(X^S)$, Proposition 4(a) implies that $X < X^S$, so that p^0 is well defined. Since the necessary and sufficient optimality conditions in (2) only involve the value of the inverse demand function at X^S, which has been unchanged, the vector $\mathbf{x}^S$ remains socially optimal for the modified model.

Let

$$A = \int_0^X p^0(q)\, dq, \qquad B = \int_X^{X^S} p(q)\, dq,$$

and

$$C = \int_X^{X^S} (p^0(q) - p(q))\, dq, \qquad D = \int_0^X (p(q) - p^0(q))\, dq.$$

See Fig. 2 for an illustration of p and a graphical interpretation of A, B, C, D. Note that since p is convex, we have $C \ge 0$ and $D \ge 0$. The efficiency of $\mathbf{x}$ in the original model with inverse demand function p, is

$$0 < \gamma(\mathbf{x}) = \frac{A + D - \sum_{n=1}^N C_n(x_n)}{A + B + D - \sum_{n=1}^N C_n(x_n^S)} \le 1,$$

where the first inequality is true because the social welfare achieved at any Cournot candidate is positive. The efficiency of $\mathbf{x}$ in the modified model is

$$\gamma^0(\mathbf{x}) = \frac{A - \sum_{n=1}^N C_n(x_n)}{A + B + C - \sum_{n=1}^N C_n(x_n^S)}.$$

Note that the denominators in the above formulas for $\gamma(\mathbf{x})$ and $\gamma^0(\mathbf{x})$ are all positive.

If $A - \sum_{n=1}^N C_n(x_n) \le 0$, then $\gamma^0(\mathbf{x}) \le 0$ and the result is clearly true. We can therefore assume that $A - \sum_{n=1}^N C_n(x_n) > 0$. We then have

$$0 < \gamma^0(\mathbf{x}) = \frac{A - \sum_{n=1}^N C_n(x_n)}{A + B + C - \sum_{n=1}^N C_n(x_n^S)}$$

$$\le \frac{A + D - \sum_{n=1}^N C_n(x_n)}{A + B + C + D - \sum_{n=1}^N C_n(x_n^S)}$$

$$\le \frac{A + D - \sum_{n=1}^N C_n(x_n)}{A + B + D - \sum_{n=1}^N C_n(x_n^S)} = \gamma(\mathbf{x}) \le 1,$$

which proves the desired result. □

Note that unless p happens to be linear on the interval $[X, X^S]$, the function p^0 is not differentiable at X and, according to Proposition 2, $\mathbf{x}$ cannot be a Cournot candidate for the modified model. Nevertheless, because p^0 preserves the first order condition at a Cournot equilibrium $\mathbf{x}$ of the original model, it can be used to derive a lower bound on the efficiency of Cournot candidates, as will be seen in the proof of Theorem 2, which is our main result.

Theorem 2. *Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and a solution to (1), respectively. Then, the following hold.*

(a) *If $p(X) = p(X^S)$, then $\gamma(\mathbf{x}) = 1$.*
(b) *If $p(X) \neq p(X^S)$, let $c = |p'(X)|$, $d = |(p(X^S) - p(X))/(X^S - X)|$, and $\overline{c} = c/d$. We have $\overline{c} \ge 1$ and*

$$f(\overline{c}) = \frac{\phi^2 + 2}{\phi^2 + 2\phi + \overline{c}} \le \gamma(\mathbf{x}) < 1, \tag{9}$$

where

$$\phi = \max\left\{\frac{2 - \overline{c} + \sqrt{\overline{c}^2 - 4\overline{c} + 12}}{2}, 1\right\}.$$

Remark 1. We do not know whether the lower bound in Theorem 2 is tight. The difficulty in proving tightness is due to the fact that the vector $\mathbf{x}$ need not be a Cournot equilibrium in the modified model.

We provide the proof of Theorem 2 in Appendix B.2. Fig. 3 shows a plot of the lower bound $f(\overline{c})$ on the efficiency of Cournot equilibria, as a function of $\overline{c} = c/d$. If p is affine, then $\overline{c} = c/d = 1$. From (9), it can be verified that $f(1) = 2/3$. We note that the lower bound $f(\overline{c})$ is monotonically decreasing in $\overline{c}$, over the domain $[1, \infty)$. When $\overline{c} \in [1, 3)$, ϕ is at least 1, and monotonically decreasing in $\overline{c}$. When $\overline{c} \ge 3$, $\phi = 1$.

5. Applications

For a given inverse demand function p, the lower bound derived in Theorem 2 requires some knowledge on the Cournot candidate and the social optimum, namely, the aggregate supplies X and X^S. Even so, for a large class of inverse demand functions, we can apply Theorem 2 to establish lower bounds on the efficiency of Cournot equilibria that do not require knowledge of X and X^S. With additional information on the suppliers' cost functions, the lower bounds can be further refined. At the end of this section, we apply our results to calculate nontrivial quantitative efficiency bounds for various convex inverse demand functions that have been considered in the economics literature.

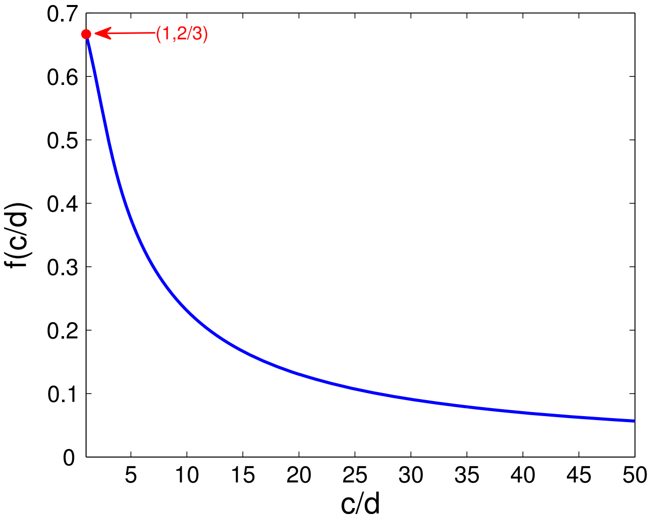

Fig. 3. Plot of the lower bound on the efficiency of a Cournot equilibrium in a Cournot oligopoly with convex inverse demand functions, as a function of the ratio c/d (cf. Theorem 2). The special case of affine demand functions corresponds to $c/d = 1$ and $f(c/d) = 2/3$.

Corollary 1. *Suppose that $p(Q) = 0$ for some $Q > 0$, and that the ratio $\mu = \partial_+ p(0)/\partial_- p(Q)$ is finite. Then, the efficiency of a Cournot candidate is at least $f(\mu)$.*

The proof of Corollary 1 can be found in Appendix B.3. For some convex inverse demand functions, e.g., for negative exponential demand, with

$$p(q) = \max\{0, \alpha - \beta \log q\}, \quad 0 < \alpha,\ 0 < \beta,\ 0 \le q,$$

Corollary 1 does not apply, because of the singularity at $q = 0$. On the other hand, we can still derive an upper bound on the total supply at a social optimum, a lower bound on the total supply at a Cournot equilibrium, and a strengthened version of Corollary 1.

Corollary 2. *Let*[3]

$$\begin{aligned} s &= \inf\{q \mid p(q) = \min_n C_n'(0)\}, \\ t &= \inf\left\{q \mid \min_n C_n'(q) \ge p(q) + q\partial_+ p(q)\right\}. \end{aligned} \tag{10}$$

If $\partial_- p(s) < 0$, then the efficiency of a Cournot candidate is at least

$$f\left(\partial_+ p(t)/\partial_- p(s)\right).$$

The result continues to hold even if we relax our assumptions, to allow $p(0)$ to be infinite.

Remark 2. If there exists a "best" supplier n such that $C_n'(x) \le C_m'(x)$, for any other supplier m and any $x > 0$, then the parameters s and t depend only on p and C_n'.

Corollary 2 is proved in Appendix B.4. We now apply Corollary 2 to three examples.

Example 2. Suppose that there is a best supplier, whose cost function is linear with a slope $c \ge 0$. Consider inverse demand functions of the form (cf. Eq. (6) in Bulow and Pfleiderer (1983))

$$p(q) = \max\{0, \alpha - \beta \log q\}, \quad 0 < q, \tag{11}$$

where α and β are positive constants. Note that Corollary 1 does not apply, because the right derivative of p at 0 is infinite.[4] Since

$$p'(q) + qp''(q) = \frac{-\beta}{q} + \frac{q\beta}{q^2} = 0, \quad \forall q \in (0, \exp(\alpha/\beta)),$$

there exists a Cournot equilibrium (Novshek, 1985). Through a simple calculation we obtain

$$s = \exp\left(\frac{\alpha - c}{\beta}\right), \qquad t = \exp\left(\frac{\alpha - \beta - c}{\beta}\right).$$

From Corollary 2 we obtain that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \ge f\left(\frac{\exp((\alpha - c)/\beta)}{\exp((\alpha - \beta - c)/\beta)}\right) = f(\exp(1)) \ge 0.5237. \tag{12}$$

Now we argue that the efficiency lower bound (12) holds even without the assumption that there is a best supplier associated with a linear cost function. From Proposition 3, the efficiency of any Cournot equilibrium $\mathbf{x}$ will not increase if the cost function of each supplier n is replaced by

$$\overline{C}_n(x) = C_n'(x_n)x, \quad \forall x \ge 0.$$

Let $c = \min_n\{C_n'(x_n)\}$. Since the efficiency lower bound in (12) holds for the modified model with linear cost functions, it also applies whenever the inverse demand function is of the form (11). □

Example 3. Suppose that there is a best supplier, whose cost function is linear with a slope $c \ge 0$. Consider inverse demand functions of the form (cf. Eq. (5) in Bulow and Pfleiderer (1983))

$$p(q) = \max\{\alpha - \beta q^\delta, 0\}, \quad 0 < \delta \le 1, \tag{13}$$

where α and β are positive constants. Note that if $\delta = 1$, then p is affine; if $0 < \delta \le 1$, then p is convex. Assumption 4 implies that $\alpha > c$. Since

$$\begin{aligned} p'(q) + qp''(q) &= -\beta\delta q^{\delta-1} - \beta\delta(\delta - 1)q^{\delta-1} \\ &= -\beta\delta^2 q^{\delta-1} \le 0, \quad 0 \le q \le \left(\frac{\alpha}{\beta}\right)^{1/\delta}, \end{aligned}$$

there exists a Cournot equilibrium (Novshek, 1985). Through a simple calculation we have

$$s = \left(\frac{\alpha - c}{\beta}\right)^{1/\delta}, \qquad t = \left(\frac{\alpha - c}{\beta(\delta + 1)}\right)^{1/\delta}.$$

From Corollary 2 we know that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \ge f\left(\frac{-\beta\delta t^{\delta-1}}{-\beta\delta s^{\delta-1}}\right) = f\left((\delta + 1)^{\frac{1-\delta}{\delta}}\right).$$

Using the argument in Example 2, we conclude that this lower bound also applies to the case of general convex cost functions. □

As we will see in the following example, it is sometimes hard to find a closed form expression for the real number t. In such cases, we can still use the fact that s is an upper bound for the aggregate supply at a social optimum and argue as in the proof of Corollary 2 (in Appendix B.4) to conclude that the efficiency of a Cournot candidate is at least $f(\partial_+ p(0)/\partial_- p(s))$. Furthermore, in terms of the aggregate supply at a Cournot equilibrium X, we know that $\gamma(\mathbf{x}) \ge f\left(p'(X)/\partial_- p(s)\right)$.

[3] Under Assumption 3, the existence of the real numbers defined in (10) is guaranteed.

[4] In fact, $p(0)$ is infinite. This turns out to not be an issue: for a small enough $\epsilon > 0$, we can guarantee that no supplier chooses a quantity below ϵ. Furthermore, $\lim_{\epsilon \downarrow 0} \int_0^\epsilon p(q)\, dq = 0$. For this reason, the details of the inverse demand function in the vicinity of zero are immaterial as far as the chosen quantities or the resulting social welfare are concerned.

Example 4. Suppose that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. p. 8 in Fabinger and Weyl (2009))

$$p(q) = \begin{cases} \alpha\,(Q-q)^{\beta}, & 0 < q \leq Q, \\ 0, & Q < q, \end{cases} \tag{14}$$

where $Q > 0, \alpha > 0$ and $\beta \geq 1$. Assumption 4 implies that $c < \alpha Q^{\beta}$. Note that when $\beta > 1$, Corollary 1 does not apply, because the left derivative of p at Q is zero. Through a simple calculation we obtain

$$s = Q - \left(\frac{c}{\alpha}\right)^{1/\beta},$$

and

$$p'(s) = \alpha\beta\left(\frac{c}{\alpha}\right)^{(\beta-1)/\beta}, \qquad \partial_+ p(0) = \alpha\beta Q^{\beta-1}.$$

Corollary 2 implies that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{\partial_+ p(0)}{p'(s)}\right) = f\left(\left(\frac{\alpha Q^{\beta}}{c}\right)^{(\beta-1)/\beta}\right) = f\left(\left(\frac{p(0)}{c}\right)^{(\beta-1)/\beta}\right).$$

Using information on the aggregate demand at the equilibrium, the efficiency bound can be refined. Since

$$p'(X) = \alpha\beta(Q-X)^{\beta-1},$$

we have

$$\gamma(\mathbf{x}) \geq f\left(\frac{p'(X)}{p'(s)}\right) = f\left(\left(\frac{\alpha(Q-X)^{\beta}}{c}\right)^{(\beta-1)/\beta}\right) = f\left(\left(\frac{p(X)}{c}\right)^{(\beta-1)/\beta}\right), \tag{15}$$

so that the efficiency bound depends only on the ratio of the equilibrium price to the marginal cost of the best supplier, and the parameter β. For affine inverse demand functions, we have $\beta = 1$ and the bound in (15) equals $f(1) = 2/3$, which agrees with Theorem 1. □

6. Monopoly and social welfare

In this section we study the special case where $N = 1$, so that we are dealing with a single monopolistic supplier. As we explain, this case also covers a setting where multiple suppliers collude to maximize their total profit. By using the additional assumption that $N = 1$, we obtain a sharper (i.e., larger) lower bound, in Theorem 3. We then establish lower bounds on the efficiency of monopoly outputs that do not require knowledge of X and X^S.

In a Cournot oligopoly, the maximum possible profit earned by all suppliers (if they collude) is an optimal solution to the following optimization problem,

$$\begin{aligned} \text{maximize} \quad & p\left(\sum_{n=1}^{N} x_n\right) \cdot \sum_{n=1}^{N} x_n - \sum_{n=1}^{N} C_n(x_n) \\ \text{subject to} \quad & x_n \geq 0, \quad n = 1, \ldots, N. \end{aligned} \tag{16}$$

We use $\mathbf{x}^P = (x_1^P, \ldots, x_N^P)$ to denote an optimal solution to (16) (a *monopoly output*), and let $X^P = \sum_{n=1}^{N} x_n^P$.

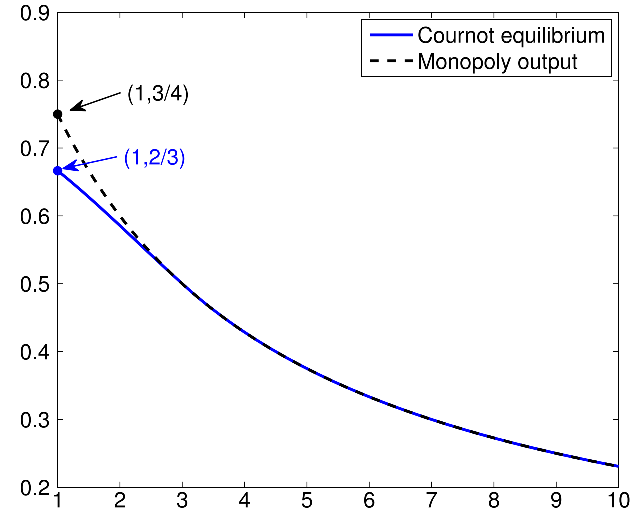

Fig. 4. Comparison of the lower bounds on the efficiency of Cournot equilibria and monopoly outputs for the case of convex inverse demand functions.

It is not hard to see that the aggregate supply at a monopoly output, X^P, is also a Cournot equilibrium in a modified model with a single supplier ($N = 1$) and a cost function $\overline{C}(X)$, defined as the optimal objective value in the optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{n=1}^{N} C_n(x_n), \\ \text{subject to} \quad & \sum_{n=1}^{N} x_n = X \\ & x_n \geq 0, \quad n = 1, \ldots, N. \end{aligned} \tag{17}$$

Note that $\overline{C}$ is convex (linear) when the C_n are convex (respectively, linear). Furthermore, the social welfare at the monopoly output $\mathbf{x}^P$, is the same as that achieved at the Cournot equilibrium, $x_1 = X^P$, in the modified model. Also, the socially optimal value of X, as well as the resulting social welfare is the same for the N-supplier model and the above defined modified model with $N = 1$. Therefore, the efficiency of the monopoly output equals the efficiency of the Cournot equilibrium of the modified model. To lower bound the efficiency of monopoly outputs resulting from multiple colluding suppliers, we can (and will) restrict to the case with $N = 1$.

Theorem 3. *Let $\mathbf{x}^S$ and $\mathbf{x}^P$ be a social optimum and a monopoly output, respectively. Then, the following hold.*

(a) *If $p(X^P) = p(X^S)$, then $\gamma(\mathbf{x}^P) = 1$.*
(b) *If $p(X^P) \neq p(X^S)$, let $c = |p'(X^P)|$, $d = |(p(X^S) - p(X^P))/(X^S - X^P)|$, and $\overline{c} = c/d$. We have $\overline{c} \geq 1$ and*

$$\gamma(\mathbf{x}^P) \geq \frac{3}{3+\overline{c}}. \tag{18}$$

(c) *The bound is tight at $\overline{c} = 1$, i.e., there exists a model with $\overline{c} = 1$ and a monopoly output whose efficiency is $3/4$.*

The proof for Theorem 3 can be found in Appendix C.1. Fig. 4 compares the efficiency lower bounds established for Cournot equilibria with that for monopoly outputs. For $\overline{c} = 1$, both efficiency bounds are tight, and it is possible for a monopoly output to achieve a higher efficiency than that of a Cournot equilibrium. This agrees with the observation in earlier works that increased competition may reduce social welfare (Comanor and Leibenstein, 1969; Crew and Rowley, 1971; Lahiria and Ono, 1988).

7. Conclusion

It is well known that Cournot oligopoly can yield arbitrarily high efficiency loss in general; for details, see Johari (2004). For a Cournot oligopoly with convex market demand and cost functions, results such as those provided in Theorem 2 show that the efficiency loss of a Cournot equilibrium can be bounded away from zero by a function of a scalar parameter that captures quantitative properties of the inverse demand function. With additional information on the cost functions, the efficiency lower bounds can be further refined. Our results apply to various convex inverse demand functions that have been considered in the economics literature.

Acknowledgments

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Appendix A. Proofs of the results in Section 2

A.1. Proof of Proposition 1

Suppose that the result is false so that there exist two optimal solutions, $\mathbf{x}^S$ and $\overline{\mathbf{x}}^S$, such that $p(X^S) \neq p(\overline{X}^S)$. Without loss of generality, we assume that $p(X^S) > p(\overline{X}^S)$. Since p is nonincreasing, we must have $X^S < \overline{X}^S$. For all n such that $\overline{x}_n^S > 0$, the optimality conditions (2) yield

$$C_n'(\overline{x}_n^S) = p(\overline{X}^S) < p(X^S) \le C_n'(x_n^S).$$

Using the convexity of the cost functions, we obtain

$$\overline{x}_n^S < x_n^S, \quad \text{if } \overline{x}_n^S > 0.$$

This contradicts the assumption that $X^S < \overline{X}^S$, and the desired result follows.

A.2. Proof of Proposition 2

Let $\mathbf{x}$ be a Cournot candidate with $X > 0$. The conditions (3)–(4) applied to some n with $x_n > 0$, imply that

$$p(X) + x_n \cdot \partial_- p(X) \ge p(X) + x_n \cdot \partial_+ p(X).$$

On the other hand, since p is convex, we have $\partial_- p(X) \le \partial_+ p(X)$. Hence, $\partial_- p(X) = \partial_+ p(X)$, as claimed.

A.3. Proof of Proposition 3

We first observe that the vector $\mathbf{x}$ satisfies the necessary conditions (3)–(4) for the modified model. Hence, the vector $\mathbf{x}$ is a Cournot candidate for the modified model. It is also not hard to see that Assumptions 1 and 2 are satisfied by the modified model. Since $\alpha_n \ge C_n'(0)$ for every n, Assumption 3 also holds in the modified model.

We now show that Assumption 4 holds in the modified model, i.e., that $p(0) > \min_n\{\alpha_n\}$. Since the vector $\mathbf{x}$ is a Cournot candidate in the original model, we have $X > 0$, so that there exists some n for which $x_n > 0$. From the necessary condition (3) we have that $\alpha_n \le p(X)$. Furthermore, if $\alpha_n = p(X)$, then $\partial_- p(X) = 0$, and the convexity of p implies that $\partial_+ p(X) = 0$. Hence, the vector $\mathbf{x}$ satisfies the optimality condition (2), and thus is socially optimal in the original model. Under our assumption that $\mathbf{x}$ is not socially optimal in the original model, we conclude that $\alpha_n < p(X)$, which implies that Assumption 4 holds in the modified model.

Let $\mathbf{x}^S$ be an optimal solution to (1) in the original model. Since $\mathbf{x}^S$ satisfies the optimality conditions in (2) for the modified model, it remains a social optimum in the modified model. In the modified model, since Assumptions 1–4 hold, the efficiency of the vector $\mathbf{x}$ is well defined and given by

$$\overline{\gamma}(\mathbf{x}) = \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N \alpha_n x_n}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \alpha_n x_n^S}. \tag{A.1}$$

Note that the denominator on the right-hand side of (A.1) is the optimal social welfare and the numerator is the social welfare achieved at the Cournot candidate $\mathbf{x}$, in the modified model. Note that both the denominator and the numerator on the right-hand side of (A.1) are positive. In particular, $\overline{\gamma}(\mathbf{x}) > 0$.

Since C_n is convex, we have

$$C_n(x_n^S) - C_n(x_n) - \alpha_n(x_n^S - x_n) \ge 0, \quad n = 1, \ldots, N.$$

Adding a nonnegative quantity to the denominator cannot increase the ratio and, therefore,

$$\begin{aligned} 1 \ge \gamma(\mathbf{x}) &= \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N C_n(x_n^S)} \\ &\ge \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right)} > 0. \end{aligned} \tag{A.2}$$

Since C_n is convex and nondecreasing, with $C_n(0) = 0$, we also have

$$A \triangleq \sum_{n=1}^N C_n(x_n) - \sum_{n=1}^N \alpha_n x_n \le 0. \tag{A.3}$$

Since the right-hand side of (A.2) is in the interval $(0, 1]$, adding the left-hand side of Eq. (A.3) (a nonpositive quantity) to both the numerator and the denominator cannot increase the ratio, as long as the numerator remains nonnegative. The numerator remains indeed nonnegative because it becomes the same as the numerator in the expression (A.1) for $\overline{\gamma}(\mathbf{x})$. We obtain

$$\begin{aligned} \gamma(\mathbf{x}) &\ge \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right)} \\ &\ge \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n) + A}{\int_0^{X^S} p(q)dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right) + A} \\ &= \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N \alpha_n x_n}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \alpha_n x_n^S} \\ &= \overline{\gamma}(\mathbf{x}), \end{aligned}$$

where A is the nonpositive constant defined in (A.3). The desired result follows.

A.4. Proof of Proposition 4(a)

By Assumption 4, $p(0) > \min_n\{C_n'(0)\}$. We also have $X > 0$. Since p is nonincreasing, the conditions in (3) imply that

$$C_n'(x_n) \le p(X), \quad \text{if } x_n > 0.$$

Suppose that $X \ge X^S$. Since the inverse demand function is nonincreasing and $p(X) \ne p(X^S)$, we have $p(X) < p(X^S)$ and $X > X^S$. For every supplier n with $x_n > 0$, we have

$$C_n'(x_n) \le p(X) < p(X^S) \le C_n'(x_n^S),$$

where the last inequality follows from (2). Since, C_n is convex, the above inequality implies that

$$x_n < x_n^S, \quad \text{if } x_n > 0,$$

from which we obtain $X < X^S$. Since we had assumed that $X \ge X^S$, we have a contradiction.

The preceding argument establishes that $X < X^S$. Since p is nonincreasing and $p(X) \ne p(X^S)$, we must have $p(X) > p(X^S)$.

A.5. Proof of Proposition 4(b)

Since Assumption 4 holds, we have $X > 0$. Since p is convex, Proposition 2 shows that p is differentiable at X and the necessary conditions in (5) are satisfied.

We will now prove that $p'(X) = 0$. Suppose not, in which case we have $p'(X) < 0$. For every n such that $x_n > 0$, from the convexity of C_n and the conditions in (5), we have

$$C_n'(0) \le C_n'(x_n) < p(X) = p(X^S).$$

Then, the social optimality conditions (2) imply that $x_n^S > 0$. It follows that

$$C_n'(x_n) < p(X^S) = C_n'(x_n^S),$$

where the last equality follows from the optimality conditions in (2). Since C_n is convex, we conclude that $x_n < x_n^S$. Since this is true for every n such that $x_n > 0$, we obtain $X < X^S$. Since the function p is nonincreasing, and we have $p'(X) < 0$ and $X < X^S$, we obtain $p(X) > p(X^S)$, which contradicts the assumption that $p(X) = p(X^S)$. The contradiction shows that $p'(X) = 0$.

Since $p'(X) = 0$ and the Cournot candidate $\mathbf{x}$ satisfies the necessary conditions in (5), it also satisfies the optimality conditions in (2). Hence, $\mathbf{x}$ is socially optimal and the desired result follows.

Appendix B. Proofs of the results in Sections 3–5

B.1. Proof of Theorem 1

We note that part (d) is an immediate consequence of the expression for $g(\beta)$, and we concentrate on the remaining parts. Since the inverse demand function is convex, Proposition 2 shows that any Cournot equilibrium satisfies the necessary conditions (5). If $X > b/a$, then $p(X) = p'(X) = 0$. In that case, the necessary conditions (5) imply the optimality conditions (2). We conclude that $\mathbf{x}$ is socially optimal.

We now assume that $X \le b/a$. Proposition 2 shows that $p'(X)$ exists, and thus $X < b/a$. Since $p'(X) = -a < 0$, Proposition 4 implies that $p(X) \ne p(X^S)$, for any social optimum $\mathbf{x}^S$. Hence, $\mathbf{x}$ is not socially optimal.

As discussed in Section 2.3, to derive a lower bound, it suffices to consider the case of linear cost functions, and obtain a lower bound on the worst case efficiency of Cournot candidates, that is, vectors that satisfy (3)–(4). We will therefore assume that $C_n(x_n) = \alpha_n x_n$ for every n. Without loss of generality, we also assume that $\alpha_1 = \min_n\{\alpha_n\}$. We consider separately the two cases where $\alpha_1 = 0$ or $\alpha_1 > 0$, respectively.

The case where $\alpha_1 = 0$

In this case, the socially optimal supply is $X^S = b/a$ and the optimal social welfare is

$$\int_0^{b/a} p(q)\,dq - 0 = \int_0^{b/a} (-ax + b)\,dx = \frac{b^2}{2a}.$$

Note also that $\beta = aX/b$.

Let $\mathbf{x}$ be a Cournot candidate. Suppose first that $x_1 = 0$. In that case, the necessary conditions $0 = \alpha_1 \ge p(X)$ imply that $p(X) = 0$. For $n \ne 1$, if $x_n > 0$, the necessary conditions yield $0 \le \alpha_n = p(X) - x_n a = -x_n a$, which implies that $x_n = 0$ for all n. But then, $X = 0$, which contradicts the fact $p(X) = 0$. We conclude that $x_1 > 0$.

Since $x_1 > 0$, the necessary conditions (5) yield $0 = \alpha_1 = b - aX - ax_1$, so that

$$x_1 = -X + \frac{b}{a}. \tag{B.1}$$

In particular, $X < b/a = X^S$, and $\beta < 1$. Furthermore,

$$0 \le \sum_{n=2}^{N} x_n = X - x_1 = 2X - \frac{b}{a},$$

from which we conclude that $\beta = aX/b \ge 1/2$.

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \ne 1$, whenever $x_n > 0$, we have $\alpha_n = p(X) - ax_n$, so that $\alpha_n x_n = (p(X) - ax_n)x_n$. The social welfare associated with $\mathbf{x}$ is

$$\begin{aligned}
&\int_0^{X} p(q)\,dq - \sum_{n=1}^{N} \alpha_n x_n \\
&= bX - \frac{1}{2}aX^2 - \sum_{n=2}^{N}(p(X) - ax_n)x_n \\
&\ge bX - \frac{1}{2}aX^2 - p(X)\sum_{n=2}^{N} x_n \\
&= bX - \frac{1}{2}aX^2 - (b - aX)(X - x_1) \\
&= bX - \frac{1}{2}aX^2 - (b - aX)\left(2X - \frac{b}{a}\right) \\
&= \frac{3}{2}aX^2 + \frac{b^2}{a} - 2bX.
\end{aligned} \tag{B.2}$$

We divide by $b^2/2a$ (the optimal social welfare) and obtain

$$\gamma(X) \ge \frac{2a}{b^2}\left(\frac{3}{2}aX^2 + \frac{b^2}{a} - 2bX\right) = 3\beta^2 - 4\beta + 2.$$

This proves the claim in part (b) of the theorem.

Tightness

We observe that the lower bound on the social welfare associated with $\mathbf{x}$ made use, in Eq. (B.2), of the inequality $\sum_{n=2}^{N} x_n^2 \ge 0$. This inequality becomes an equality, asymptotically, if we let $N \to \infty$ and $x_n = O(1/N)$ for $n \ne 1$. This motivates the proof of tightness (part (c) of the theorem) given below.

We are given some $\beta \in [1/2, 1)$ and construct an N-supplier model ($N \ge 2$) with $a = b = 1$, and the following linear cost functions:

$$C_1^N(x_1) = 0,$$

$$C_n^N(x_n) = \left(p(X) - \frac{2X - 1}{N - 1}\right)x_n, \quad n = 2, \ldots, N.$$

It can be verified that the variables

$$x_1 = -X + b/a, \qquad x_n = \frac{2X - b/a}{N-1}, \quad n = 2, \dots, N,$$

form a Cournot equilibrium. A simple calculation (consistent with the intuition given earlier) shows that as N increases to infinity, the sum $\sum_{n=2}^{N} x_n^2$ goes to zero and the associated efficiency converges to $g(\beta)$.

The case where $\alpha_1 > 0$

We now consider the case where $\alpha_n > 0$ for every n. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n\{\alpha_n\} = \alpha_1 = 1$. By Assumption 4, we have $b > 1$.

At the social optimum, we must have $p(X^S) = \alpha_1 = 1$ and thus $X^S = (b-1)/a$. The optimal social welfare is

$$\frac{(b-1)(b+1)}{2a} - \frac{b-1}{a} = \frac{(b-1)^2}{2a}.$$

Note also that $\beta = aX/(b-1)$.

Similar to the proof for the case where $\alpha_1 = 0$, we can show that $x_1 > 0$ and therefore $1 = p(X) - ax_1 = b - aX - ax_1$, so that

$$x_1 = -X + \frac{b-1}{a} > 0,$$

which implies that $\beta < 1$. In particular,

$$X < \frac{b-1}{a} = X^S.$$

Furthermore,

$$0 \le \sum_{n=2}^{N} x_n = X - x_1 = 2X - \frac{b-1}{a},$$

from which we conclude that $\beta = aX/(b-1) \ge 1/2$.

A calculation similar to the one for the case where $\alpha_1 = 0$ yields

$$\begin{aligned}
&\int_0^X p(q)\,dq - \sum_{n=1}^{N} \alpha_n x_n \\
&= bX - \frac{1}{2}aX^2 - x_1 - \sum_{n=2}^{N}(p(X) - ax_n)x_n \\
&\ge bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - p(X)\sum_{n=2}^{N} x_n \\
&= bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - (b - aX)(X - x_1) \\
&= bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - (b - aX)\left(2X - \frac{b-1}{a}\right) \\
&= \frac{3}{2}aX^2 + \frac{(b-1)^2}{a} - 2(b-1)X.
\end{aligned}$$

After dividing with the value of the social welfare, we obtain $g(\beta)$, as desired.

B.2. Proof of Theorem 2

Let $\mathbf{x}$ be a Cournot candidate. According to Proposition 4(b), if $p(X) = p(X^S)$, then the efficiency of the Cournot candidate must equal one, which proves part (a). To prove part (b), we assume that $p(X) \ne p(X^S)$. By Proposition 1, the Cournot candidate $\mathbf{x}$ cannot be socially optimal, and, therefore, $\gamma(\mathbf{x}) < 1$.

We have shown in Proposition 3 that if all cost functions are replaced by linear ones, the vector $\mathbf{x}$ remains a Cournot candidate, and Assumptions 1–4 still hold. Further, the efficiency of $\mathbf{x}$ cannot increase after all cost functions are replaced by linear ones. Thus, to lower bound the worst case efficiency loss, it suffices to derive a lower bound for the efficiency of Cournot candidates for the case of linear cost functions. We therefore assume that $C_n(x_n) = \alpha_n x_n$ for each n. Without loss of generality, we further assume that $\alpha_1 = \min_n\{\alpha_n\}$. Note that, by Assumption 4, we have $p(0) > \alpha_1$. We will prove the theorem by considering separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$.

We will rely on Proposition 5, according to which the efficiency of a Cournot candidate $\mathbf{x}$ is lower bounded by the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in a model involving the piecewise linear and convex inverse demand function of the form in the definition of p^0. Note that since $p(X) \ne p(X^S)$, we have that $d > 0$. For conciseness, we let $y = p(X)$ throughout the proof.

The case $\alpha_1 = 0$

Let $\mathbf{x}$ be a Cournot candidate in the original model with linear cost functions and the inverse demand function p. By Proposition 2, $\mathbf{x}$ satisfies the necessary conditions (5), with respect to the original inverse demand function p. Suppose first that $x_1 = 0$. The second inequality in (5) implies that $p(X) = 0$. On the other hand, Assumption 4 implies that $X > 0$. Thus, there exists some n such that $x_n > 0$. The first equality in (5) yields,

$$0 \le \alpha_n = p(X) + x_n p'(X) = x_n p'(X) \le 0,$$

which implies that $p'(X) = 0$. Then, the vector $\mathbf{x}$ satisfies the optimality conditions in (2), and is thus socially optimal in the original model. This contradicts the fact that $p(X) \ne p(X^S)$ and shows that we must have $x_1 > 0$.

If $p'(X)$ were equal to zero, then the necessary conditions (5) would imply the optimality conditions (2), and $\mathbf{x}$ would be socially optimal in the original model. Hence, we must have $p'(X) < 0$ and $c > 0$. The first equality in (5) yields $y > 0$, $x_1 = y/c$, and $X \ge y/c$. We also have

$$0 \le \sum_{n=2}^{N} x_n = X - \frac{y}{c}. \tag{B.3}$$

From Proposition 5, the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in the modified model cannot be more than its efficiency $\gamma(\mathbf{x})$ in the original model. Hence, to prove the second part of the theorem, it suffices to show that $\gamma^0(\mathbf{x}) \ge f(\overline{c})$, for any Cournot candidate with $c/d = \overline{c}$.

The optimal social welfare in the modified model is

$$\begin{aligned}
\int_0^\infty p^0(q)dq - 0 &= \int_0^{X+y/d} p^0(q)dq - 0 \\
&= \frac{y^2}{2d} + \frac{(2y + cX)X}{2}.
\end{aligned} \tag{B.4}$$

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \ge 2$, whenever $x_n > 0$, from the first equality in (5) we have $\alpha_n = y - x_n c$ and $\alpha_n x_n = (y - x_n c)x_n$. Hence, in the modified model, the social welfare associated with $\mathbf{x}$ is

$$\begin{aligned}
\int_0^X p^0(q)dq - \sum_{n=1}^{N} \alpha_n x_n &= \frac{(2y + cX)X}{2} - \sum_{n=2}^{N}(y - x_n c)x_n \\
&\ge \frac{(2y + cX)X}{2} - y\sum_{n=2}^{N} x_n \\
&= \frac{(2y + cX)X}{2} - y(X - y/c) \\
&= cX^2/2 + y^2/c.
\end{aligned}$$

Therefore,

$$\gamma^0(\mathbf{x}) \ge \frac{cX^2/2 + y^2/c}{y^2/(2d) + (2y + cX)X/2}. \tag{B.5}$$

Note that c, d, X, and y are all positive. Substituting $\bar{c} = c/d$ and $\bar{y} = cX/y$ in (B.5), we obtain

$$\begin{aligned}\gamma^0(\mathbf{x}) &\geq \frac{cX^2/2 + y^2/c}{y^2/(2d) + (2y + cX)X/2} \\ &= \frac{c^2X^2/2 + y^2}{y^2c/(2d) + cXy + c^2X^2/2} \\ &= \frac{\bar{y}^2 + 2}{\bar{c} + 2\bar{y} + \bar{y}^2}. \end{aligned} \tag{B.6}$$

We have shown earlier that $X \geq y/c$, so that $\bar{y} \geq 1$. On the interval $\bar{y} \in [1, \infty)$, the minimum value of the right hand side of (B.6) is attained at

$$\bar{y} = \max\left\{\frac{2 - \bar{c} + \sqrt{\bar{c}^2 - 4\bar{c} + 12}}{2}, 1\right\} \triangleq \phi,$$

and thus,

$$\gamma^0(\mathbf{x}) \geq \frac{\phi^2 + 2}{\phi^2 + 2\phi + \bar{c}} = f(\bar{c}).$$

The case $\alpha_1 > 0$

We now consider the case where $\alpha_n > 0$ for every n. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n\{\alpha_n\} = \alpha_1 = 1$. Suppose first that $x_1 = 0$. The second inequality in (5) implies that $p(X) \leq 1$. We also have that $X > 0$ so that there exists some n for which $x_n > 0$. The first equality in (5) yields,

$$\alpha_n = p(X) + x_n p'(X) \leq p(X) \leq 1.$$

Since $\alpha_n \geq 1$, we obtain $p(X) = 1$ and $p'(X) = 0$. Then, the vector $\mathbf{x}$ satisfies the optimality conditions in (2), and thus is socially optimal in the original model. But this would contradict the fact that $p(X) \neq p(X^S)$. We conclude that $x_1 > 0$.

If $p'(X)$ were equal to zero, then the necessary conditions (5) would imply the optimality conditions (2), and $\mathbf{x}$ would be socially optimal in the modified game. Therefore, we must have $p'(X) < 0$ and $c > 0$. The first equality in (5) yields $y > 1$, $x_1 = (y-1)/c$, and $X \geq (y-1)/c$. We also have

$$0 \leq \sum_{n=2}^{N} x_n = X - \frac{y-1}{c}, \tag{B.7}$$

from which we conclude that $X \geq (y-1)/c$.

From Proposition 5, the efficiency $\gamma^0(\mathbf{x})$ of the vector $\mathbf{x}$ in the modified model cannot be more than its efficiency $\gamma(\mathbf{x})$ in the original model. So, it suffices to consider the efficiency of $\mathbf{x}$ in the modified model. From the optimality conditions (2), we have that $p^0(X^S) = 1$, and thus, using the definition of d,

$$X^S = X + \frac{y-1}{d}.$$

The optimal social welfare in the modified model is

$$\int_0^{X^S} p^0(q)\,dq - X^S = \frac{(y-1)^2}{2d} + X(y-1) + \frac{cX^2}{2}.$$

Note that for $n = 1$ we have $\alpha_n x_n = x_1$. For $n \geq 2$ and whenever $x_n > 0$, from the first equality in (5) we have $\alpha_n = y - x_n c$ and $\alpha_n x_n = (y - x_n c)x_n$. Hence, in the modified model, the social welfare associated with $\mathbf{x}$ is

$$\begin{aligned}\int_0^{X} p^0(q)\,dq - \sum_{n=1}^{N}\alpha_n x_n &= Xy + cX^2/2 - x_1 - \sum_{n=2}^{N}(y - x_n c)x_n \\ &\geq Xy + cX^2/2 - x_1 - y\sum_{n=2}^{N} x_n \\ &= X(y-1) + cX^2/2 - (y-1)\sum_{n=2}^{N} x_n \\ &= X(y-1) + cX^2/2 - (y-1)(X - (y-1)/c) \\ &= cX^2/2 + (y-1)^2/c.\end{aligned}$$

Therefore,

$$\gamma^0(\mathbf{x}) \geq \frac{cX^2/2 + (y-1)^2/c}{(y-1)^2/(2d) + X(y-1) + cX^2/2}. \tag{B.8}$$

Note that c, d, X, and $y - 1$ are all positive. Substituting $\bar{c} = c/d$ and $\bar{y} = (cX)/(y-1)$ in (B.8), we obtain

$$\gamma^0(\mathbf{x}) \geq \frac{2\bar{y}^2 + 1}{\bar{c}\bar{y}^2 + 2\bar{y} + 1}. \tag{B.9}$$

From (B.7) we have that $\bar{y} \geq 1$. On the interval $\bar{y} \in [1, \infty)$, the minimum value of the right hand side of (B.9) is attained at

$$\bar{y} = \min\left\{\frac{2 - \bar{c} + \sqrt{\bar{c}^2 - 4\bar{c} + 12}}{2}, 1\right\} \triangleq \phi,$$

and thus,

$$\gamma^0(\mathbf{x}) \geq \frac{\phi^2 + 2}{\phi^2 + 2\phi + \bar{c}} = f(\bar{c}).$$

B.3. Proof of Corollary 1

Let $\mathbf{x}$ be a Cournot candidate. Since the inverse demand function is convex, we have that $\mu \geq 1$. If $X > Q$, then $p(X) = p'(X) = 0$. The necessary conditions (3)–(4) imply the optimality condition in (2), and thus $\gamma(\mathbf{x}) = 1 > f(\mu)$.

Now consider the case $X \leq Q$. If $p(X) = p(X^S)$ for some social optimum $\mathbf{x}^S$, then Proposition 4(b) implies that $\gamma(\mathbf{x}) = 1 > f(\mu)$. Otherwise, for any social optimum $\mathbf{x}^S$, we have that $\bar{c} = c/d \leq \mu$. Theorem 2 shows that the efficiency of every Cournot candidate cannot be less than $f(\bar{c})$. The desired result then follows from the fact that $f(\bar{c})$ is decreasing in $\bar{c}$.

B.4. Proof of Corollary 2

Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and a social optimum, respectively. If $p(X) = p(X^S)$, for some social optimum $\mathbf{x}^S$, then $\gamma(\mathbf{x}) = 1$ and the desired result holds trivially. Now suppose that $p(X) \neq p(X^S)$. We first derive an upper bound on the aggregate supply at a social optimum, and then establish a lower bound on the aggregate supply at a Cournot candidate. The desired results will follow from the fact that the function f is strictly decreasing.

Step 1: *There exists a social optimum with an aggregate supply no more than s.*

We first have $X^S > 0$ and there exists a supplier n such that $x_n^S > 0$. From the optimality conditions (2) we have $p(X^S) = C_n'(x_n^S)$, which implies that $p(X^S) \geq C_n'(0)$, due to the convexity of the cost functions. We conclude that

$$p(X^S) = C_n'(x_n^S) \geq C_n'(0) \geq \min_n C_n'(0). \tag{B.10}$$

If $p(X^S) > \min_n C_n'(0)$, then from the definition of s in (10), and the assumption that p is nonincreasing, we have that $X^S < s$.

If $p(X^S) = \min_n C_n'(0)$, by (B.10) we know that for any n such that $x_n^S > 0$, we must have $C_n'(x_n^S) = C_n'(0) = p(X^S)$. Since C_n is convex, we conclude that C_n is actually linear on the interval $[0, x_n^S]$. We now argue that there exists a social optimum $\mathbf{x}^S$ such that $X^S \leq s$. If $X^S \leq s$, then we are done. Otherwise, we have $X^S > s$. Let $\mathcal{N}$ be the set of the indices of suppliers who produce a positive quantity at $\mathbf{x}^S$. Since p is nonincreasing, and $p(s) = \min_n C_n'(0) = p(X^S)$, we know that for any $q \in [s, X^S]$, $p(q) = C_n'(0)$ for every $n \in \mathcal{N}$. Combining with the fact that for each supplier n in the set $\mathcal{N}$, C_n is linear on the interval $[0, x_n^S]$, we have

$$\int_s^{X^S} p(q)\,dq = (X^S - s)C_n'(x), \quad \forall n \in \mathcal{N},\ \forall x \in [0, x_n^S),$$

from which we conclude that the vector, $(s/X^S) \cdot \mathbf{x}^S$, yields the same social welfare as $\mathbf{x}^S$, and thus is socially optimal. Note that the aggregate supply at $(s/X^S) \cdot \mathbf{x}^S$ is s.

If $p(X) = p(X^S)$, then $\gamma(\mathbf{x}) = 1$ and the desired result holds trivially. Otherwise, since p is nonincreasing and convex, we have

$$|\partial_- p(s)| \le \left|(p(X^S) - p(X))/(X^S - X)\right| = d. \tag{B.11}$$

Step 2: The aggregate supply at a Cournot candidate $\mathbf{x}$ *is at least* t.

Since p is convex, $\mathbf{x}$ satisfies the necessary conditions in (5). Therefore,

$$C_n'(x_n) \ge p(X) + x_n p'(X), \quad \forall n. \tag{B.12}$$

Since $X \ge x_n$, we have

$$C_n'(x_n) \le C_n'(X), \qquad Xp'(X) \le x_n p'(X), \tag{B.13}$$

where the first inequality follows from the convexity of the cost functions, and the second one is true because $p'(X) \le 0$. Combining (B.12) and (B.13), we have

$$C_n'(X) \ge p(X) + Xp'(X), \quad \forall n,$$

which implies that $X \ge t$. Since p is nonincreasing and convex, we have

$$c = \left|p'(X)\right| \le |\partial_+ p(t)|\,. \tag{B.14}$$

Since $\partial_- p(s) < 0$, Eqs. (B.11) and (B.14) yield

$$\bar{c} = c/d \le \partial_+ p(t)/\partial_- p(s).$$

The desired result follows from Theorem 2, and the fact that f is strictly decreasing.

Appendix C. Proofs of the results in Section 6

C.1. Proof of Theorem 3

According to the discussion in Section 6, we only need to lower bound the efficiency of a Cournot equilibrium $\mathbf{x}$ in a model with $N = 1$. Since $N = 1$, we can identify the vectors $\mathbf{x}$ and $\mathbf{x}^S$ with the scalars X and X^S. If $p(X) = p(X^S)$, then according to Proposition 4(b), the efficiency of the Cournot equilibrium, X, must equal one, which establishes part (a).

We now turn to the proof of part (b), and we assume that $p(X) \ne p(X^S)$. According to Proposition 1, we know that $\mathbf{x}$ cannot be socially optimal. We will consider separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$.

We will again rely on Proposition 5, according to which the efficiency of a Cournot candidate $\mathbf{x}$ is lower bounded by the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in a model involving the piecewise linear and convex inverse demand function p^0. Note that since $p(X) \ne p(X^S)$, we have that $d > 0$. As shown in the proof of Theorem 2, we have $p'(X) < 0$, i.e., $c > 0$. For conciseness, we let $y = p(X)$ throughout the proof.

The case $\alpha_1 = 0$

Applying conditions (5) to the supplier we have $X = y/c$. From Proposition 5, it suffices to show that $\gamma^0(\mathbf{x}) \ge 3/(3 + \bar{c})$. The optimal social welfare in the modified model is

$$\int_0^\infty p^0(q)\,dq - 0 = \int_0^{X+y/d} p^0(q)\,dq - 0 = \frac{y^2}{2d} + \frac{(2y + cX)X}{2}. \tag{C.1}$$

In the modified model, the social welfare associated with $\mathbf{x}$ is

$$\int_0^{X^P} p^0(q)\,dq - 0 = \frac{(2y + cX)X}{2}.$$

Therefore,

$$\gamma^0(\mathbf{x}) = \frac{(2y + cX)X/2}{y^2/(2d) + (2y + cX)X/2} = \frac{3}{3 + \bar{c}},$$

where the last equality is true because $xc = y$.

Tightness

Consider a model in which the inverse demand function is $p(q) = \max\{1 - q, 0\}$ and the supplier's cost function is identically zero, i.e., $C_1(x_1) = 0$. The profit maximizing output is $x_1 = 1/2$. We observe that $\gamma(\mathbf{x}) = 3/4$.

The case $\alpha_1 > 0$

We now consider the case where $\alpha_1 > 0$. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\alpha_1 = 1$. Applying conditions (5) to the supplier, we obtain $X = (y - 1)/c$.

According to Proposition 5, it suffices to show that the efficiency of $\mathbf{x}$ in the modified model, $\gamma^0(\mathbf{x})$, is at least $3/(3 + \bar{c})$. From the optimality conditions (2) we have that $p^0(X^S) = 1$, and therefore,

$$X^S = X + (y - 1)/d.$$

The optimal social welfare achieved in the modified model is

$$\int_0^{X^S} p^0(q)\,dq - X^S = \frac{(y-1)^2}{2d} + X^P(y - 1) + \frac{c(X)^2}{2}.$$

In the modified model, the social welfare associated with $\mathbf{x}$ is

$$\int_0^{X} p^0(q)\,dq - X = X(y - 1) + \frac{cX^2}{2}.$$

Since $cX = y - 1$, we have

$$\gamma^0(\mathbf{x}) = \frac{3}{3 + \bar{c}}.$$

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