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NP-hardness of deciding convexity of quartic polynomials and related problems

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Abstract We show that unless P = NP, there exists no polynomial time (or even pseudo-polynomial time) algorithm that can decide whether a multivariate polynomial of degree four (or higher even degree) is globally convex. This solves a problem that has been open since 1992 when N. Z. Shor asked for the complexity of deciding convexity for quartic polynomials. We also prove that deciding strict convexity, strong convexity, quasiconvexity, and pseudoconvexity of polynomials of even degree four or higher is strongly NP-hard. By contrast, we show that quasiconvexity and pseudoconvexity of odd degree polynomials can be decided in polynomial time.

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1 Introduction

The role of *convexity* in modern day mathematical programming has proven to be remarkably fundamental, to the point that tractability of an optimization problem is nowadays assessed, more often than not, by whether or not the problem benefits from some sort of underlying convexity. In the famous words of Rockafellar [41]:

> "In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."

But how easy is it to distinguish between convexity and nonconvexity? Can we decide in an efficient manner if a given optimization problem is convex?

A class of optimization problems that allow for a rigorous study of this question from a computational complexity viewpoint is the class of polynomial optimization problems. These are optimization problems where the objective is given by a polynomial function and the feasible set is described by polynomial inequalities. Our research in this direction was motivated by a concrete question of N. Z. Shor that appeared as one of seven open problems in complexity theory for numerical optimization put together by Pardalos and Vavasis in 1992 [38]:

> "Given a degree-4 polynomial in n variables, what is the complexity of determining whether this polynomial describes a convex function?"

As we will explain in more detail shortly, the reason why Shor's question is specifically about degree 4 polynomials is that deciding convexity of odd degree polynomials is trivial and deciding convexity of degree 2 (quadratic) polynomials can be reduced to the simple task of checking whether a constant matrix is positive semidefinite. So, the first interesting case really occurs for degree 4 (quartic) polynomials. Our main contribution in this paper (Theorem 2.1 in Sect. 2.3) is to show that deciding convexity of polynomials is strongly NP-hard already for polynomials of degree 4.

The implication of NP-hardness of this problem is that unless P = NP, there exists no algorithm that can take as input the (rational) coefficients of a quartic polynomial, have running time bounded by a polynomial in the number of bits needed to represent the coefficients, and output correctly on every instance whether or not the polynomial is convex. Furthermore, the fact that our NP-hardness result is in the strong sense (as opposed to weakly NP-hard problems such as KNAPSACK) implies, roughly speaking, that the problem remains NP-hard even when the magnitude of the coefficients of the polynomial are restricted to be "small." For a strongly NP-hard problem, even a pseudo-polynomial time algorithm cannot exist unless P = NP. See [19] for precise definitions and more details.

There are many areas of application where one would like to establish convexity of polynomials. Perhaps the simplest example is in global minimization of polynomials, where it could be very useful to decide first whether the polynomial to be optimized is convex. Once convexity is verified, then every local minimum is global and very basic techniques (e.g., gradient descent) can find a global minimum—a task that is in general NP-hard in the absence of convexity [35,39]. As another example, if we can certify that a homogeneous polynomial is convex, then we define a gauge (or Minkowski) norm based on its convex sublevel sets, which may be useful in many applications. In several other problems of practical relevance, we might not just be interested in checking whether a given polynomial is convex, but to *parameterize* a family of convex polynomials and perhaps search or optimize over them. For example we might be interested in approximating the convex envelope of a complicated nonconvex function with a convex polynomial, or in fitting a convex polynomial to a set of data points with minimum error [30]. Not surprisingly, if testing membership to the set of convex polynomials is hard, searching and optimizing over that set also turns out to be a hard problem.

We also extend our hardness result to some variants of convexity, namely, the problems of deciding *strict convexity*, *strong convexity*, *pseudoconvexity*, and *quasiconvexity* of polynomials. Strict convexity is a property that is often useful to check because it guarantees uniqueness of the optimal solution in optimization problems. The notion of strong convexity is a common assumption in convergence analysis of many iterative Newton-type algorithms in optimization theory; see, e.g., [9, Chaps. 9–11]. So, in order to ensure the theoretical convergence rates promised by many of these algorithms, one needs to first make sure that the objective function is strongly convex. The problem of checking quasiconvexity (convexity of sublevel sets) of polynomials also arises frequently in practice. For instance, if the feasible set of an optimization problem is defined by polynomial inequalities, by certifying quasiconvexity of the defining polynomials we can ensure that the feasible set is convex. In several statistics and clustering problems, we are interested in finding minimum volume convex sets that contain a set of data points in space. This problem can be tackled by searching over the set of quasiconvex polynomials [30]. In economics, quasiconcave functions are prevalent as desirable utility functions [5,28]. In control and systems theory, it is useful at times to search for quasiconvex Lyapunov functions whose convex sublevel sets contain relevant information about the trajectories of a dynamical system [3,11]. Finally, the notion of pseudoconvexity is a natural generalization of convexity that inherits many of the attractive properties of convex functions. For example, every stationary point or every local minimum of a pseudoconvex function must be a global minimum. Because of these nice features, pseudoconvex programs have been studied extensively in nonlinear programming [13,31].

As an outcome of close to a century of research in convex analysis, numerous necessary, sufficient, and exact conditions for convexity and all of its variants are available; see, e.g., [9, Chap. 3], [14,18,28,32,33] and references therein for a by no means exhaustive list. Our results suggest that none of the exact characterizations of these notions can be efficiently checked for polynomials. In fact, when turned upside down, many of these equivalent formulations reveal new NP-hard problems; see, e.g., Corollary 2.6 and 2.8.

1.1 Related literature

There are several results in the literature on the complexity of various special cases of polynomial optimization problems. The interested reader can find many of these results in the edited volume of Pardalos [37] or in the survey papers of de Klerk [16], and Blondel and Tsitsiklis [8]. A very general and fundamental concept in certifying feasibility of polynomial equations and inequalities is the Tarski–Seidenberg quantifier elimination theory [42,43], from which it follows that all of the problems that we consider in this paper are algorithmically *decidable*. This means that there are algorithms that on all instances of our problems of interest halt in finite time and always output the correct yes–no answer. Unfortunately, algorithms based on quantifier elimination or similar decision algebra techniques have running times that are at least exponential in the number of variables [6], and in practice can only solve problems with very few parameters.

When we turn to the issue of polynomial time solvability, perhaps the most relevant result for our purposes is the NP-hardness of deciding nonnegativity of quartic polynomials and biquadratic forms (see Definition 2.2); the reduction that we give in this paper will in fact be from the latter problem. As we will see in Sect. 2.3, it turns out that deciding convexity of quartic forms is equivalent to checking nonnegativity of a special class of biquadratic forms, which are themselves a special class of quartic forms. The NP-hardness of checking nonnegativity of quartic forms follows, e.g., as a direct consequence of NP-hardness of testing matrix copositivity, a result proven by Murty and Kabadi [35]. As for the hardness of checking nonnegativity of biquadratic forms, we know of two different proofs. The first one is due to Gurvits [22], who proves that the entanglement problem in quantum mechanics (i.e., the problem of distinguishing separable quantum states from entangled ones) is NP-hard. A dual reformulation of this result shows directly that checking nonnegativity of biquadratic forms is NP-hard; see [17]. The second proof is due to Ling et al. [29], who use a theorem of Motzkin and Straus to give a very short and elegant reduction from the maximum clique problem in graphs.

The only work in the literature on the hardness of deciding polynomial convexity that we are aware of is the work of Guo on the complexity of deciding convexity of quartic polynomials over simplices [21]. Guo discusses some of the difficulties that arise from this problem, but he does not prove that deciding convexity of polynomials over simplices is NP-hard. Canny shows in [10] that the existential theory of the real numbers can be decided in PSPACE. From this, it follows that testing several properties of polynomials, including nonnegativity and convexity, can be done in polynomial space. In [36], Nie proves that the related notion of *matrix convexity* is NP-hard for polynomial matrices whose entries are quadratic forms.

On the algorithmic side, several techniques have been proposed both for testing convexity of sets and convexity of functions. Rademacher and Vempala present and analyze randomized algorithms for testing the relaxed notion of *approximate convexity* [40]. In [27], Lasserre proposes a semidefinite programming hierarchy for testing convexity of basic closed semialgebraic sets; a problem that we also prove to be NP-hard (see Corollary 2.8). As for testing convexity of functions, an approach that some convex optimization parsers (e.g., `CVX` [20]) take is to start with some ground set of convex functions and then check whether the desired function can be obtained by applying a set of convexity preserving operations to the functions in the ground set [15], [9, p. 79]. Techniques of this type that are based on the calculus of convex functions are successful for a large range of applications. However, when applied to general polynomial functions, they can only detect a subclass of convex polynomials.

Table 1 Summary of our complexity results

Property versus degree	1	2	Odd ≥ 3	Even ≥ 4
Strong convexity	No	P	No	Strongly NP-hard
Strict convexity	No	P	No	Strongly NP-hard
Convexity	Yes	P	No	Strongly NP-hard
Pseudoconvexity	Yes	P	P	Strongly NP-hard
Quasiconvexity	Yes	P	P	Strongly NP-hard

A yes (no) entry means that the question is trivial for that particular entry because the answer is always yes (no) independent of the input. By P, we mean that the problem can be solved in polynomial time

Related to convexity of polynomials, a concept that has attracted recent attention is the algebraic notion of *sos-convexity* (see Definition 2.4) [3,11,23,25,26,30]. This is a powerful sufficient condition for convexity that relies on an appropriately defined sum of squares decomposition of the Hessian matrix, and can be efficiently checked by solving a single semidefinite program. However, in [2,4], Ahmadi and Parrilo gave an explicit counterexample to show that not every convex polynomial is sos-convex. The NP-hardness result in this work certainly justifies the existence of such a counterexample and more generally suggests that *any* polynomial time algorithm attempted for checking polynomial convexity is doomed to fail on some hard instances.

1.2 Contributions and organization of the paper

The main contribution of this paper is to establish the computational complexity of deciding convexity, strict convexity, strong convexity, pseudoconvexity, and quasiconvexity of polynomials for any given degree. See Table 1 for a quick summary. The results are mainly divided in three sections, with Sect. 2 covering convexity, Sect. 3 covering strict and strong convexity, and Sect. 4 covering quasiconvexity and pseudoconvexity. These three sections follow a similar pattern and are each divided into three parts: first, the definitions and basics, second, the degrees for which the questions can be answered in polynomial time, and third, the degrees for which the questions are NP-hard.

Our main reduction, which establishes NP-hardness of checking convexity of quartic forms, is given in Sect. 2.3. This hardness result is extended to strict and strong convexity in Sect. 3.3, and to quasiconvexity and pseudoconvexity in Sect. 4.3. By contrast, we show in Sect. 4.2 that quasiconvexity and pseudoconvexity of odd degree polynomials can be decided in polynomial time. Finally, a summary of our results and some concluding remarks are presented in Sect. 5.

2 Complexity of deciding convexity

2.1 Definitions and basics

A (multivariate) *polynomial* $p(x)$ in variables $x := (x_1, \ldots, x_n)^T$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$ that is a finite linear combination of monomials:

$$p(x) = \sum_{\alpha} c_\alpha x^\alpha = \sum_{\alpha_1,\dots,\alpha_n} c_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{1}$$

where the sum is over n-tuples of nonnegative integers α_i. An algorithm for testing some property of polynomials will have as its input an ordered list of the coefficients c_α. Since our complexity results are based on models of digital computation, where the input must be represented by a finite number of bits, the coefficients c_α for us will always be rational numbers, which upon clearing the denominators can be taken to be integers. So, for the remainder of the paper, even when not explicitly stated, we will always have $c_\alpha \in \mathbb{Z}$.

The *degree* of a monomial x^α is equal to $\alpha_1 + \cdots + \alpha_n$. The degree of a polynomial $p(x)$ is defined to be the highest degree of its component monomials. A simple counting argument shows that a polynomial of degree d in n variables has $\binom{n+d}{d}$ coefficients. A *homogeneous polynomial* (or a *form*) is a polynomial where all the monomials have the same degree. A form $p(x)$ of degree d is a homogeneous function of degree d (since it satisfies $p(\lambda x) = \lambda^d p(x)$), and has $\binom{n+d-1}{d}$ coefficients.

A polynomial $p(x)$ is said to be *nonnegative* or *positive semidefinite (psd)* if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. Clearly, a necessary condition for a polynomial to be psd is for its degree to be even. We say that $p(x)$ is a *sum of squares (sos)*, if there exist polynomials $q_1(x), \dots, q_m(x)$ such that $p(x) = \sum_{i=1}^m q_i^2(x)$. Every sos polynomial is obviously psd. A *polynomial matrix* $P(x)$ is a matrix with polynomial entries. We say that a polynomial matrix $P(x)$ is *PSD* (denoted $P(x) \succeq 0$) if it is positive semidefinite in the matrix sense for every value of the indeterminates x. (Note the upper case convention for matrices.) It is easy to see that $P(x)$ is PSD if and only if the scalar polynomial $y^T P(x) y$ in variables $(x; y)$ is psd.

We recall that a polynomial $p(x)$ is convex if and only if its Hessian matrix, which will be generally denoted by $H(x)$, is PSD.

2.2 Degrees that are easy

The question of deciding convexity is trivial for odd degree polynomials. Indeed, it is easy to check that linear polynomials ($d = 1$) are always convex and that polynomials of odd degree $d \geq 3$ can never be convex. The case of quadratic polynomials ($d = 2$) is also straightforward. A quadratic polynomial $p(x) = \frac{1}{2} x^T Q x + q^T x + c$ is convex if and only if the constant matrix Q is positive semidefinite. This can be decided in polynomial time for example by performing Gaussian pivot steps along the main diagonal of Q [35] or by computing the characteristic polynomial of Q exactly and then checking that the signs of its coefficients alternate [24, p. 403].

Unfortunately, the results that come next suggest that the case of quadratic polynomials is essentially the only nontrivial case where convexity can be efficiently decided.

2.3 Degrees that are hard

The main hardness result of the paper is the following theorem.

Theorem 2.1 *Deciding convexity of degree four polynomials is strongly NP-hard. This is true even when the polynomials are restricted to be homogeneous.*

We will give a reduction from the problem of deciding nonnegativity of biquadratic forms. We start by recalling some basic facts about biquadratic forms and sketching the idea of the proof.

Definition 2.2 A *biquadratic form* $b(x; y)$ is a form in the variables $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_m)^T$ that can be written as

$$b(x; y) = \sum_{i \leq j,\, k \leq l} \alpha_{ijkl} x_i x_j y_k y_l. \tag{2}$$

Note that for fixed x, $b(x; y)$ becomes a quadratic form in y, and for fixed y, it becomes a quadratic form in x. Every biquadratic form is a quartic form, but the converse is of course not true. It follows from a result of Ling et al. [29] that deciding nonnegativity of biquadratic forms is strongly NP-hard. For the benefit of the reader, let us briefly summarize the proof from [29] before we proceed, as this result underlies everything that follows.

The argument in [29] is based on a reduction from CLIQUE (given a graph $G(V, E)$ and a positive integer $k \leq |V|$, decide whether G contains a clique of size k or more) whose (strong) NP-hardness is well-known [19]. For a given graph $G(V, E)$ on n nodes, if we define the biquadratic form $b_G(x; y)$ in the variables $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ by

$$b_G(x; y) = -2 \sum_{(i,j) \in E} x_i x_j y_i y_j,$$

then Ling et al. [29] use a theorem of Motzkin and Straus [34] to show

$$\min_{||x||=||y||=1} b_G(x; y) = -1 + \frac{1}{\omega(G)}. \tag{3}$$

Here, $\omega(G)$ denotes the clique number of the graph G, i.e., the size of a maximal clique.[1] From this, we see that for any value of k, $\omega(G) \leq k$ if and only if

$$\min_{||x||=||y||=1} b_G(x; y) \geq \frac{1-k}{k},$$

which by homogenization holds if and only if the biquadratic form

$$\hat{b}_G(x; y) = -2k \sum_{(i,j) \in E} x_i x_j y_i y_j - (1-k) \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right)$$

[1] Equation (3) above is stated in [29] with the stability number $\alpha(G)$ in place of the clique number $\omega(G)$. This seems to be a minor typo.

is nonnegative. Hence, by checking nonnegativity of $\hat{b}_G(x; y)$ for all values of $k \in \{1, \ldots, n-1\}$, we can find the exact value of $\omega(G)$. It follows that deciding nonnegativity of biquadratic forms is NP-hard, and in view of the fact that the coefficients of $\hat{b}_G(x; y)$ are all integers with absolute value at most $2n-2$, the NP-hardness claim is in the strong sense. Note also that the result holds even when $n = m$ in Definition 2.2. In the sequel, we will always have $n = m$.

It is not difficult to see that any biquadratic form $b(x; y)$ can be written in the form

$$b(x; y) = y^T A(x) y \tag{4}$$

(or of course as $x^T B(y) x$) for some symmetric polynomial matrix $A(x)$ whose entries are quadratic forms. Therefore, it is strongly NP-hard to decide whether a symmetric polynomial matrix with quadratic form entries is PSD. One might hope that this would lead to a quick proof of NP-hardness of testing convexity of quartic forms, because the Hessian of a quartic form is exactly a symmetric polynomial matrix with quadratic form entries. However, the major problem that stands in the way is that not every polynomial matrix is a *valid Hessian*. Indeed, if any of the partial derivatives between the entries of $A(x)$ do not commute (e.g., if $\frac{\partial A_{11}(x)}{\partial x_2} \neq \frac{\partial A_{12}(x)}{\partial x_1}$), then $A(x)$ cannot be the matrix of second derivatives of some polynomial. This is because all mixed third partial derivatives of polynomials must commute.

Our task is therefore to prove that even with these additional constraints on the entries of $A(x)$, the problem of deciding positive semidefiniteness of such matrices remains NP-hard. We will show that any given symmetric $n \times n$ matrix $A(x)$, whose entries are quadratic forms, can be embedded in a $2n \times 2n$ polynomial matrix $H(x, y)$, again with quadratic form entries, so that $H(x, y)$ is a valid Hessian and $A(x)$ is PSD if and only if $H(x, y)$ is. In fact, we will directly construct the polynomial $f(x, y)$ whose Hessian is the matrix $H(x, y)$. This is done in the next theorem, which establishes the correctness of our main reduction. Once this theorem is proven, the proof of Theorem 2.1 will become immediate.

Theorem 2.3 *Given a biquadratic form $b(x; y)$, define the the $n \times n$ polynomial matrix $C(x, y)$ by setting*

$$[C(x, y)]_{ij} := \frac{\partial b(x; y)}{\partial x_i \partial y_j}, \tag{5}$$

and let γ be the largest coefficient, in absolute value, of any monomial present in some entry of the matrix $C(x, y)$. Let f be the form given by

$$f(x, y) := b(x; y) + \frac{n^2\gamma}{2}\left(\sum_{i=1}^{n} x_i^4 + \sum_{i=1}^{n} y_i^4 + \sum_{\substack{i,j=1,\ldots,n \\ i<j}} x_i^2 x_j^2 + \sum_{\substack{i,j=1,\ldots,n \\ i<j}} y_i^2 y_j^2\right). \tag{6}$$

Then, $b(x; y)$ is psd if and only if $f(x, y)$ is convex.

Proof Before we prove the claim, let us make a few observations and try to shed light on the intuition behind this construction. We will use $H(x, y)$ to denote the Hessian of f. This is a $2n \times 2n$ polynomial matrix whose entries are quadratic forms. The polynomial f is convex if and only if $z^T H(x, y)z$ is psd. For bookkeeping purposes, let us split the variables z as $z := (z_x, z_y)^T$, where z_x and z_y each belong to $\mathbb{R}^n$. It will also be helpful to give a name to the second group of terms in the definition of $f(x, y)$ in (6). So, let

$$g(x, y) := \frac{n^2\gamma}{2}\left(\sum_{i=1}^{n} x_i^4 + \sum_{i=1}^{n} y_i^4 + \sum_{\substack{i,j=1,\dots,n\\ i<j}} x_i^2 x_j^2 + \sum_{\substack{i,j=1,\dots,n\\ i<j}} y_i^2 y_j^2\right). \tag{7}$$

We denote the Hessian matrices of $b(x, y)$ and $g(x, y)$ with $H_b(x, y)$ and $H_g(x, y)$ respectively. Thus, $H(x, y) = H_b(x, y) + H_g(x, y)$. Let us first focus on the structure of $H_b(x, y)$. Observe that if we define

$$[A(x)]_{ij} = \frac{\partial b(x; y)}{\partial y_i \partial y_j},$$

then $A(x)$ depends only on x, and

$$\frac{1}{2} y^T A(x) y = b(x; y). \tag{8}$$

Similarly, if we let

$$[B(y)]_{ij} = \frac{\partial b(x; y)}{\partial x_i \partial x_j},$$

then $B(y)$ depends only on y, and

$$\frac{1}{2} x^T B(y) x = b(x; y). \tag{9}$$

From Eq. (8), we have that $b(x; y)$ is psd if and only if $A(x)$ is PSD; from Eq. (9), we see that $b(x; y)$ is psd if and only if $B(y)$ is PSD.

Putting the blocks together, we have

$$H_b(x, y) = \begin{bmatrix} B(y) & C(x, y) \\ C^T(x, y) & A(x) \end{bmatrix}. \tag{10}$$

The matrix $C(x, y)$ is not in general symmetric. The entries of $C(x, y)$ consist of square-free monomials that are each a multiple of $x_i y_j$ for some i, j, with $1 \le i, j \le n$; (see (2) and (5)).

The Hessian $H_g(x, y)$ of the polynomial $g(x, y)$ in (7) is given by

$$H_g(x, y) = \frac{n^2\gamma}{2}\begin{bmatrix} H_g^{11}(x) & 0 \\ 0 & H_g^{22}(y) \end{bmatrix}, \tag{11}$$

where

$$H_g^{11}(x) = \begin{bmatrix} 12x_1^2 + 2\sum\limits_{\substack{i=1,\ldots,n \\ i\neq 1}} x_i^2 & 4x_1x_2 & \cdots & 4x_1x_n \\ 4x_1x_2 & 12x_2^2 + 2\sum\limits_{\substack{i=1,\ldots,n \\ i\neq 2}} x_i^2 & \cdots & 4x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ 4x_1x_n & \cdots & 4x_{n-1}x_n & 12x_n^2 + 2\sum\limits_{\substack{i=1,\ldots,n \\ i\neq n}} x_i^2 \end{bmatrix}, \tag{12}$$

and

$$H_g^{22}(y) = \begin{bmatrix} 12y_1^2 + 2\sum\limits_{\substack{i=1,\ldots,n \\ i\neq 1}} y_i^2 & 4y_1y_2 & \cdots & 4y_1y_n \\ 4y_1y_2 & 12y_2^2 + 2\sum\limits_{\substack{i=1,\ldots,n \\ i\neq 2}} y_i^2 & \cdots & 4y_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 4y_1y_n & \cdots & 4y_{n-1}y_n & 12y_n^2 + 2\sum\limits_{\substack{i=1,\ldots,n \\ i\neq n}} y_i^2 \end{bmatrix}. \tag{13}$$

Note that all diagonal elements of $H_g^{11}(x)$ and $H_g^{22}(y)$ contain the square of every variable $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively.

We fist give an intuitive summary of the rest of the proof. If $b(x; y)$ is not psd, then $B(y)$ and $A(x)$ are not PSD and hence $H_b(x, y)$ is not PSD. Moreover, adding $H_g(x, y)$ to $H_b(x, y)$ cannot help make $H(x, y)$ PSD because the dependence of the diagonal blocks of $H_b(x, y)$ and $H_g(x, y)$ on x and y runs backwards. On the other hand, if $b(x; y)$ is psd, then $H_b(x, y)$ will have PSD diagonal blocks. In principle, $H_b(x, y)$ might still not be PSD because of the off-diagonal block $C(x, y)$. However, the squares in the diagonal elements of $H_g(x, y)$ will be shown to dominate the monomials of $C(x, y)$ and make $H(x, y)$ PSD.

Let us now prove the theorem formally. One direction is easy: if $b(x; y)$ is not psd, then $f(x, y)$ is not convex. Indeed, if there exist $\bar{x}$ and $\bar{y}$ in $\mathbb{R}^n$ such that $b(\bar{x}; \bar{y}) < 0$, then

$$z^T H(x, y)z\Big|_{z_x=0, x=\bar{x}, y=0, z_y=\bar{y}} = \bar{y}^T A(\bar{x})\bar{y} = 2b(\bar{x}; \bar{y}) < 0.$$

For the converse, suppose that $b(x; y)$ is psd; we will prove that $z^T H(x, y)z$ is psd and hence $f(x, y)$ is convex. We have

$$\begin{aligned} z^T H(x, y)z &= z_y^T A(x)z_y + z_x^T B(y)z_x + 2z_x^T C(x, y)z_y \\ &\quad + \frac{n^2\gamma}{2} z_x^T H_g^{11}(x)z_x + \frac{n^2\gamma}{2} z_y^T H_g^{22}(y)z_y. \end{aligned} \tag{14}$$

Because $z_y^T A(x)z_y$ and $z_x^T B(y)z_x$ are psd by assumption (see (8) and (9)), it suffices to show that $z^T H(x, y)z - z_y^T A(x)z_y - z_x^T B(y)z_x$ is psd. In fact, we will show that $z^T H(x, y)z - z_y^T A(x)z_y - z_x^T B(y)z_x$ is a sum of squares.

After some regrouping of terms we can write

$$z^T H(x, y)z - z_y^T A(x)z_y - z_x^T B(y)z_x = p_1(x, y, z) + p_2(x, z_x) + p_3(y, z_y), \tag{15}$$

where

$$p_1(x, y, z) = 2z_x^T C(x, y)z_y + n^2\gamma\left(\sum_{i=1}^n z_{x,i}^2\right)\left(\sum_{i=1}^n x_i^2\right) + n^2\gamma\left(\sum_{i=1}^n z_{y,i}^2\right)\left(\sum_{i=1}^n y_i^2\right), \tag{16}$$

$$p_2(x, z_x) = n^2\gamma z_x^T \begin{bmatrix} 5x_1^2 & 2x_1x_2 & \cdots & 2x_1x_n \\ 2x_1x_2 & 5x_2^2 & \cdots & 2x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ 2x_1x_n & \cdots & 2x_{n-1}x_n & 5x_n^2 \end{bmatrix} z_x, \tag{17}$$

and

$$p_3(y, z_y) = n^2\gamma z_y^T \begin{bmatrix} 5y_1^2 & 2y_1y_2 & \cdots & 2y_1y_n \\ 2y_1y_2 & 5y_2^2 & \cdots & 2y_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 2y_1y_n & \cdots & 2y_{n-1}y_n & 5y_n^2 \end{bmatrix} z_y. \tag{18}$$

We show that (15) is sos by showing that p_1, p_2, and p_3 are each individually sos. To see that p_2 is sos, simply note that we can rewrite it as

$$p_2(x, z_x) = n^2\gamma \left[3\sum_{k=1}^{n} z_{x,k}^2 x_k^2 + 2\left(\sum_{k=1}^{n} z_{x,k}x_k\right)^2 \right].$$

The argument for p_3 is of course identical. To show that p_1 is sos, we argue as follows. If we multiply out the first term $2z_x^T C(x, y)z_y$, we obtain a polynomial with monomials of the form

$$\pm 2\beta_{i,j,k,l} z_{x,k} x_i y_j z_{y,l}, \tag{19}$$

where $0 \le \beta_{i,j,k,l} \le \gamma$, by the definition of γ. Since

$$\pm 2\beta_{i,j,k,l} z_{x,k} x_i y_j z_{y,l} + \beta_{i,j,k,l} z_{x,k}^2 x_i^2 + \beta_{i,j,k,l} y_j^2 z_{y,l}^2 = \beta_{i,j,k,l}(z_{x,k}x_i \pm y_j z_{y,l})^2, \tag{20}$$

by pairing up the terms of $2z_x^T C(x, y)z_y$ with fractions of the squared terms $z_{x,k}^2 x_i^2$ and $z_{y,l}^2 y_j^2$, we get a sum of squares. Observe that there are more than enough squares for each monomial of $2z_x^T C(x, y)z_y$ because each such monomial $\pm 2\beta_{i,j,k,l} z_{x,k} x_i y_j z_{y,l}$ occurs at most once, so that each of the terms $z_{x,k}^2 x_i^2$ and $z_{y,l}^2 y_j^2$ will be needed at most n^2 times, each time with a coefficient of at most γ. Therefore, p_1 is sos, and this completes the proof. □

We can now complete the proof of strong NP-hardness of deciding convexity of quartic forms.

Proof of Theorem 2.1 As we remarked earlier, deciding nonnegativity of biquadratic forms is known to be strongly NP-hard [29]. Given such a biquadratic form $b(x; y)$, we can construct the polynomial $f(x, y)$ as in (6). Note that $f(x, y)$ has degree four and is homogeneous. Moreover, the reduction from $b(x; y)$ to $f(x, y)$ runs in polynomial time as we are adding to $b(x; y)$ only $2n + 2\binom{n}{2}$ new monomials with coefficient $\frac{n^2\gamma}{2}$, and the size of γ is by definition only polynomially larger than the size of any coefficient of $b(x; y)$. Since by Theorem 2.3 convexity of $f(x, y)$ is equivalent to nonnegativity of $b(x; y)$, we conclude that deciding convexity of quartic forms is strongly NP-hard. □

An algebraic version of the reduction. Before we proceed further with our results, we make a slight detour and present an algebraic analogue of this reduction, which relates sum of squares biquadratic forms to sos-convex polynomials. Both of these concepts are well-studied in the literature, in particular in regards to their connection to semidefinite programming; see, e.g., [4,29], and references therein.

Definition 2.4 A polynomial $p(x)$, with its Hessian denoted by $H(x)$, is *sos-convex* if the polynomial $y^T H(x) y$ is a sum of squares in variables (x;y).[2]

Theorem 2.5 *Given a biquadratic form $b(x; y)$, let $f(x, y)$ be the quartic form defined as in (6). Then $b(x; y)$ is a sum of squares if and only if $f(x, y)$ is sos-convex.*

Proof The proof is very similar to the proof of Theorem 2.3 and is left to the reader. □

Perhaps of independent interest, Theorems 2.3 and 2.5 imply that our reduction gives an explicit way of constructing convex but not sos-convex quartic forms (see [4]), starting from any example of a psd but not sos biquadratic form (see [12]).

Some NP-hardness results, obtained as corollaries. NP-hardness of checking convexity of quartic forms directly establishes NP-hardness[3] of several problems of interest. Here, we mention a few examples.

Corollary 2.6 *It is NP-hard to decide nonnegativity of a homogeneous polynomial q of degree four, of the form*

$$q(x, y) = \frac{1}{2}p(x) + \frac{1}{2}p(y) - p\left(\tfrac{x+y}{2}\right),$$

for some homogeneous quartic polynomial p.

Proof Nonnegativity of q is equivalent to convexity of p, and the result follows directly from Theorem 2.1. □

Definition 2.7 A set $\mathcal{S} \subset \mathbb{R}^n$ is *basic closed semialgebraic* if it can be written as

$$\mathcal{S} = \{x \in \mathbb{R}^n |\ f_i(x) \geq 0, \quad i = 1, \ldots, m\}, \tag{21}$$

for some positive integer m and some polynomials $f_i(x)$.

Corollary 2.8 *Given a basic closed semialgebraic set $\mathcal{S}$ as in (21), where at least one of the defining polynomials $f_i(x)$ has degree four, it is NP-hard to decide whether $\mathcal{S}$ is a convex set.*

Proof Given a quartic polynomial $p(x)$, consider the basic closed semialgebraic set

$$\mathcal{E}_p = \{(x, t) \in \mathbb{R}^{n+1} |\ t - p(x) \geq 0\},$$

describing the epigraph of $p(x)$. Since $p(x)$ is convex if and only if its epigraph is a convex set, the result follows.[4] □

[2] See [3] for three other equivalent definitions of sos-convexity.

[3] All of our NP-hardness results in this paper are in the strong sense. For the sake of brevity, from now on we refer to strongly NP-hard problems simply as NP-hard problems.

[4] Another proof of this corollary is given by the NP-hardness of checking convexity of sublevel sets of quartic polynomials (Theorem 4.10 in Sect. 4.3).

Convexity of polynomials of even degree larger than four. We end this section by extending our hardness result to polynomials of higher degree.

Corollary 2.9 *It is NP-hard to check convexity of polynomials of any fixed even degree $d \geq 4$.*

Proof We have already established the result for polynomials of degree four. Given such a degree four polynomial $p(x) := p(x_1, \ldots, x_n)$ and an even degree $d \geq 6$, consider the polynomial

$$q(x, x_{n+1}) = p(x) + x_{n+1}^d$$

in $n+1$ variables. It is clear (e.g., from the block diagonal structure of the Hessian of q) that $p(x)$ is convex if and only if $q(x)$ is convex. The result follows. □

Remark 2.1 Corollary 2.9 does not establish NP-hardness of checking convexity for *forms* of fixed even degree $d \geq 6$. If needed, such a refinement is possible. One approach, which we just sketch, is to give a reduction from the problem of deciding nonnegativity of forms of fixed even degree $d \geq 4$. Given such a form $p(x)$, one can construct a form $q(x)$ of degree $d+2$ in such a way that $p(x)$ is a diagonal element of the Hessian of $q(x)$, and $p(x)$ is nonnegative if and only if $q(x)$ is convex. A construction of this type, although for a different purpose, is given in [1, Theorem 3.18].

3 Complexity of deciding strict convexity and strong convexity

3.1 Definitions and basics

Definition 3.1 A function $f : \mathbb{R}^n \to \mathbb{R}$ is *strictly convex* if for all $x \neq y$ and all $\lambda \in (0, 1)$, we have

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y). \tag{22}$$

Definition 3.2 A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is *strongly convex* if its Hessian $H(x)$ satisfies

$$H(x) \succeq mI, \tag{23}$$

for a scalar $m > 0$ and for all x.

We have the standard implications

$$\text{strong convexity} \implies \text{strict convexity} \implies \text{convexity}, \tag{24}$$

but none of the converse implications is true.

3.2 Degrees that are easy

From the implications in (24) and our previous discussion, it is clear that odd degree polynomials can never be strictly convex or strongly convex. We cover the case of quadratic polynomials in the following straightforward proposition.

Proposition 3.3 *For a quadratic polynomial* $p(x) = \frac{1}{2}x^T Qx + q^T x + c$*, the notions of strict convexity and strong convexity are equivalent, and can be decided in polynomial time.*

Proof Strong convexity always implies strict convexity. For the reverse direction, assume that $p(x)$ is not strongly convex. In view of (23), this means that the matrix Q is not positive definite. If Q has a negative eigenvalue, $p(x)$ is not convex, let alone strictly convex. If Q has a zero eigenvalue, let $\bar{x} \neq 0$ be the corresponding eigenvector. Then $p(x)$ restricted to the line from the origin to $\bar{x}$ is linear and hence not strictly convex.

To see that these properties can be checked in polynomial time, note that $p(x)$ is strongly convex if and only if the symmetric matrix Q is positive definite. By Sylvester's criterion, positive definiteness of an $n \times n$ symmetric matrix is equivalent to positivity of its n leading principal minors, each of which can be computed in polynomial time. □

3.3 Degrees that are hard

With little effort, we can extend our NP-hardness result in the previous section to address strict convexity and strong convexity.

Proposition 3.4 *It is NP-hard to decide strong convexity of polynomials of any fixed even degree* $d \geq 4$.

Proof We give a reduction from the problem of deciding convexity of quartic forms. Given a homogenous quartic polynomial $p(x) := p(x_1, \ldots, x_n)$ and an even degree $d \geq 4$, consider the polynomial

$$q(x, x_{n+1}) := p(x) + x_{n+1}^d + \tfrac{1}{2}(x_1^2 + \cdots + x_n^2 + x_{n+1}^2) \tag{25}$$

in $n + 1$ variables. We claim that p is convex if and only if q is strongly convex. The only if direction should be obvious. For the converse, suppose $p(x)$ is not convex. Let us denote the Hessians of p and q respectively by H_p and H_q. If p is not convex, then there exists a point $\bar{x} \in \mathbb{R}^n$ such that

$$\lambda_{\min}(H_p(\bar{x})) < 0,$$

where $\lambda_{\min}$ here denotes the minimum eigenvalue. Because $p(x)$ is homogenous of degree four, we have

$$\lambda_{\min}(H_p(c\bar{x})) = c^2 \lambda_{\min}(H_p(\bar{x})),$$

for any scalar $c \in \mathbb{R}$. Pick c large enough such that $\lambda_{\min}(H_p(c\bar{x})) < 1$. Then it is easy to see that $H_q(c\bar{x}, 0)$ has a negative eigenvalue and hence q is not convex, let alone strongly convex. □

Remark 3.1 It is worth noting that homogeneous polynomials of degree $d > 2$ can never be strongly convex (because their Hessians vanish at the origin). Not surprisingly, the polynomial q in the proof of Proposition 3.4 is not homogeneous.

Proposition 3.5 *It is NP-hard to decide strict convexity of polynomials of any fixed even degree* $d \geq 4$.

Proof The proof is almost identical to the proof of Proposition 3.4. Let q be defined as in (25). If p is convex, then we established that q is strongly convex and hence also strictly convex. If p is not convex, we showed that q is not convex and hence also not strictly convex. □

4 Complexity of deciding quasiconvexity and pseudoconvexity

4.1 Definitions and basics

Definition 4.1 A function $f : \mathbb{R}^n \to \mathbb{R}$ is *quasiconvex* if its sublevel sets

$$\mathcal{S}(\alpha) := \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}, \tag{26}$$

for all $\alpha \in \mathbb{R}$, are convex.

Definition 4.2 A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is *pseudoconvex* if the implication

$$\nabla f(x)^T (y - x) \geq 0 \implies f(y) \geq f(x) \tag{27}$$

holds for all x and y in $\mathbb{R}^n$.

The following implications are well-known (see e.g. [7, p. 143]):

$$\text{convexity} \implies \text{pseudoconvexity} \implies \text{quasiconvexity}, \tag{28}$$

but the converse of neither implication is true in general.

4.2 Degrees that are easy

As we remarked earlier, linear polynomials are always convex and hence also pseudoconvex and quasiconvex. Unlike convexity, however, it is possible for polynomials of odd degree $d \geq 3$ to be pseudoconvex or quasiconvex. We will show in this section that somewhat surprisingly, quasiconvexity and pseudoconvexity of polynomials of any fixed odd degree can be decided in polynomial time. Before we present these results, we will cover the easy case of quadratic polynomials.

Proposition 4.3 *For a quadratic polynomial $p(x) = \frac{1}{2}x^T Qx + q^T x + c$, the notions of convexity, pseudoconvexity, and quasiconvexity are equivalent, and can be decided in polynomial time.*

Proof We argue that the quadratic polynomial $p(x)$ is convex if and only if it is quasiconvex. Indeed, if $p(x)$ is not convex, then Q has a negative eigenvalue; letting $\bar{x}$ be a corresponding eigenvector, we have that $p(t\bar{x})$ is a quadratic polynomial in t, with negative leading coefficient, so $p(t\bar{x})$ is not quasiconvex, as a function of t. This, however, implies that $p(x)$ is not quasiconvex.

We have already argued in Sect. 2.2 that convexity of quadratic polynomials can be decided in polynomial time. □

4.2.1 Quasiconvexity of polynomials of odd degree

In this subsection, we provide a polynomial time algorithm for checking whether an odd-degree polynomial is quasiconvex. Towards this goal, we will first show that quasiconvex polynomials of odd degree have a very particular structure (Proposition 4.6).

Our first lemma concerns quasiconvex polynomials of odd degree in one variable. The proof is easy and left to the reader. A version of this lemma is provided in [9, p. 99], though there also without proof.

Lemma 4.4 *Suppose that $p(t)$ is a quasiconvex univariate polynomial of odd degree. Then, $p(t)$ is monotonic.*

Next, we use the preceding lemma to characterize the complements of sublevel sets of quasiconvex polynomials of odd degree.

Lemma 4.5 *Suppose that $p(x)$ is a quasiconvex polynomial of odd degree d. Then the set $\{x \mid p(x) \geq \alpha\}$ is convex.*

Proof Suppose not. In that case, there exist x, y, z such that z is on the line segment connecting x and y, and such that $p(x), p(y) \geq \alpha$ but $p(z) < \alpha$. Consider the polynomial

$$q(t) = p(x + t(y - x)).$$

This is, of course, a quasiconvex polynomial with $q(0) = p(x), q(1) = p(y)$, and $q(t') = p(z)$, for some $t' \in (0, 1)$. If $q(t)$ has degree d, then, by Lemma 4.4, it must be monotonic, which immediately provides a contradiction.

Suppose now that $q(t)$ has degree less than d. Let us attempt to perturb x to $x + x'$, and y to $y + y'$, so that the new polynomial

$$\hat{q}(t) = p\left(x + x' + t(y + y' - x - x')\right)$$

has the following two properties: (i) $\hat{q}(t)$ is a polynomial of degree d, and (ii) $\hat{q}(0) > \hat{q}(t'), \hat{q}(1) > \hat{q}(t')$. If such perturbation vectors x', y' can be found, then we obtain a contradiction as in the previous paragraph.

To satisfy condition (ii), it suffices (by continuity) to take x', y' with $\|x'\|$, $\|y'\|$ small enough. Thus, we only need to argue that we can find arbitrarily small x', y' that satisfy condition (i). Observe that the coefficient of t^d in the polynomial $\hat{q}(t)$ is a nonzero polynomial in $x+x'$, $y+y'$; let us denote that coefficient as $r(x+x', y+y')$. Since r is a nonzero polynomial, it cannot vanish at all points of any given ball. Therefore, even when considering a small ball around (x, y) (to satisfy condition (ii)), we can find $(x+x', y+y')$ in that ball, with $r(x+x', y+y') \neq 0$, thus establishing that the degree of $\hat{q}$ is indeed d. This completes the proof. □

We now proceed to a characterization of quasiconvex polynomials of odd degree.

Proposition 4.6 *Let $p(x)$ be a polynomial of odd degree d. Then, $p(x)$ is quasiconvex if and only if it can be written as*

$$p(x) = h(\xi^T x), \tag{29}$$

for some nonzero $\xi \in \mathbb{R}^n$, and for some monotonic univariate polynomial $h(t)$ of degree d. If, in addition, we require the nonzero component of ξ with the smallest index to be equal to unity, then ξ and $h(t)$ are uniquely determined by $p(x)$.

Proof It is easy to see that any polynomial that can be written in the above form is quasiconvex. In order to prove the converse, let us assume that $p(x)$ is quasiconvex. By the definition of quasiconvexity, the closed set $\mathcal{S}(\alpha) = \{x \mid p(x) \leq \alpha\}$ is convex. On the other hand, Lemma 4.5 states that the closure of the complement of $\mathcal{S}(\alpha)$ is also convex. It is not hard to verify that, as a consequence of these two properties, the set $\mathcal{S}(\alpha)$ must be a halfspace. Thus, for any given α, the sublevel set $\mathcal{S}(\alpha)$ can be written as $\{x \mid \xi(\alpha)^T x \leq c(\alpha)\}$ for some $\xi(\alpha) \in \mathbb{R}^n$ and $c(\alpha) \in \mathbb{R}$. This of course implies that the level sets $\{x \mid p(x) = \alpha\}$ are hyperplanes of the form $\{x \mid \xi(\alpha)^T x = c(\alpha)\}$.

We note that the sublevel sets are necessarily nested: if $\alpha < \beta$, then $\mathcal{S}(\alpha) \subseteq \mathcal{S}(\beta)$. An elementary consequence of this property is that the hyperplanes must be collinear, i.e., that the vectors $\xi(\alpha)$ must be positive multiples of each other. Thus, by suitably scaling the coefficients $c(\alpha)$, we can assume, without loss of generality, that $\xi(\alpha) = \xi$, for some $\xi \in \mathbb{R}^n$, and for all α. We then have that $\{x \mid p(x) = \alpha\} = \{x \mid \xi^T x = c(\alpha)\}$. Clearly, there is a one-to-one correspondence between α and $c(\alpha)$, and therefore the value of $p(x)$ is completely determined by $\xi^T x$. In particular, there exists a function $h(t)$ such that $p(x) = h(q^T x)$. Since $p(x)$ is a polynomial of degree d, it follows that $h(t)$ is a univariate polynomial of degree d. Finally, we observe that if $h(t)$ is not monotonic, then $p(x)$ is not quasiconvex. This proves that a representation of the desired form exists. Note that by suitably scaling ξ, we can also impose the condition that the nonzero component of ξ with the smallest index is equal to one.

Suppose that now that $p(x)$ can also be represented in the form $p(x) = \bar{h}(\bar{\xi}^T x)$ for some other polynomial $\bar{h}(t)$ and vector $\bar{\xi}$. Then, the gradient vector of $p(x)$ must be proportional to both ξ and $\bar{\xi}$. The vectors ξ and $\bar{\xi}$ are therefore collinear. Once we impose the requirement that the nonzero component of ξ with the smallest index is equal to one, we obtain that $\xi = \bar{\xi}$ and, consequently, $h = \bar{h}$. This establishes the claimed uniqueness of the representation. □

Remark It is not hard to see that if $p(x)$ is homogeneous and quasiconvex, then one can additionally conclude that $h(t)$ can be taken to be $h(t) = t^d$, where d is the degree of $p(x)$.

Theorem 4.7 *For any fixed odd degree d, the quasiconvexity of polynomials of degree d can be checked in polynomial time.*

Proof The algorithm consists of attempting to build a representation of $p(x)$ of the form given in Proposition 4.6. The polynomial $p(x)$ is quasiconvex if and only if the attempt is successful.

Let us proceed under the assumption that $p(x)$ is quasiconvex. We differentiate $p(x)$ symbolically to obtain its gradient vector. Since a representation of the form given in Proposition 4.6 exists, the gradient is of the form $\nabla p(x) = \xi h'(\xi^T x)$, where $h'(t)$ is the derivative of $h(t)$. In particular, the different components of the gradient are polynomials that are proportional to each other. (If they are not proportional, we conclude that $p(x)$ is not quasiconvex, and the algorithm terminates.) By considering the ratios between different components, we can identify the vector ξ, up to a scaling factor. By imposing the additional requirement that the nonzero component of ξ with the smallest index is equal to one, we can identify ξ uniquely.

We now proceed to identify the polynomial $h(t)$. For $k = 1, \ldots, d+1$, we evaluate $p(k\xi)$, which must be equal to $h(\xi^T \xi k)$. We thus obtain the values of $h(t)$ at $d + 1$ distinct points, from which $h(t)$ is completely determined. We then verify that $h(\xi^T x)$ is indeed equal to $p(x)$. This is easily done, in polynomial time, by writing out the $O(n^d)$ coefficients of these two polynomials in x and verifying that they are equal. (If they are not all equal, we conclude that $p(x)$ is not quasiconvex, and the algorithm terminates.)

Finally, we test whether the above constructed univariate polynomial h is monotonic, i.e., whether its derivative $h'(t)$ is either nonnegative or nonpositive. This can be accomplished, e.g., by quantifier elimination or by other well-known algebraic techniques for counting the number and the multiplicity of real roots of univariate polynomials; see [6]. Note that this requires only a constant number of arithmetic operations since the degree d is fixed. If h fails this test, then $p(x)$ is not quasiconvex. Otherwise, our attempt has been successful and we decide that $p(x)$ is indeed quasiconvex. □

4.2.2 *Pseudoconvexity of polynomials of odd degree*

In analogy to Proposition 4.6, we present next a characterization of odd degree pseudoconvex polynomials, which gives rise to a polynomial time algorithm for checking this property.

Corollary 4.8 *Let $p(x)$ be a polynomial of odd degree d. Then, $p(x)$ is pseudoconvex if and only if $p(x)$ can be written in the form*

$$p(x) = h(\xi^T x), \tag{30}$$

for some $\xi \in \mathbb{R}^n$ and some univariate polynomial h of degree d such that its derivative $h'(t)$ has no real roots.

Remark Observe that polynomials h with h' having no real roots comprise a subset of the set of monotonic polynomials.

Proof Suppose that $p(x)$ is pseudoconvex. Since a pseudoconvex polynomial is quasiconvex, it admits a representation $h(\xi^T x)$ where h is monotonic. If $h'(t) = 0$ for some t, then picking $a = t \cdot \xi / \|\xi\|_2^2$, we have that $\nabla p(a) = 0$, so that by pseudoconvexity, $p(x)$ is minimized at a. This, however, is impossible since an odd degree polynomial is never bounded below. Conversely, suppose $p(x)$ can be represented as in Eq. (30). Fix some x, y, and define the polynomial $u(t) = p(x + t(y - x))$. Since $u(t) = h(\xi^T x + t\xi^T (y - x))$, we have that either (i) $u(t)$ is constant, or (ii) $u'(t)$ has no real roots. Now if $\nabla p(x)(y - x) \geq 0$, then $u'(0) \geq 0$. Regardless of whether (i) or (ii) holds, this implies that $u'(t) \geq 0$ everywhere, so that $u(1) \geq u(0)$ or $p(y) \geq p(x)$. □

Corollary 4.9 *For any fixed odd degree d, the pseudoconvexity of polynomials of degree d can be checked in polynomial time.*

Proof This is a simple modification of our algorithm for testing quasiconvexity (Theorem 4.7). The first step of the algorithm is in fact identical: once we impose the additional requirement that the nonzero component of ξ with the smallest index should be equal to one, we can uniquely determine the vector ξ and the coefficients of the univariate polynomial $h(t)$ that satisfy Eq. (30). (If we fail, $p(x)$ is not quasiconvex and hence also not pseudoconvex.) Once we have $h(t)$, we can check whether $h'(t)$ has no real roots e.g. by computing the signature of the Hermite form of $h'(t)$; see [6]. □

Remark 4.1 Homogeneous polynomials of odd degree $d \geq 3$ are never pseudoconvex. The reason is that the gradient of these polynomials vanishes at the origin, but yet the origin is not a global minimum since odd degree polynomials are unbounded below.

4.3 Degrees that are hard

The main result of this section is the following theorem.

Theorem 4.10 *It is NP-hard to check quasiconvexity/pseudoconvexity of degree four polynomials. This is true even when the polynomials are restricted to be homogeneous.*

In view of Theorem 2.1, which established NP-hardness of deciding convexity of *homogeneous* quartic polynomials, Theorem 4.10 follows immediately from the following result.[5]

Theorem 4.11 *For a homogeneous polynomial $p(x)$ of even degree d, the notions of convexity, pseudoconvexity, and quasiconvexity are all equivalent.*[6]

[5] A slight variant of Theorem 4.11 has appeared in [3].

[6] The result is more generally true for differentiable functions that are homogeneous of even degree. Also, the requirements of homogeneity and having an even degree both need to be present. Indeed, x^3 and $x^4 - 8x^3 + 18x^2$ are both quasiconvex but not convex, the first being homogeneous of odd degree and the second being nonhomogeneous of even degree.

We start the proof of this theorem by first proving an easy lemma.

Lemma 4.12 *Let $p(x)$ be a quasiconvex homogeneous polynomial of even degree $d \geq 2$. Then $p(x)$ is nonnegative.*

Proof Suppose, to derive a contradiction, that there exist some $\epsilon > 0$ and $\bar{x} \in \mathbb{R}^n$ such that $p(\bar{x}) = -\epsilon$. Then by homogeneity of even degree we must have $p(-\bar{x}) = p(\bar{x}) = -\epsilon$. On the other hand, homogeneity of p implies that $p(0) = 0$. Since the origin is on the line between $\bar{x}$ and $-\bar{x}$, this shows that the sublevel set $\mathcal{S}(-\epsilon)$ is not convex, contradicting the quasiconvexity of p. □

Proof of Theorem 4.11 We show that a quasiconvex homogeneous polynomial of even degree is convex. In view of implication (28), this proves the theorem.

Suppose that $p(x)$ is a quasiconvex polynomial. Define $\mathcal{S} = \{x \in \mathbb{R}^n \mid p(x) \leq 1\}$. By homogeneity, for any $a \in \mathbb{R}^n$ with $p(a) > 0$, we have that

$$\frac{a}{p(a)^{1/d}} \in \mathcal{S}.$$

By quasiconvexity, this implies that for any a, b with $p(a), p(b) > 0$, any point on the line connecting $a/p(a)^{1/d}$ and $b/p(b)^{1/d}$ is in $\mathcal{S}$. In particular, consider

$$c = \frac{a+b}{p(a)^{1/d} + p(b)^{1/d}}.$$

Because c can be written as

$$c = \left(\frac{p(a)^{1/d}}{p(a)^{1/d} + p(b)^{1/d}}\right)\left(\frac{a}{p(a)^{1/d}}\right) + \left(\frac{p(b)^{1/d}}{p(a)^{1/d} + p(b)^{1/d}}\right)\left(\frac{b}{p(b)^{1/d}}\right),$$

we have that $c \in \mathcal{S}$, i.e., $p(c) \leq 1$. By homogeneity, this inequality can be restated as

$$p(a+b) \leq (p(a)^{1/d} + p(b)^{1/d})^d,$$

and therefore

$$p\left(\frac{a+b}{2}\right) \leq \left(\frac{p(a)^{1/d} + p(b)^{1/d}}{2}\right)^d \leq \frac{p(a)+p(b)}{2}, \tag{31}$$

where the last inequality is due to the convexity of x^d.

Finally, note that for any polynomial p, the set $\{x \mid p(x) \neq 0\}$ is dense in $\mathbb{R}^n$ (here we again appeal to the fact that the only polynomial that is zero on a ball of positive radius is the zero polynomial); and since p is nonnegative due to Lemma 4.12, the set $\{x \mid p(x) > 0\}$ is dense in $\mathbb{R}^n$. Using the continuity of p, it follows that Eq. (31) holds not only when a, b satisfy $p(a), p(b) > 0$, but for all a, b. Appealing to the continuity of p again, we see that for all a, b, $p(\lambda a + (1-\lambda)b) \leq \lambda p(a) + (1-\lambda)p(b)$, for all $\lambda \in [0, 1]$. This establishes that p is convex. □

Quasiconvexity/pseudoconvexity of polynomials of even degree larger than four.

Corollary 4.13 *It is NP-hard to decide quasiconvexity of polynomials of any fixed even degree $d \geq 4$.*

Proof We have already proved the result for $d = 4$. To establish the result for even degree $d \geq 6$, recall that we have established NP-hardness of deciding convexity of homogeneous quartic polynomials. Given such a quartic form $p(x) := p(x_1, \ldots, x_n)$, consider the polynomial

$$q(x_1, \ldots, x_{n+1}) = p(x_1, \ldots, x_n) + x_{n+1}^d. \tag{32}$$

We claim that q is quasiconvex if and only if p is convex. Indeed, if p is convex, then obviously so is q, and therefore q is quasiconvex. Conversely, if p is not convex, then by Theorem 4.11, it is not quasiconvex. So, there exist points $a, b, c \in \mathbb{R}^n$, with c on the line connecting a and b, such that $p(a) \leq 1$, $p(b) \leq 1$, but $p(c) > 1$. Considering points $(a, 0)$, $(b, 0)$, $(c, 0)$, we see that q is not quasiconvex. It follows that it is NP-hard to decide quasiconvexity of polynomials of even degree four or larger. □

Corollary 4.14 *It is NP-hard to decide pseudoconvexity of polynomials of any fixed even degree $d \geq 4$.*

Proof The proof is almost identical to the proof of Corollary 4.13. Let q be defined as in (32). If p is convex, then q is convex and hence also pseudoconvex. If p is not convex, we showed that q is not quasiconvex and hence also not pseudoconvex. □

5 Summary and conclusions

We studied the computational complexity of testing convexity and some of its variants, for polynomial functions. The notions that we considered and the implications among them are summarized below:

strong convexity $\Longrightarrow$ strict convexity $\Longrightarrow$ convexity $\Longrightarrow$ pseudoconvexity $\Longrightarrow$ quasiconvexity.

Our complexity results as a function of the degree of the polynomial are listed in Table 1 in Sect. 1.

We gave polynomial time algorithms for checking pseudoconvexity and quasiconvexity of odd degree polynomials that can be useful in many applications. Our negative results, on the other hand, imply (under P≠NP) the impossibility of a polynomial time (or even pseudo-polynomial time) algorithm for testing any of the properties listed in Table 1 for polynomials of even degree four or larger. Although the implications of convexity are very significant in optimization theory, our results suggest that unless additional structure is present, ensuring the mere presence of convexity is likely an intractable task. It is therefore natural to wonder whether there are other properties of optimization problems that share some of the attractive consequences of convexity, but are easier to check.

Of course, NP-hardness of a problem does not stop us from studying it, but on the contrary, stresses the need for finding good approximation algorithms that can deal with a large number of instances efficiently. As an example, semidefinite programming based relaxations relying on algebraic concepts such as sum of squares decomposition of polynomials currently seem to be very promising techniques for recognizing convexity of polynomials and basic semialgebraic sets. It would be useful to identify special cases where these relaxations are exact or give theoretical bounds on their performance guarantees.

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