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Three problems on the decidability and complexity of stability

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1 Decidability and complexity

We describe three simply-stated problems that deal with the notion of stability.

All three problems are yes-no *decision* problems; upon input of the data associated with an instance of the problem, we wish to *decide* whether a certain property is satisfied by the instance. Many results are available in the literature for these three problems, but no satisfactory answers are yet available. We suggest looking at the decidability and at the computational complexity of these three problems.

We say that a problem is *decidable* if there is an algorithm which, upon input of the data associated with an instance of the problem, provides a yes-no answer after finitely many steps. The precise definition of what is meant by an *algorithm* is not critical; most algorithm models proposed so far are known to be equivalent from the point of view of their computing capabilities (see [Hopcroft and Ullman, 1969]).

V. Blondel et al. (eds.), *Open Problems in Mathematical Systems and Control Theory*

We say that a problem can be decided in *polynomial time*, or that it can be decided *efficiently*, if there is a polynomial p and an algorithm which, upon input of an instance Σ of the problem, provides a yes-no answer after at most $p(size(\Sigma))$ computational steps. Again, the precise definition of *size*(Σ), and of *computational step* are not critical. The property of being decidable in polynomial time is robust across all reasonable definitions. The class P is the class of problems that can be decided in polynomial time.
The class NP is a class of problems that includes all problems in P and includes a large number of problems of practical interest for which no polynomial time algorithms have yet been found. It is widely believed that $P \neq NP$. A problem is NP-hard if it is at least as hard as any problem in NP. A polynomial time algorithm for an NP-hard problem would immediately result in polynomial time algorithms for all problems in NP. Finally, a problem is NP-complete if it is NP-hard and belongs to NP. See [Papadimitriou, 1994] for more details. The reference [Blondel and Tsitsiklis, 1998] surveys complexity results available for systems and control problems.

2 Static output feedback

We are given an input-output linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

and we consider a static feedback control law $u = Ky$. The resulting closed loop is

$$\dot{x} = (A + BKC)x.$$

The problem is to find conditions on the triplet of real matrices (A, B, C) under which there exists a feedback gain matrix K such that $A + BKC$ is stable, i.e., has all its eigenvalues in the left half plane.

STABILIZATION BY STATIC OUTPUT FEEDBACK.
Instance: Matrices A, B and C.
Problem: Does there exists a matrix K such that $A + BKC$ has all its eigenvalues in the left half plane?

Let n be the dimension of A. In the case of full state feedback ($C = I$) a necessary and sufficient condition for the the system to be stabilizable by static output feedback is that the rank of the matrix

$$(B, AB, A^2B, \ldots, A^{n-1}B)$$

is equal to n. This condition can be checked in a number of operations that is polynomial in the dimension of the matrices A and B; see, e.g., [Schrijver, 1986]. When C is invertible, a similar condition can be obtained easily. When C is not invertible, no general tractable necessary and sufficient conditions are known. After more than two decades of research it seems now unlikely that a closed-form solution exists to this problem. In [Anderson, Bose and Jury, 1975] it is shown that an *algorithmic* solution is possible. The algorithm proposed in this reference is based on the Tarski-Seidenberg elimination algorithm and uses, in the worst case, a number of operations that grows faster than any polynomial in the number of input and output variables.

Open Problem 1: Can STABILIZATION BY STATIC OUTPUT FEEDBACK be solved in time polynomial in the size of the matrices A, B and C? Is the problem NP-hard?

In [Blondel and Tsitsiklis, 1997] it is shown that the following related *constrained* problem is indeed NP-hard.

STABILIZATION BY CONSTRAINED STATIC OUTPUT FEEDBACK.
Instance: Matrices A, B and C, rational numbers $\underline{k}_{ij}, \overline{k}_{ij}$.
Problem: Does there exists a matrix $K = (k_{ij})$ satisfying $\underline{k}_{ij} \leq k_{ij} \leq \overline{k}_{ij}$ and such that $A + BKC$ has all its eigenvalues in the left half plane?

There does not seem to be any easy extension of the proof of this result for the unconstrained case.

3 Stability of all infinite products

Let $\Omega = \{A_1, \ldots, A_m\}$ be a set of $n \times n$ real matrices. Given a system of the form

$$x_{t+1} = A_t x_t \tag{11.1}$$

suppose that it is known that $A_t \in \Omega$, for each t, but that the exact value of A_t is not a priori known, because of exogenous conditions or changes in the operating point of the system. This system can also be thought of as a time-varying system. We say that such a system is *stable* if

$$\lim_{t \to \infty} x_t = 0$$

for all initial states x_0 and all sequences of matrix products. This condition is equivalent to the requirement

$$\lim_{t \to \infty} A_{i_t} \cdots A_{i_1} A_{i_0} = 0$$

for all sequences of indices i_j.

STABILITY OF ALL INFINITE PRODUCTS.
Instance: A finite set of $n \times n$ matrices $\Omega = \{A_1, \ldots, A_m\}$.
Problem: Do the products

$$A_{i_t} \cdots A_{i_1} A_{i_0}$$

converge to zero for all sequences of indices i_j?

This problem is obviously decidable when $n = 1$ and when $m = 1$. No general decision algorithms are known for any other values of n and m.

Open Problem 2: For what values of n and m is STABILITY OF ALL INFINITE PRODUCTS decidable?

The problem is known to be related to the *finiteness conjecture* on the generalized spectral radius of matrices. Let $\rho(A)$ denote the *spectral radius* of a real matrix A,

$$\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Let $\Omega = \{A_1, \ldots, A_m\}$ be a finite set of matrices. The *generalized spectral radius* $\rho'(\Omega)$ is defined in [Daubechies and Lagarias, 1992] by

$$\rho'(\Omega) = \limsup_{k \to \infty} \rho'_k(\Omega), \tag{11.2}$$

where

$$\rho'_k(\Omega) = \max\{(\rho(A_1 A_2 \cdots A_k))^{1/k} : \text{each } A_i \in \Omega\}$$

for each $k \geq 1$. It is conjectured in [Lagarias and Wang, 1995] that the equality $\rho'(\Omega) = \rho'_k(\Omega)$ always occur for some finite k. If this conjecture is true, then STABILITY OF ALL INFINITE PRODUCTS is decidable. Conversely, if the problem is undecidable for some n and m, then the finiteness conjecture must be false.

The related problem for which the matrices in (11.1) occur with a certain probability is studied in [Tsitsiklis and Blondel, 1997]. The corresponding stability problem is then undecidable. See also [Gurvits, 1995].

4 Stability of systems of the neural type

Let $\sigma : \mathbf{R} \mapsto \mathbf{R}$ be a fixed scalar function and consider the system

$$x_{t+1} = \sigma(Ax_t) \tag{11.3}$$

where σ is defined componentwise and A is a real matrix. The system is *stable* if

$$\lim_{t\to\infty} x_t = 0$$

for all initial states x_0.

STABILITY OF σ-SYSTEMS.
Instance: A $n \times n$ matrices A.
Problem: Is the system $x_{t+1} = \sigma(Ax_t)$ stable?

The dynamics of such systems depends on the function σ. When σ is linear, the system is linear and stability is easy to check. When σ has finite range, stability can be decided by simple enumeration since there are only finitely many possible states.

Open problem 3: What are the functions σ for which stability of $x_{t+1} = \sigma(Ax_t)$ is undecidable?

The systems $x_{t+1} = \sigma(Ax_t)$ arise in a wide variety of situations. In particular, recurrent artificial neural networks are modeled by such equations where the function σ is the activation function used in the network. In [Siegelmann and Sontag, 1991] it is shown that, when σ is the saturated linear function, systems of this type are capable of simulating arbitrary Turing machines. Thus, as computational devices, linear saturated systems are as powerful as Turing machines. From this result it is easy to prove that the problem of deciding whether a given initial state of a saturated linear system eventually reaches a certain state (that encodes a halting configuration), is undecidable (see [Sontag, 1995]). Similar simulations are given in [Koiran, 1996] for a large class of other functions σ. These results do however not have direct implications for the decidability of stability of such systems. Undecidability of stability for saturated linear systems was conjectured in [Sontag, 1995].

Acknowledgments

This research was sponsored by the European Commission (TMR (Alapedes), network contract ERBFMRXCT960074) and by the NATO under grant CRG-961115.

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