8.808/8.308 IAP 2026 Recitation 9: Ballistic deposition

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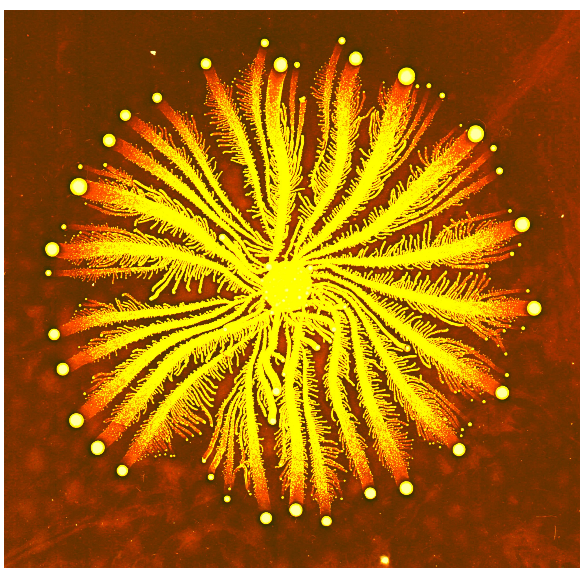
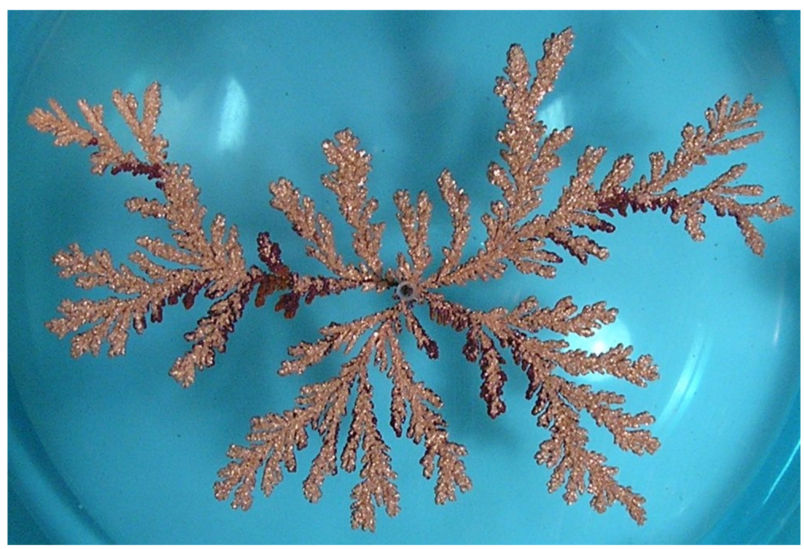
Contents

Today we'll look at a model of interface growth that is straightforward and fun to numerically simulate.

The natural world is full of interfaces with wildly different morphologies. Sometimes they are almost completely flat (e.g. the liquid-gas interface at the surface of a still water droplet), but sometimes they are very rough. For instance, the interfaces of some bacterial colonies (left) and some electrodeposition processes (right) form fractal-like structures:



https://commons.wikimedia.org/wiki/File:Paenibacillus_vortex_colony.jpg

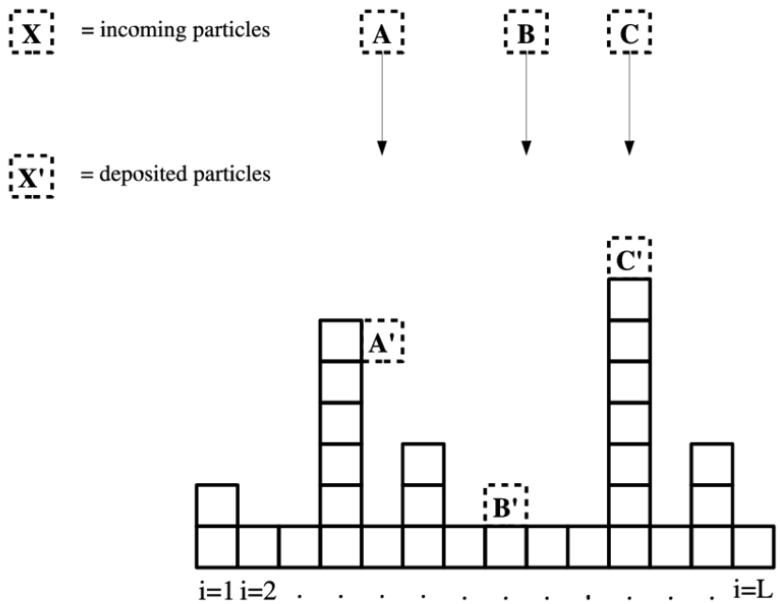


https://commons.wikimedia.org/wiki/File:DLA_Cluster.JPG

1 A simple model of interface growth

One popular model of interface growth called "ballistic deposition" can be numerically implemented in a straightforward way that is similar to some of our previous simulations. We will consider a 1-dimensional, flat substrate with L sites. Our interface will be represented by "blocks" that will be deposited onto the substrate. The blocks will fall at some uniform rate r, randomly selecting a lattice site each time.

However, we will treat the blocks as sticky: when a particle falls on a site that is next to another site with a taller stack of blocks, it will stick to the highest block in that neighboring site. This is visualized in the figure below (from Cakir et al. https://arxiv.org/abs/cond-mat/0610245):



We will implement this numerically.

2 Observables

Let's look at the behavior of the height of the interface over time:

$$h_i(t) \equiv \text{height of highest block at site } i \ . \tag{1}$$

It is straightforward to convince yourself that the average height $\overline{h}(t)$ should grow at a steady rate, since the blocks are deposited at a steady rate:

$$\overline{h}(t) \sim t \ . \tag{2}$$

We can study the structure of this moving interface through properties like the roughness $w(t)$

$$w(t) \equiv \sqrt{\frac{1}{L}\sum_{i=1}^{N}\left[h_i(t) - \overline{h}(t)\right]^2} \ . \tag{3}$$

It is unclear how this will evolve over time. We can imagine that it might follow some power law

$$w(t) \sim t^{\beta} \ . \tag{4}$$

But what is β? We will measure it in our simulations.

3 Numerics

You will simulate a ballistic deposition process using `Rec9_ballistic_deposition.jl` and `Rec9_ballistic_deposition_module.jl`. After filling in the ballistic deposition algorithm, running the code will generate a video of the dynamics (the object `state.img` stores a binary image of the filled and empty squares). Then, it will make a plot of the interface roughness over time and fit it to a power-law, averaging it over multiple independent realizations (`n_seeds`). You can adjust the system size L, the duration of the movie, the spacing of points in the plot, number of points in the plot, and number of seeds.

Note that if you plot the roughness over a long enough time, it will no longer obey a simple power law. If you have time, you can try to characterize this change, and see how it changes with the system size.

4 Finding a growth equation

Though this model has simple rules, it is difficult to characterize analytically. We will now use various approximations to write it in a more familiar form: a Langevin equation, but for the infinite-dimensional continuum height field $h(x,t)$. We will now work with a continuous domain $x \in [0, L]$.

The equation should be of the form

$$\partial_t h(x,t) = G(h(x,t), x, t) + \sqrt{2D}\eta(x,t) \tag{5}$$

for some noise η. Using the following symmetry principles, we can constrain the form of G (See Barabasi & Stanley ch. 5):

- Time-homogeneity
  $\implies$ G has no explicit time-dependence
- Translational symmetry with respect to the h-axis
  $\implies$ G no explicit dependence on h, only on its derivatives
- Translational symmetry with respect to the x-axis
  $\implies$ G has no explicit dependence on x
- Inversion symmetry with respect to the x-axis
  $\implies$ G has no odd derivatives of h.

Thus, we should have something like

$$\partial_t h = v_0 + \nu \partial_x^2 h + \nu' \partial_x^4 h + \ldots + \frac{\lambda}{2}(\partial_x h)^2 + \ldots + \sqrt{2D}\eta \tag{6}$$

Moreover, we can use the following simplifications to truncate our equation:

- Large-scale/long-wavelength limit
  $\implies$ no spatial derivatives beyond the lowest order, $\mathcal{O}(\partial_x^2)$
  $\implies$ whatever form the noise takes, we can take it to be δ-correlated in space
- Long-time/low-frequency limit
  $\implies$ no time derivatives beyond the lowest order, ∂_t
  $\implies$ whatever form the noise takes, we can take it to be δ-correlated in time
- Small-fluctuation limit of h
  $\implies$ smaller powers of h
- Only worry about *relative* fluctuations of h, not overall displacement
  $\implies$ move to co-moving frame and take $v_0 = 0$.

Of course, if our truncated equation turns out to be insufficient to describe the behavior of the ballistic deposition model, we will have to include some higher-order terms.

4.1 The linear model

To lowest order, we have a linear equation

$$\partial_t h = \nu \partial_x^2 h + \sqrt{2D}\eta(x,t) \; , \qquad \text{where} \tag{7}$$

$$\langle \eta(x,t) \rangle = 0 \; , \qquad \langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t') \; . \tag{8}$$

This equation is explicitly solvable by Fourier transform. Use

$$h(k,t) = \int dx e^{ikx} h(x,t) \; , \qquad \eta(k,t) = \int dx e^{ikx} \eta(x,t) \tag{9}$$

where

$$\langle \eta \rangle = 0 \; , \quad \langle \eta(k,t)\eta(k',t') \rangle = \int_0^L dx \int_0^L dx' e^{ikx+ik'x'} \underbrace{\langle \eta(x,t)\eta(x',t') \rangle}_{\delta(x-x')\delta(t-t')} = \delta(t-t') \int_0^L dx e^{i(k+k')x} = \delta(t-t') L \delta_{k+k',0} \tag{10}$$

and $\delta_{k+k',0}$ is a Kroenecker delta, since our space $[0, L]$ is finite. We can immediately see that Eq. (7) simplifies to

$$\partial_t h(k,t) = -\nu k^2 h(k,t) + \sqrt{2D}\eta(k,t) \; . \tag{11}$$

If we look at each wavelength k separately, we see that they all behave like independent particles of diffusivity D in a harmonic well with stiffness νk^2. We know the explicit solution of this system:

$$h(k,t) = h(0,t)e^{-\nu k^2 t} + \sqrt{2D}\int_0^t ds e^{-\nu k^2(t-s)}\eta(s) \; . \tag{12}$$

If we start with a flat interface at $h = 0$ (as we do in our simulations), we have

$$h(k,t) = \sqrt{2D}\int_0^t ds e^{-\nu k^2(t-s)}\eta(k,s) \; . \tag{13}$$

We can then immediately calculate the roughness. Before launching into the calculation, we note that

$$w(t)^2 = \frac{1}{L}\int_0^L dx\langle h(x,t)^2\rangle = \frac{1}{L}\underbrace{\int_0^L dx \sum_{k,q} e^{-ikx-iqx}}_{L\delta_{k+q,0}}\langle h(k,t)h(q,t)\rangle = \sum_k \langle h(k,t)h(-k,t)\rangle \; . \tag{14}$$

Thus it is sufficient to calculate, and then sum, $\langle h(k,t)h(-k,t)\rangle$. Eq. (13) gives us

$$\langle h(k,t)h(-k,t)\rangle = 2D\int_0^t ds_1 \int_0^t ds_2 e^{-\nu k^2(2t-s_1-s_2)} \underbrace{\langle \eta(k,s_1)\eta(-k,s_2)\rangle}_{=L\delta(s_1-s_2)} \tag{15}$$

$$= 2DL\int_0^t ds_1 e^{-2\nu k^2(t-s_1)} = \frac{DL}{\nu k^2}\left[1 - e^{-2\nu k^2 t}\right] \; . \tag{16}$$

Finally, we sum this to find w:

$$w(t)^2 = \sum_k \frac{DL}{\nu k^2}\left[1 - e^{-2\nu k^2 t}\right] = \text{const.} - \frac{DL}{\nu}\sum_k \frac{e^{-2\nu k^2 t}}{k^2} \; . \tag{17}$$

The time-dependence is contained in the second term. For a large enough system, we can approximate it as an integral, and find

$$w(t)^2 \sim \int \frac{dk}{2\pi}\frac{e^{-2\nu k^2 t}}{k^2} \; . \tag{18}$$

The time-scaling can be found by directly scaling time by a factor γ:

$$w(\gamma t)^2 \sim \int \frac{dk}{2\pi}\frac{e^{-2\nu k^2 \gamma t}}{k^2} = \int \frac{d\tilde{k}/\sqrt{\gamma}}{2\pi}\frac{e^{-2\nu \tilde{k}^2 t}}{\tilde{k}^2/\gamma} = \sqrt{\gamma}\int \frac{d\tilde{k}}{2\pi}\frac{e^{-2\nu \tilde{k}^2 t}}{\tilde{k}^2} \sim \sqrt{\gamma}w(t)^2 \; . \tag{19}$$

We conclude that $w(t)^2 \sim t^{1/2}$, and thus the roughness should scale as

$$w(t) \sim t^{1/4} \; . \tag{20}$$

But in simulations, with a large enough system, we see

$$w(t) \sim t^{0.33} \; . \tag{21}$$

What is missing?

4.2 The nonlinear model

It turns out that if we want to describe the ballistic deposition model, we have truncated our phenomenological growth equation too early. We need to include a nonlinear term in the growth equation, the lowest order of which is:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 + \sqrt{2D}\eta \; . \tag{22}$$

This is the Kardar-Parisi-Zhang (KPZ) equation. Note that it breaks the up-down symmetry observed by the linear model. This is important for modeling the ballistic deposition model, which exhibits lateral growth due to the "stickiness" of the blocks.

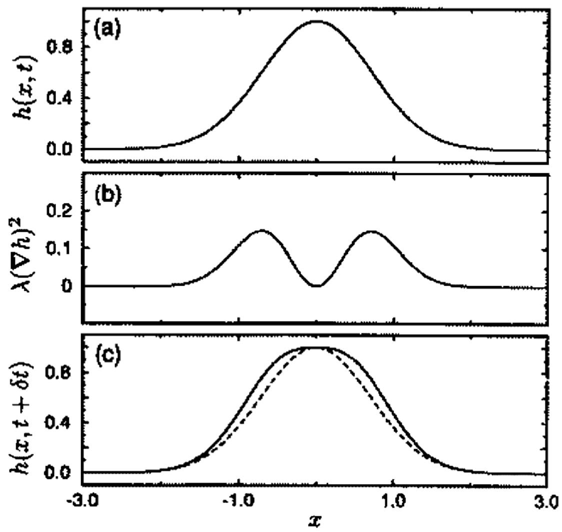


Fig 6.2 of Barabasi & Stanley (1991)

But does this equation predict the correct roughness exponent? It is much harder to show this explicitly. One option is to instead directly simulate the KPZ equation using numerical PDE integration. This is a big numerical problem of its own, which we will look at in the next recitation.

5 References

- Barabasi & Stanley (1991), *Fractal concepts in surface growth*
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