8.808/8.308 IAP 2026 Recitation 6: Blowtorch ratchets

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January 21, 2026

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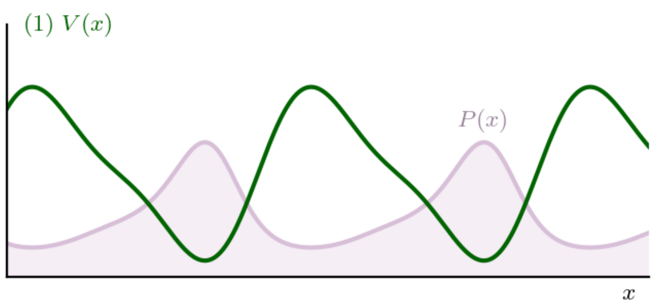
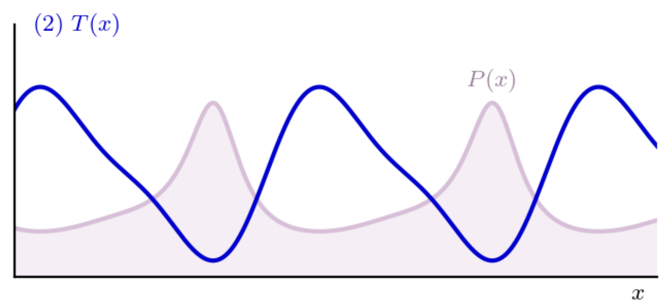
1 Introduction

We've now spent some time studying two different types of system:
(1) Overdamped Brownian particles in an external potential $V(x)$
(2) Overdamped Brownian particle in a spatially-varying temperature $T(x)$, with discretization α



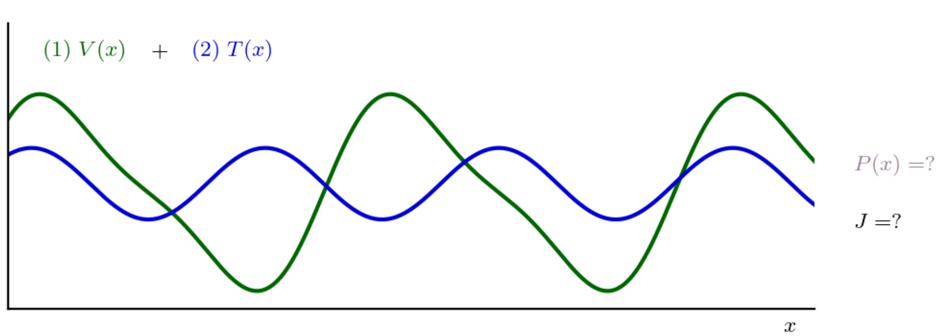
$$P(x) \propto e^{-\beta V(x)} \quad \Longrightarrow \quad J = -P\partial_x V - T\partial_x P = 0$$



$$P(x) \propto T(x)^{\alpha-1} \quad \Longrightarrow \quad J = -T^{\alpha}\partial_x(T^{1-\alpha}P) = 0$$

Both of these systems have no steady-state current, regardless of the $V(x)$ or $T(x)$ landscape chosen. We can see this at the level of the steady-state probability distributions and current at the level of the Fokker-Planck equation, but also from time-reversal symmetry considerations. Case (1) with a potential is in thermal equilibrium. Case (2) has a hidden time-reversal symmetry, which we proved using the operator method.

But what happens when we combine cases (1) and (2)?



We will study this system today, and show that it does, in fact, display ratchet currents.

2 Brownian particle with $V(x)$ and $T(x)$

Throughout today, we will work in $d = 1$ spatial dimension, and with unit mobility $\mu = 1$. We will use units so the Boltzmann constant $k_B = 1$. We will work in the Itō discretization, i.e. $\alpha = 0$. The Fokker-Planck equation is then

$$\dot{P} = -\partial_x J \ , \qquad J = -P\partial_x V - \partial_x(PT) \ . \tag{1}$$

There are 2 reasons to use the Itō discretization:

1. There is no loss of generality in fixing a discretization now. If we started with a different discretization α, the FPE would then be

$$J = -P\partial_x V - T^{\alpha}\partial_x(PT^{1-\alpha}) = -P\partial_x V - \partial_x(TP) + \alpha P\partial_x T \equiv -P\partial_x \tilde{V} - \partial_x(TP) \ , \tag{2}$$

   where we have defined new effective $\tilde{V} \equiv V - \alpha T$. (This force $\alpha\partial_x T$ is what is often called the "spurious drift".) Since we are working with an arbitrary V, this is the same problem.

2. The Itō discretization is physically-motivated: it is the overdamped limit of an underdamped Brownian particle in a spatially-varying temperature. This is shown in the Appendix.

Clearly some straightforward generalization of the Boltzmann distribution, e.g. $e^{-V(x)/T(x)}$, won't solve Eq. (1). Thus we will start from the beginning and systematically derive the solution.

3 Solving the FPE in 1 dimension

We will now solve the FPE [Eq. (1)]. Work in periodic boundary conditions with a system of length L. Van Kampen gave the solution to this equation for arbitrary $\mu(x)$, $T(x)$, and $V(x)$.[1] We will go through part of the solution here.

First, note that in the steady-state, $0 = \dot{\rho} = -J'$ and thus the current J must be constant, i.e.

$$\rho(T' + V') + T\rho' = -J = \text{const.} \ . \tag{3}$$

If the steady-state is current-free, i.e. $J = 0$, then we would have

$$\text{current-free, i.e. } \ J = 0 \qquad \Longrightarrow \qquad \frac{\rho'}{\rho} = \frac{V' + T'}{T} \qquad \Longrightarrow \qquad \rho(x) \propto \frac{e^{-\Phi(x)}}{T(x)} \tag{4}$$

where we have defined

$$\Phi(x) \equiv \int_0^x du \frac{V'(u)}{T(u)} \ . \tag{5}$$

Note that this "function" may not be periodic on our domain. If it isn't periodic, then ρ is not periodic, which is not impossible. We conclude that when $\Phi(x)$ is aperiodic, the current J is nonzero. To find the solution in this case, we try out the new ansatz

$$\rho(x) = \frac{e^{-\Phi(x)}}{T(x)} g(x) \tag{6}$$

which, inserted into Eq. (3), gives

$$g'(x)e^{-\Phi(x)} = -J = \text{const.} \qquad \Longrightarrow \qquad g(x) = C - J\int_0^x e^{\Phi(u)} du \ . \tag{7}$$

The constants C and J are then determined by enforcing periodicity and normalization of ρ. In summary, we have

$$\rho(x) = \frac{e^{-\Phi(x)}}{T(x)} \left[\rho(0)T(0) - J\int_0^x e^{\Phi(u)} du\right] \tag{8}$$

where we have made the appropriate substitution $C = \rho(0)T(0)$, so that now $\rho(0)$ and J are the constants to be determined.

[1] N. G. van Kampen, *Relative stability in nonuniform temperature*, IBM J. Res. Dev. (1988).

Periodicity of ρ enforces

$$\rho(0) = \rho(L) = \frac{e^{-\Phi(L)}}{T(L)}\left[\rho(0)T(0) - J\int_0^L e^{\Phi(u)}\right] = e^{-\Phi(L)}\left[\rho(0) - \frac{J}{T(0)}\int_0^L e^{\Phi(u)}\right] \tag{9}$$

$$\implies J = \frac{\rho(0)T(0)\big(1 - e^{\Phi(L)}\big)}{\int_0^L e^{\Phi(x)}dx} \ . \tag{10}$$

We now see that the current is nonzero *if and only if* Φ is aperiodic, i.e. $\Phi(L) \neq \Phi(0) = 0$. Thus, an inhomogeneous temperature field, working together with an external potential, can generate steady-state currents. This is the celebrated "blowtorch ratchet".[2]

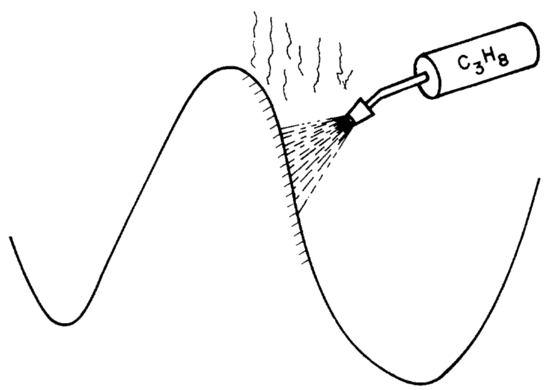
4 Blowtorch ratchets

Let's consider the case where

$$V(x) \quad \text{arbitrary} \ , \tag{11}$$

$$T(x) = \begin{cases} T_0 \ , & 0 \leq x < \lambda \\ T_1 \ , & \lambda \leq x < L \end{cases} \ . \tag{12}$$

One possible physical realization of this scenario was sketched in Landauer's 1988 paper (Fig. 3):



Then, we can directly calculate $\Phi(x)$:

$$\Phi(x) = \begin{cases} \frac{V(x)-V(0)}{T_0} \ , & 0 \leq x < \lambda \\ \frac{V(\lambda)-V(0)}{T_0} + \frac{V(x)-V(\lambda)}{T_1} \ , & \lambda \leq x < L \end{cases} \tag{13}$$

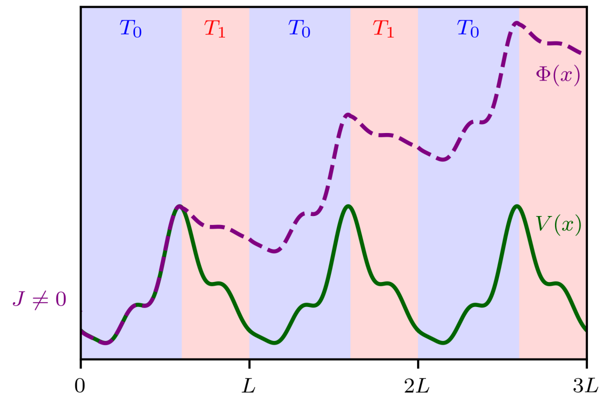
and thus

$$\Phi(L) = \big[V(\lambda) - V(0)\big]\left(\frac{1}{T_0} - \frac{1}{T_1}\right) \ . \tag{14}$$

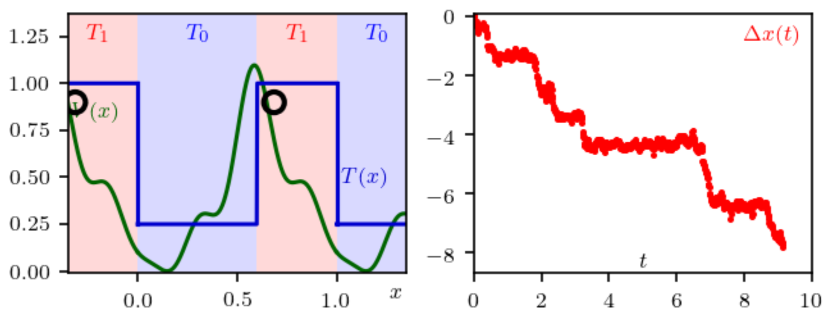
By the condition derived above, the current is nonzero only if $T_0 \neq T_1$. Moreover, the points where the temperature switches must have different V.

I have chosen some periodic $V(x)$, along with some T_0, T_1, and λ, and plotted the resulting $\Phi(x)$ alongside V. For x_1 and x_2 both inside the same region of constant $T(x) = T_0$, the integration simply tells us that $\Phi(x_2) - \Phi(x_1) = [V(x_2) - V(x_1)]/T_0$, i.e. differences in Φ are proportional to differences in V. However, upon crossing into the region where $T = T_1$, the constant of proportionality changes. This change in slope (below, a softening of the slope) causes $\Phi(L)$ to deviate from $\Phi(0)$. The resulting Φ increases (or decreases) indefinitely as you loop around the system.

[2]M. Büttiker, Z. Phys. B: Condens. Matter 68, 161 1987; R. Landauer, J. Stat. Phys. 53, 233 1988



I've done some simulations of a particle in a periodic potential $V(x)$ with a temperature $T_0 = 0.25$ for x between 0 and 0.6, and $T_1 = 1$ for x between 0.6 and 1. The resulting system indeed shows net motion towards the left.



When watching an animation of the system, the physical mechanism for the current is clear. When the particle is in the low-temperature region, it mostly sits near the minimum of V. When it moves into the higher-temperature region (which is accessible to a particle at the minimum of V), its position fluctuates wildly, sometimes reaching the top of the potential. When it crosses the top of the potential, its temperature goes back down to T_0, and it falls back down into the minimum. At this point, it has circled around the system once. This continues indefinitely, and after long times, the particle has circled the system many times.

4.1 Thermodynamic interpretation

The steady-state probability distribution (8) sort of looks like a Boltzmann distribution, if the weight is changed from $V(x)/T$ to $\Phi(x) + \ln T(x)$ and we multiply it by the correction $g(x)$ (necessary to maintain periodicity). This meshes well with a thermodynamic interpretation due to Landauer. Forgetting about the discretization ambiguities of stochastic integrals for a moment, the particle moving a small distance in the system will absorb some heat dQ from the reservoir, such that

$$dQ = dV + dT \, . \tag{15}$$

The change in the particle's entropy after an interval of time will then be

$$\Delta S = \oint dS = \oint \frac{dQ}{T} = \oint \frac{dV}{T} + \oint \frac{dT}{T} \, . \tag{16}$$

Now consider a trajectory that begins and ends at the same location. If we're allowed to use the standard rules of calculus, the last term becomes zero. Likewise, if we average ΔS over a long trajectory, the final term will remain bounded as long as T is bounded, and the leading contribution with be due to dV/T. We are then left with

$$\Delta S \sim \oint \frac{dV}{T} \, . \tag{17}$$

If we identify this with Φ, we see that the condition to have nonzero current is the condition to have nonzero entropy production.

This hand-wavey argument can be made rigorous by directly calculating the system's entropy production rate, while being careful of the discretization. The result is indeed

$$\sigma = \lim_{t_\mathrm{f} \to \infty} \frac{1}{t_\mathrm{f}} \int_0^{t_\mathrm{f}} dt \dot{x}(t) \frac{V'(x(t))}{T(x(t))} , \tag{18}$$

where the integral is a **Stratonovich integral**. We can then identify this with

$$\sigma = \lim_{t_\mathrm{f} \to \infty} \frac{1}{t_\mathrm{f}} \int_0^{t_\mathrm{f}} dt \frac{1}{T(x(t))} \frac{d}{dt} V(x(t)) = \lim_{t_\mathrm{f} \to \infty} \frac{1}{t_\mathrm{f}} \oint \frac{dV}{T} \tag{19}$$

where the path of the contour integral is stochastic. This can only be nonzero if $V'(x)/T(x)$ does not integrate to zero, i.e. $\Phi(x)$ is not periodic — which is true if and only if the current is nonzero. In this case, the particle falls down the "entropy landscape" defined by Φ as if it's inside a tilted potential.

Landauer erroneously argued that the condition for a nonzero current shouldn't be nonzero entropy production, because in the inhomogeneous temperature case without a potential there is no current, and he assumed an inhomogeneous temperature must cause entropy production. But as we showed in Recitation 3, the inhomogeneous temperature system genuinely is time-reversal symmetric, i.e. $\sigma = 0$, as long as we adopt the information-theoretic definition of entropy which doesn't worry about the external processes that cause the temperature gradients.