

8.08/8.S308 - Problem Set 4 - IAP 2026**Due before January 27 23:59**

Anything marked as “graduate” count as bonus problems for undergraduate students.

1- Active spins on lattice

We consider an active particle on a square lattice in an arbitrary dimension d . The particle hops in a given direction at rate h and changes direction at rate α , picking its new direction among the $2d$ possibilities uniformly at random. Such a direction change is called a ‘tumble’. The purpose of the next few questions is to compute the diffusivity of the particle. A trajectory of duration t can be decomposed into n successive runs, defined as the period between two tumbles. Note that the number n is itself a random variable, which fluctuates between different trajectories of the same duration t . During each run, the particle undergoes a displacement \vec{a}_i along one of the lattice direction (the lattice spacing is taken equal to 1).

1.1) Draw a schematic trajectory with $n = 6$ for a two-dimensional lattice.

1.2) Show that, if the particle starts at the origin, its displacement $\vec{r}(t)$ at time t satisfies

$$\langle \vec{r}(t)^2 \rangle = \left\langle \sum_{i=1}^n \vec{a}_i^2 + 2 \sum_{i < j} \vec{a}_i \cdot \vec{a}_j \right\rangle \quad (1)$$

where brackets represent averages over many trajectories.

1.3) Using simple symmetry arguments, explain why $\langle \vec{a}_i \cdot \vec{a}_j \rangle = 0$ for $i \neq j$.

1.4) We now assume that $\langle \sum_{i=1}^n \vec{a}_i^2 \rangle \underset{t \rightarrow \infty}{\sim} \langle n \rangle \langle a^2 \rangle$, where $\langle n \rangle$ is the average number of run phases during a trajectory of length t and $\langle a^2 \rangle$ is the second moment of the run length. This is called the Wald identity; explain briefly the underlying assumption and why it is valid in the large time limit.

1.5) Compute the mean time between two tumbles. Deduce the mean number $\langle n \rangle$ of run phases during a large time t .

1.6) We now want to compute $\langle a^2 \rangle$. At any time, what is the probability that the next configuration change is a hopping event? A tumble? Show that the probability $P(a)$ that the particle covers a distance a during a run phase is given by

$$P(a) = \frac{\alpha h^a}{(\alpha + h)^{a+1}} \quad (2)$$

1.7) We introduce the generating function $G(z) = \sum_{a=0}^{\infty} z^a P(a)$. Compute $G(z)$, for $z < 1 + \alpha/h$, in terms of α , h and z .

1.8) Compute $G'(1)$ and $G''(1)$. Deduce the first and second moment of the run length a :

$$\langle a \rangle = \frac{h}{\alpha}; \quad \langle a^2 \rangle = 2 \frac{h^2}{\alpha^2} + \frac{h}{\alpha} \quad (3)$$

1.9) Using these results and Eq. (1), compute the diffusivity D as:

$$D \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\langle \vec{r}(t)^2 \rangle}{2d}. \quad (4)$$

2- Rectification of bacterial density by anisotropic tumbles

Let us consider a self-propelled particle on a one-dimensional lattice of L sites with periodic boundary conditions (in the following, we silently identify site $L + 1$ and site 1, as well as site 0 and site L). $P(i, +; t)$ and $P(i, -; t)$ are the probabilities to find the particle on site i going to the right and to the left, respectively. We silently omit the time dependency in the following and simply write $P(i, \pm)$. On site i , the particle may

- hop to the site $i + 1$ at rate d_i^+ if it is in the right-going state.
- hop to the site $i - 1$ at rate d_i^- if it is in the left-going state.
- change its orientation from right-going to left-going at rate α_i^+ .
- change its orientation from left-going to right-going at rate α_i^- .

2.1) The configurations corresponding to having the particle at site i in a right-going or left-going states are noted $(i, +)$ and $(i, -)$, respectively. List all possible transitions into and out of configuration $(i, +)$ and the corresponding rates, and show that the evolution of $P(i, +)$ is given by the master equation

$$\partial_t P(i, +) = d_{i-1}^+ P(i-1, +) - d_i^+ P(i, +) + \alpha_i^- P(i, -) - \alpha_i^+ P(i, +) \quad (5)$$

What is the master equation yielding the evolution of $P(i, -)$?

2.2) We note $P(i)$ the probability to find the particle at site i , irrespectively of its direction. How are $P(i)$, $P(i, +)$, and $P(i, -)$ related mathematically? Show the evolution of $P(i)$ to be of the form

$$\partial_t P(i) = J_{i-1, i} - J_{i, i+1} \quad (6)$$

Give the expression of $J_{i, i+1}$ and its physical interpretation. If you were to simulate this system, how would you measure $J_{i, i+1}$?

2.3) We now consider the system in **steady state**. Show that the probability current is constant and that its value is given by

$$J = P(i, +)(d_i^+ + \alpha_i^+) - P(i, -)(d_i^- + \alpha_i^-) \quad (7)$$

2.4) *Graduate*. Express $P(i, +)$ as a function of $P(i-1, +)$, J and of the microscopic rates.

2.5) *Graduate*. We now consider a closed system of L sites ($d_1^- = d_L^+ = 0$). What is the value of J in the steady-state? Show that

$$\forall i \geq 2, \quad P(i, +) = \left(\prod_{j=2}^i \frac{d_{j-1}^+}{d_j^-} \frac{d_j^- + \alpha_j^-}{d_j^+ + \alpha_j^+} \right) P(1, +); \quad P(i, -) = \frac{d_i^+ + \alpha_i^+}{d_i^- + \alpha_i^-} P(i, +) \quad (8)$$

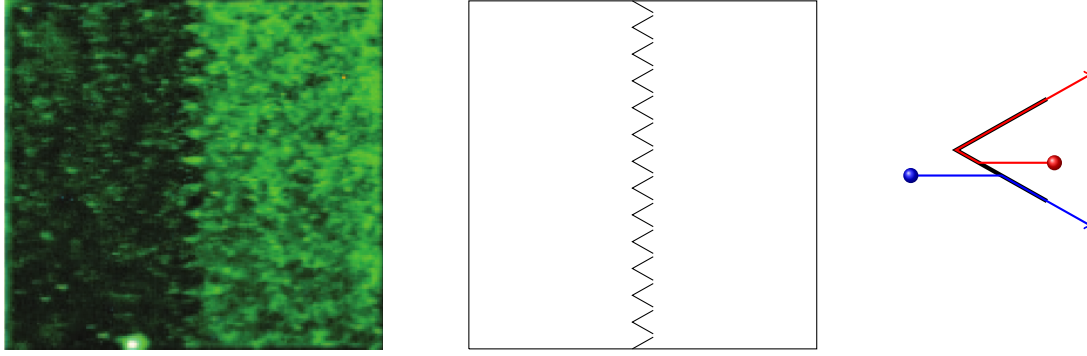


Figure 1: *Left:* Steady-state density of bacteria in a cavity with asymmetric obstacles in the middle. *Center:* Sketch of the cavity. The distance between obstacles is larger than the size of the bacteria. *Right:* Description of the bacteria trajectories when they collide with an obstacle.

2.6) Graduate. We consider a system with uniform, isotropic hopping rates $d_i^+ = d_i^- = d$. For $1 \leq i \leq \ell$ and for $\ell + h < i \leq L$, $\alpha_i^+ = \alpha_i^- = \alpha$. For $\ell < i \leq \ell + h$, on the other hand, we take $\alpha_i^+ = \alpha$ and $\alpha_i^- = \alpha + \varepsilon$. Compute $P(i, +)$, $P(i, -)$ and $P(i)$ for all i . Make a sketch of $\log P(i)$ as a function of i .

2.7) Graduate. In Fig. 1, the steady-state density distribution of bacteria swimming in a two-dimensional cavity containing at its center a one-dimensional line of asymmetric obstacles is shown. Explain this experimental result. (Feel free to model the experiment in any way you like.)

3- Zero-range process

We consider a one-dimensional lattice with L sites and periodic boundary conditions. N particles hop stochastically on the lattice and a configuration of the system is thus characterised by the occupancy numbers $\{\mathbf{n}\} \equiv (n_1, \dots, n_L)$ for all sites. We consider $u(n)$ an arbitrary positive function of n such that $u(0) = 0$. With a rate $u(n_i)$, one particle on site i is transferred to site $i + 1$, so that $n_i \rightarrow n_i - 1$ and $n_{i+1} \rightarrow n_{i+1} + 1$.

3.1) Consider a configuration $\{\mathbf{n}\}$. What are the configurations $\{\mathbf{n}'\}$ connected to $\{\mathbf{n}\}$ by a single-particle displacement? What are the corresponding transition rates from $\{\mathbf{n}\}$ to $\{\mathbf{n}'\}$ and from $\{\mathbf{n}'\}$ to $\{\mathbf{n}\}$?

3.2) Explain the principle of detailed balance for a model with a discrete set of configurations. Can this model satisfy detailed balance? Show that the master equation can be written as

$$\partial_t P(\{\mathbf{n}\}) = \sum_{k=1}^L [u(n_{k-1} + 1)P(\{n_{k-1} + 1, n_k - 1\}) - u(n_k)P(\{\mathbf{n}\})] \quad (9)$$

where $\{n_{k-1} + 1, n_k - 1\}$ is the configuration obtained from $\{\mathbf{n}\}$ by adding a particle at site $k - 1$ and subtracting a particle from site k .

3.3) We look for a factorized steady-state:

$$P(\{\mathbf{n}\}) = Z_{L,N}^{-1} \prod_{k=1}^L f(n_k) \quad (10)$$

Show that a sufficient condition for (10) to be a solution of Eq. (9) is

$$u(n_{k-1} + 1) \frac{f(n_{k-1} + 1)}{f(n_{k-1})} = u(n_k) \frac{f(n_k)}{f(n_k - 1)} \quad (11)$$

Explain why this imposes both left-hand side and right-hand side to be equal to a constant λ independent of n_k and n_{k-1} .

3.4) We assume $\lambda = f(0) = 1$ without loss of generality. Compute $f(n_k)$ and show that

$$P(\{\mathbf{n}\}) = Z_{L,N}^{-1} \prod_{k=1}^L \prod_{j=1}^{n_k} \frac{1}{u(j)}, \quad (12)$$

where $Z_{L,N}^{-1}$ is a normalization constant.

3.5) Show that

$$Z_{L,N} = \sum_{\{\mathbf{n}\}} \prod_{k=1}^L f(n_k) \delta\left(\sum_{i=1}^L n_i - N\right) \quad (13)$$

where $\delta(p = 0) = 1$ and $\delta(p \neq 0) = 0$.

3.6) Graduate. Let $p(n)$ be the (marginal) probability that n particles are at site 1, irrespectively of the other occupancy numbers, but given that there are N particles in total. Show that

$$p(n) = f(n) \frac{Z_{L-1, N-n}}{Z_{L,N}} \quad (14)$$

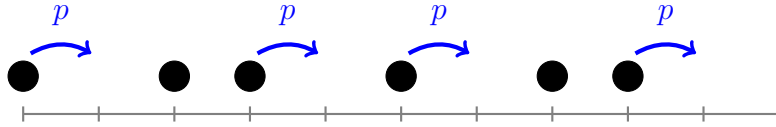


Figure 2: In a TASEP, all particles hop forward at rate p provided the arrival site is empty. (Conversely, all the particles could be hoping towards the left.)

3.7) Graduate. We now consider the following mapping, valid for periodic boundary conditions, from a lattice with L sites and N particles to a lattice with $N + L$ sites and L particles. Each *site* of the system 1 corresponds to a *particle* of the system 2. The occupancy number of site i in system 1 then corresponds to the number of empty sites between the $(i - 1)^{\text{th}}$ particle and the i^{th} particle in system 2. Draw two configurations of your choice, that you find illustrative, for $L = 6$ and $N = 5$. Show that if system 1 is a zero-range process with $u(n) = p$, then the system 2 corresponds to a TASEP. To do so, one will represent all possible representative transitions in one system and the corresponding transitions in the second system.

3.8) Graduate. Consider the TASEP illustrated in Figure 2. Write down the master equation for a system of $M = N + L$ sites and N particles. Show that in steady state, all configurations have equal probability. Given the mapping discussed at the question 3.7, is this compatible with (12)?