Math Beyond Competitions

for advanced competitors

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My Biases

I'm giving just one perspective

- Attended MIT
- Bad at geometry
- Only know a fraction of the math community, not necessarily representative of population

This is a sampler of topics and resources, not comprehensive

High School Math Contest Subjects

Algebra: polynomials, sequences, inequalities, functional equations

Combinatorics: counting, probability, graph theory

Geometry: Euclidean, projective, inversion, complex, vectors

Number Theory: integers, primes, factorization

Why these 4? Idk but IMO Shortlist uses these too

Other High School Contests

- Linguistics (NACLO)
- Computer science (USACO)
- Physics (USAPhO)
- Chemistry (USACO)
- Biology (USABO)
- Science Bowl

Philosophy

Textbooks are often long and comprehensive—requires high motivation and commitment

Think about how each individual learns best—watching videos, reading notes, classroom setting

Choice of topic(s) not too important—any will develop breadth and "mathematical maturity"

General Resources

- MIT OpenCourseWare: Video lectures and lecture notes
- edX, Coursera
- Youtube: 3Blue1Brown, Richard Borcherds, ... lots more
- Evan Chen's "Napkin"
- Summer math programs: I attended HCSSiM and Canada/USA Mathcamp

Combinatorics

Probability theory: Random variables, probability distributions (especially continuous distributions once you know calculus)

- 18.600 lecture notes by Scott Sheffield
- 18.650 statistics video lectures, edX course

Algebraic combinatorics

- 18.212 lecture notes

Extremal combinatorics, probabilistic method

- Yufei Zhao's Sunday talk tomorrow!
- 18.226 lecture notes

Adjacent Subjects

Computer science: Not just coding, also includes algorithms, complexity, information theory, cryptography

Set theory and logic: Infinite cardinals and ordinals, Banach-Tarski, Gödel theorems

Game theory: Combinatorial games, surreal numbers, Nash equilibrium, voting theory, economics

Number Theory

Formal treatment of elementary number theory

- Can you prove Fermat, Euler, & Wilson? Quadratic reciprocity?
- PROMYS and Ross summer programs focus on number theory

Number theory beyond integers: Gaussian integers, Gauss sums, towards algebraic

- *A Classical Introduction to Modern Number Theory* by Ireland and Rosen
- Good intuition and segue for abstract algebra
- WARNING: Algebraic number theory proper is very advanced
- Evan Chen's *Napkin* has good examples for basic algebraic number theory

If you know calculus, have a go at analytic number theory

Pure Math Classification

MIT: "Traditionally, pure mathematics has been classified into three general fields: **analysis**, which deals with continuous aspects of mathematics; **algebra**, which deals with discrete aspects; and **geometry**."[1]

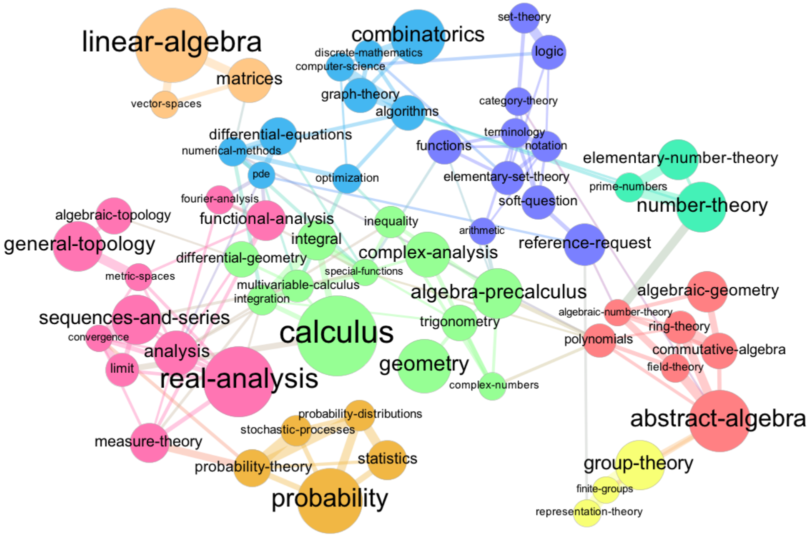
[1] "Course 18 Option 3:" https://math.mit.edu/academics/undergrad/major/course18/pure.php

Pure Math Classification

Harvard: "Perhaps the most important concept of mathematics is that of function, which provides us with the means to study dependence and change. The study of real functions of a real variable (and later complex functions), particularly in connection with the limit concept, is called **analysis**. The most effective tool for this study is the infinitesimal calculus that analyzes the relation between functions and their derivatives. The study of number systems and their generalizations is called **algebra**. Here the primary concepts are group, ring, field, and module. The last great branch of mathematics is **geometry**, which now goes far beyond the classical study of the space we live in to include spaces of high dimension and topology, the abstract theory of shape."[2]

[2] Jacob Lurie, "Mathematics – 2015-2016." https://people.math.harvard.edu/undergrad/handbook/handbook2015-16.pdf

An Incomplete Map



("A Graph Map of Math.SE." http://meta.math.stackexchange.com/questions/6479/a-graph-map-of-math-se)

Calculus and Analysis

Start by learning "calculus with proofs"

- Michael Spivak's *Calculus*

Complex variables: contour integration very cool!

Move to analysis over $\mathbb{R}$

- 18.100A video lectures and notes
- Key is to understand role of compactness and uniform convergence, lots of inequality bounds

Full "real analysis" over metric spaces is more advanced

- I do **not** recommend the standard text *Principles of Mathematical Analysis* by Rudin—too dry
- *Princeton Lectures in Analysis* series by Stein and Shakarchi good breadth

Algebra

Start with linear algebra

- Most can learn earlier than you think—do not need calculus
- 18.06 video lectures by Gilbert Strang
- Do **not** have "solving linear equations" as sole motivation — need to understand linearity, independence, vector spaces, and operators
- *Linear Algebra Done Right* very good but more abstract (very few matrices written out)

Abstract Algebra

Group theory and ring theory: use number theory intuition

- Most textbooks are massive—I like Artin's exposition but do problems from Dummit and Foote
- AoPS has Group Theory course over summer

Field theory: algebraic numbers, field extensions, Galois theory

- Often at end of textbooks/courses but can be mostly self-contained; number theory helps
- Galois theory needs some group theory

Geometry and Topology

Point set topology: compactness and continuity with open sets

- *Topology* by Munkres
- If you get bored, stop and move on

Algebraic topology: Fundamental group, homotopy, homology

- Computation techniques are accessible, proofs may involve more algebra
- Popular DRP and summer camp topic, look for self-contained notes

Differential geometry, differential topology

My Recommendations

Learn probability: especially random variables and distributions

Learn linear algebra: operators and vectors

Lots of cool math—look up phrases and pursue what's interesting

Thank you!

Thoughts, comments, and feedback: `www.yellkey.com/show`