Secretary Problems with Random Number of Candidates: How Prior Distributional Information Helps

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Abstract

We study variants of the secretary problem, where N, the number of candidates, is a random variable, and the decision maker wants to maximize the probability of success – picking the largest number among the N candidates – using only the relative ranks of the candidates revealed so far.

We consider three forms of prior information about \mathbf{p} , the probability distribution of N. In the *full* information setting, we assume \mathbf{p} to be fully known. In that case, we show that single-threshold type of strategies can achieve 1/e-approximation to the maximum probability of success among all possible strategies. In the upper bound setting, we assume that $N \leq \overline{n}$ (or $\mathbb{E}[N] \leq \overline{\mu}$), where \overline{n} (or $\overline{\mu}$) is known. In that case, we show that randomization over single-threshold type of strategies can achieve the optimal worst case probability of success of $\frac{1}{\log(\overline{n})}$ (or $\frac{1}{\log(\overline{\mu})}$) asymptotically. Surprisingly, there is a singlethreshold strategy (depending on \overline{n}) that can succeed with probability $2/e^2$ for all but an exponentially small fraction of distributions supported on $[\overline{n}]$. In the sampling setting, we assume that we have access to m samples $N^{(1)}, \ldots, N^{(m)} \sim_{iid} \mathbf{p}$. In that case, we show that if $N \leq T$ with probability at least $1 - O(\epsilon)$ for some $T \in \mathbb{N}$, $m \gtrsim \frac{1}{\epsilon^2} \max(\log(\frac{1}{\epsilon}), \epsilon \log(\frac{\log(T)}{\epsilon}))$ is enough to learn a strategy that is at least ϵ -suboptimal, and we provide a lower bound of $\Omega(\frac{1}{\epsilon^2})$, showing that the sampling algorithm is optimal when $\epsilon = O(\frac{1}{\log\log(T)})$.

We also discuss applications of some of our techniques to the matroid secretary problems, iid prophet inequalities, and online linear optimization problems when the time horizon N is a random variable.

1 Introduction

Online optimization has gained increasing attention due to its application in a variety of areas such as online revenue management ([33, 21]), online resource allocation ([2, 20]), online auctions and advertising ([34, 6]), to name a few. In the typical setting, information about a constrained optimization problem is revealed sequentially, and a decision maker must make irrevocable decisions (about some variables of the problem) at each time period, with the final goal to maximize/minimize the objective function without violating the constraints. Different models have been proposed to capture this online nature, from perhaps the purest model of decision making under uncertainty represented by the classical secretary problem [9, 12, 18, 31], to more elaborate ones such as general online linear optimization problems [2, 35, 30]. However, all of these models assume that the number of decisions to be made, i.e., the total number of arrivals, or the time horizon of the problem, is known a priori, and the (successful) algorithms that have been proposed usually require the time horizon as an essential part of their input.

In many realistic applications, however, the exact size of the sequence of arrivals, or the time horizon, is hardly known a priori. For example, when allocating advertisement slots to online advertisers subject to daily budget limit, the decision maker usually doesn't know the exact number of advertisers that will show up each day to compete in the successive auctions. Instead, it would be more natural to model it as an unknown, possibly as a random variable with some known or learnable distribution.

In this work, we focus on the secretary problem when the number of candidates, N, is a random variable independent of the process, and the decision maker is given prior information about its distribution \mathbf{p} in the form of *full information* of \mathbf{p} , upper bound on N or $\mathbb{E}[N]$, or *iid samples* following \mathbf{p} . Due to the

randomness in N, a constant success probability, such as 1/e which is attainable in the classical setting, may be impossible, and the optimal strategy may be much more complex than the classical single-threshold type of strategies: that is, reject the first l candidates, then accept the first candidate that is the best among all candidates that have shown up so far [38]. Nevertheless, in this paper, we provide three positive results.

a) In the full information setting, \mathbf{p} is fully known. We show that single-threshold type of strategies can achieve 1/e-approximation to the maximum probability of success among all possible strategies.

b) In the upper bound setting, $N \leq \overline{n}$ (or $\mathbb{E}[N] \leq \overline{\mu}$) and \overline{n} (or $\overline{\mu}$) is known. We show that randomization over single-threshold type of strategies can achieve the optimal worst case probability of success of $\frac{1}{\log(\overline{n})}$ (or $\frac{1}{\log(\overline{\mu})}$) as $\overline{n} \to \infty$ (or $\overline{\mu} \to \infty$). Surprisingly, there is a single-threshold strategy (depending on \overline{n}) that can succeed with probability $2/e^2$ for all but an exponentially small fraction of distributions supported on $[\overline{n}]$.

c) In the sampling setting, m samples $N^{(1)}, \ldots, N^{(m)} \sim_{iid} \mathbf{p}$ are given. We provide an algorithm which finds an ϵ -suboptimal strategy using $m \gtrsim \frac{1}{\epsilon^2} \max(\log(\frac{1}{\epsilon}), \epsilon \log(\frac{\log(T)}{\epsilon}))$ samples where $N \leq T$ with probability at least $1 - O(\epsilon)$, and show a lower bound of $\Omega(\frac{1}{\epsilon^2})$.

In addition, we extend some of the techniques above to other online optimization problems including the matroid secretary problem, iid prophet inequality, and online packing linear optimization. When the time horizon $\underline{n} \leq N \leq \overline{n}$ is a random variable with known upper and lower bound, we provide randomization strategies over algorithms designed for problem instances when the time horizon is known (which we use as black boxes). The performance of these randomization strategies depends on how "sensitive" the black-box algorithms are to the error in the time horizon in the input. For iid prophet inequality, we also show that our algorithm is in fact optimal up to a constant factor (Theorem B.1).

1.1 Comparison with previous work

There have been extensive studies on the secretary problem and its variants, and interested readers can look at [23] for a survey. We are particularly interested in the variant where the number of candidates is a random variable, which was first studied by [38].

Full information. In [38], with full information about the distribution \mathbf{p} , the problem is reformulated into an optimal stopping problem for a Markov chain whose transition probability and payoff function depend on the distribution \mathbf{p} of the time horizon N. Classical results from optimal stopping theory then implies an optimal solution of the form $\tau := \min\{i|i\text{-th} \text{ number} \text{ is the largest seen so far}, i \in \Gamma\}$ for some index set $\Gamma \subset \mathbb{N}$. When \mathbf{p} is a delta, uniform, Poisson, or geometric distribution, Γ is of the simple form $\{n^*, n^*+1, \ldots\}$ for some n^* that depends on the parameters of \mathbf{p} , and the probability of success does not depend on the parameters of \mathbf{p} asymptotically¹. However, for general \mathbf{p} , the set Γ might be composed of infinite number of intervals, leading to a rather complicated strategy. We show that the much simpler single-threshold type of strategies, i.e. $\Gamma = \{n^*, n^* + 1, \ldots\}$ for some $n^* \in \mathbb{N}$, already achieves a 1/e-approximation to the optimal, which echoes a recent line of research on efficient approximations to optimal online algorithms [3, 37, 8].

Another important work on this random time horizon variant is [1], which studies a particular family of strategies parameterized by $\mathbf{q} \in [0, 1]^{\mathbb{N}}$, where q_i denotes the probability of selecting the *i*-th candidate given that no candidate has been selected so far and the current candidate is the best among all that have shown up. The focus of [1], however, is conditions under which \mathbf{q} is admissible, i.e. not dominated by another \mathbf{q}' , for every \mathbf{p} . Although their goal is different from ours, we adopt their framework, and show that any secretary strategy is dominated by some strategy in this family (Lemma 2.1). Thus, in order to maximize the probability of success (given prior information), it suffices to consider strategies in this family.

Upper bound. When $N \leq n$ is upper bounded, [1] gives a distribution $\mathbf{p}_*^{(n)}$ supported only on [n] which we restate in Equation (1) below, such that the optimal success probability parameterized by some $\mathbf{q} \in [0,1]^{\mathbb{N}}$ is $H_n^{-1} \approx \frac{1}{\log(n)}$ where H_n is the *n*-th Harmonic number. In fact, [28] shows that the minimazoptimal strategy for this problem succeeds with probability approximately $\frac{1}{\log(n)}$, and [36] shows that H_n^{-1} is the worst case scenario, even when N is picked by a *non-adaptive* adversary², and gives a $\mathbf{q} \in [0,1]^{\mathbb{N}}$ which succeeds with probability $\frac{1}{H_{n-1}+1}$. In addition, [36] shows that when there is no information about

¹ "Asymptotic" refers to the case $\lambda \to \infty$, where λ represents the expected value of the aforementioned distributions.

²That is, the adversary picks an $N \leq n$ before the numbers arrive. In fact, this "non-adaptiveness" is crucial here, since we provide an adaptive strategy for the adversary such that no algorithm can succeed with probability more than $\frac{1}{\sqrt{n}}$. See discussion in Section 2 and more details in Appendix A.3 Lemma A.2.

N at all, for any $\epsilon > 0$, there is an algorithm which has probability of success $\Omega(\frac{\epsilon}{\log^{1+\epsilon}(N)})$. When $\mathbb{E}[N] \leq \overline{\mu}$ has known first moment upper bound, [26] gives the exact formula for the minimax optimal strategy and probability of success, but does not give asymptotic analysis.

We extend these bounds in two directions: first, we show that the $\frac{1}{H_{n-1}+1}$ guarantee can be achieved by a randomization over the single-threshold types of algorithm, and similar $\frac{1}{\log(\overline{\mu})}$ bounds hold when $\mathbb{E}[N] \leq \overline{\mu}$; second, we look beyond worst case by providing a single-threshold strategy which achieves a constant probability of success for all but an exponential small fraction of distributions supported on [n].

Samples. When prior information about N comes in the form of samples, the problem resembles a recent line of work on sample based secretary problems and prophet inequalities [17, 18, 19, 16, 4, 39]. For instance, in the sample-driven secretary problem in [18], each number out of n numbers is, independently, included in the sample set with probability p. Then the secretary problem restricted to the rest of the numbers is played, with the only difference being the revealed information: at time t, the relative ranking of the t-th arriving number among the previous t - 1 numbers together with the sample set is revealed. [18] provides algorithms for each $p \in [0, 1)$, which succeeds with probability 1/e when p = 0 and 0.58 as $p \to 1$, matching the optimal probability of success in the classical and full information³ secretary problem respectively. The setting we consider is different from theirs. We investigate the number of iid samples of N needed in order to find a strategy which is ϵ suboptimal as compared to knowing the full distribution of N. When $N \leq n$, the naive approach to use the samples to estimate the distribution of N up to accuracy ϵ in l_1 norm (details in Section 3.4) requires $O(\frac{n}{\epsilon^2})$ samples in general[25]. In contrast, we provide a block based algorithm which only requires $O(\frac{1}{\epsilon^2} \max(\log(\frac{1}{\epsilon}), \epsilon \log(\frac{\log(n)}{\epsilon})))$ samples, and provide a sampling lower bound of $\Omega(\frac{1}{\epsilon^2})$.

For the matroid secretary problem, prophet inequality, and online linear optimization, we defer detailed comparisons with prior work to Section 4 and Appendix B.

1.2 Contribution

On a high level, the extra randomness in N makes the problem harder than the classical problem. However, due to the inherent random permutation, the distribution \mathbf{p} of N affects the performance of strategies only through the sequence $\lambda_i(\mathbf{p}) := \sum_{j=i}^{\infty} \frac{1}{j} p_j$, $i \in \mathbb{N}$, interpreted as the (marginal) probability that the *i*-th arriving candidate is the correct one to pick (Lemma 2.2). In particular, this allows us to improve upon previous works from the following perspectives:

- strategy simplicity: in the full information setting, we "approximate" the sequence $(\lambda_i(\mathbf{p}))_{i \in \mathbb{N}}$ with the sequence $(\lambda_i(\boldsymbol{\delta}^{(n)}))_{i \in \mathbb{N}}$ generated by the delta distribution supported at $n \in \mathbb{N}$ (equivalent to the classical secretary problem with *n* candidates), and show that single-threshold type of strategies can achieve at least 1/e of the optimal probability of success among all strategies;
- average case analysis: in the known upper bound setting, we use the linearity of the probability of success in **p** and concentration inequalities to show that an optimal strategy when N is uniformly distributed on [n] (which is single-threshold and succeeds with probability $2/e^2$ as $n \to \infty$) can succeed with probability at least $2/e^2$ for all but an exponentially small fraction of distributions supported on [n], despite the worst case distribution supported on [n] where no strategy can succeed with probability more than $1/H_n$;
- sampling algorithm: we provide a "block based" algorithm which can find an ϵ suboptimal strategy (as compared to knowing the distribution) with high probability, using $O(\frac{1}{\epsilon^2} \max(\log(\frac{1}{\epsilon}), \epsilon \log(\frac{\log(T)}{\epsilon})))$ iid samples of N, where $N \leq T$ with probability $1 O(\epsilon)$. Our algorithm estimates the sequence $(i\lambda_i)_{i\in[T]}$ using a blocked version of the distribution \mathbf{p} , thereby reducing the number of parameters to be estimated from T to $\log(T)$. Fortunately, due to the random permutation, the performance of any strategy is insensitive to "local" changes to the distribution and so the bias caused by the blocked version of the distribution is small. We also provide a matching lower bound of $\Omega(\frac{1}{\epsilon^2})$, showing that our algorithm is optimal when $\epsilon = O(\frac{1}{\log \log(T)})$.

We also consider the random time horizon variant of three classical online optimization problems: the matroid secretary problem, the iid prophet inequality, and the online packing LP. We investigate the performance

 $^{^{3}}$ In the full information secretary problem, the numbers are iid sampled from a known distribution, and each time the value of the number, instead of its relative rank, is revealed.

of randomizing over classical algorithms, i.e. those designed for fixed known time horizon problems, in the setting where N is a random variable with known upper and lower bound. For the iid prophet inequality, our algorithm is optimal up to a constant factor when the sequence is supported on [x, 1] for some $x \in (0, 1]$.

2 Preliminary

In the classical secretary problem, a decision maker wants to maximize the probability of picking the largest number out of n numbers that arrive in a uniform random order. Each time a number arrives, its relative ranking among all numbers that have arrived is revealed, and the decision maker must decide whether to pick it or not immediately and irrevocably. In this work, we consider the variant where, instead of a fixed, known n, the total number to be randomly permuted is a random variable N independent of the random permutation, and the decision maker's goal is to maximize the probability of picking the largest number before the process stops (at time N).

Before formally stating the problem in Section 2.1, we briefly discuss the impacts due to the randomness in N. The secretary problem and the variant studied in this work can be viewed as optimal stopping problems, where stopping at time t means picking the t-th arriving number. A decision maker who knows the value of N (as in the classical setting) is equivalent to a stopping time that is adapted to the filtration generated by the observations (relative rankings) together with N. By contrast, in our setting, the stopping time is adapted to the filtration generated by the relative rankings with only $\mathbf{1}[N \ge i]$ for $i = 1, 2, \ldots$, which are sub- σ -algebras of those in the previous case, thereby leading to an optimal value which can never be greater than the previous case (more details in Appendix A.2).

On the positive side, modeling N as a random variable is more realistic than some other common ways to model parameter uncertainty. Consider the setting where N has an upper bound n, and N is picked by an adversary who has access to the same "rolling" information as the decision maker. That is, at time t, the adversary is given the relative ranking of the t-th arriving number, as well as the decision maker's decision at time t - 1, and then needs to decide immediately and irrevocably if N = t. We show then that the adversary has a strategy such that the decision maker cannot succeed with probability more than $\frac{1}{\sqrt{n}}$ (more details in Appendix A.3). As a comparison, if $N \leq n$ is a random variable, as opposed to be picked by an adversary, the decision maker has a strategy which guarantees a $\frac{1}{1+H_{n-1}} \approx \frac{1}{\log(n)}$ probability of success in the worst case (Theorem 3.2). In addition, modeling N as a random variable allows us to explore how prior information about its distribution (such as upper bound or samples) might help in choosing strategies.

2.1 Set up

We assume that there is a sequence of numbers $(x_i)_{i\in\mathbb{N}}$ (assuming no tie), and a distribution $\mathbf{p} \in \Delta$, where $\Delta := \{\mathbf{y} = (y_1, y_2, \ldots) | \sum_{i=1}^{\infty} y_i = 1, y_i \ge 0 \ \forall i \in \mathbb{N} \}$. First, nature generates N according to the distribution \mathbf{p} such that $\mathbb{P}[N = i] = p_i$. On the event $\{N = i\}$,

First, nature generates N according to the distribution **p** such that $\mathbb{P}[N=i] = p_i$. On the event $\{N=i\}$, independent of N, a permutation $\sigma : [i] \to [i]$ is picked uniformly at random from all permutations on [i]. At time $t = 1, 2, \ldots, i, R_t$, the relative ranking of $x_{\sigma(t)}$ among $\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(t)}\}$, is revealed. That is, $R_t = |\{i \in [t] | x_{\sigma(i)} \ge x_{\sigma(t)}\}|$. After observing R_1, R_2, \ldots, R_t , the decision maker using the strategy Str must decide whether to accept or reject $x_{\sigma(t)}$ immediately, and the decision cannot be changed in the future. Due to the random permutation, conditioning on N = i, $\{R_t\}_{t \le i}$ are independent, and $R_t \sim Uniform([t])$.

The goal of the decision maker is to maximize the probability of success, i.e. picking the largest number before the process stops. We denote this event under strategy Str as $Success(Str) := \bigcup_{i=1}^{\infty} Success_i(Str)$, where $Success_i(Str)$ is the following event

$$\mathsf{Success}_i(\mathsf{Str}) := \{N = i\} \cap \{\mathsf{Str picks} \ x_{\sigma(t)} \text{ for some } t \in [i]\} \cap \{x_{\sigma(t)} = \max_{1 \le j \le i} x_{\sigma(j)}\}.$$

With these notations, the classical secretary problem corresponds to the case when the decision maker knows, before picking the strategy, that $\mathbf{p} = \boldsymbol{\delta}^{(n)}$ is the delta distribution supported on $\{n\}$, i.e. $\boldsymbol{\delta}_i^{(n)} = \mathbf{1}[n = i]$. By contrast, in this work, we don't assume that \mathbf{p} is a delta distribution, or is fully known. Instead, the decision maker is given prior information about \mathbf{p} in the form of the distribution \mathbf{p} itself, an upper bound on N or $\mathbb{E}[N]$, or iid samples under \mathbf{p} .

To avoid confusion, we distinguish between *strategy* and *algorithm*. By *strategy*, we refer to strategies a decision maker can use for the secretary problem with unknown time horizon alone. By *algorithm*, we refer to learning algorithms which, given prior information about \mathbf{p} such as samples, output a strategy Str that the decision maker can use.

2.2 A parameterized family of strategies

Let \overline{S} denote all strategies for the secretary problem with random N. That is, conditioning on $N \geq t$, the decision whether to pick the t-th number depends only on the first t relative rankings $(R_i)_{i\leq t} \in \prod_{i=1}^t [i]$ and potential randomness due to the strategy itself. However, it makes no sense to pick any number whose relative rank is not 1, and the relative ranks of the first t-1 numbers give no information about the true rank of the t-th number among all N numbers. Thus, without sacrificing performance, we can in fact restrict the strategies to a family of strategies $Str(\mathbf{q})$ parameterized by $\mathbf{q} \in \mathcal{S} := \{(q'_i)_{i\in\mathbb{N}} | q'_i \in [0,1], i \in \mathbb{N}\}$, where

 $q_i = \mathbb{P}[\text{ pick } x_{\sigma(i)} | N \ge i, R_i = 1, \text{ no number has been picked}].$

The strategy $Str(\mathbf{q})$ works in the following way: for t = 1, 2, ..., if $N \ge t$, observe the relative rank R_t . If $R_t = 1$, generate $Y_t \sim Bernoulli(q_t)$ independent of any previously generated $Y_{t'}$. If $Y_t = 1$, accept the t-th number (and the process stops); otherwise continue. The strategy parameterized by $\Gamma \subset \mathbb{N}$ in [38] correspond to $Str(\mathbf{q})$ where $q_i = 1$ if and only if $i \in \Gamma$.

With a slight abuse of notation, we abbreviate Success(Str(q)) as Success(q). We have the following guarantee, the proof of which is deferred to Appendix C.

Lemma 2.1. For any strategy $Str \in \overline{S}$ for the secretary problem with random N, there exists $q^{Str} \in S$, such that for any $\mathbf{p} \in \Delta$, and $N \sim \mathbf{p}$, $\mathbb{P}[Success(q^{Str})] \geq \mathbb{P}[Success(Str)]$.

For convenience, for each $l \in \mathbb{N}$, we let $\mathbf{q}^{(l)} \in \mathcal{S}$, $q_i^{(l)} := \mathbf{1}[l \leq i]$, which represents the single-threshold strategy that rejects the first l-1 numbers, and accepts the first number $t \geq l$ whose relative rank $R_t = 1$.

The family of strategies parameterized by $\mathbf{q} \in S$ is in fact studied in [1], but the focus there is on conditions under which \mathbf{q} is an admissible policy (not dominated by another \mathbf{q}' for every $\mathbf{p} \in \Delta$). Below, we state its performance, which can be found in [1], and can be verified by direct calculation:

Lemma 2.2. [from [1]] If $N \sim \mathbf{p} \in \Delta$, strategy $\mathbf{q} \in S$ has probability of success

$$\mathbb{P}[\mathsf{Success}(\mathbf{q})] = A(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{\infty} p_i(\frac{1}{i} \sum_{l=1}^{i} U_{l-1}(\mathbf{q})q_l) = \sum_{i=1}^{\infty} U_{i-1}(\mathbf{q})q_i\lambda_i(\mathbf{p})$$

where $U_0(\mathbf{q}) = 1$, $U_i(\mathbf{q}) := \prod_{l=1}^{i} (1 - q_l/l)$ is the probability that the strategy has not accepted any number among the first *i* numbers conditioned on $N \ge i$. $\lambda_i(\mathbf{p}) := \sum_{l=i}^{\infty} p_l/l$.

For any $n \in \mathbb{N}$, [1] constructs a distribution supported on [n] (we denote as $\mathbf{p}_*^{(n)}$) such that $A(\mathbf{p}_*^{(n)}, \mathbf{q}) = 1/H_n(1 - U_n(\mathbf{q})) \leq 1/H_n$ for any $\mathbf{q} \in \mathcal{S}$ where $H_n = \sum_{i=1}^n 1/i$ is the *n*-the harmonic number:

$$p_{*,i}^{(n)} = \frac{1/H_n}{i+1}, \ i = 1, \dots, n-1, \quad p_{*,n}^{(n)} = 1/H_n, \quad p_{*,i}^{(n)} = 0, \quad i \ge n+1.$$
(1)

Together with Lemma 2.1, we get the following result:

Corollary 2.1. When $N \sim \mathbf{p}_*^{(n)}$ defined in Equation 1, for any $\mathsf{Str} \in \overline{\mathcal{S}}$, $\mathbb{P}[\mathsf{Success}(\mathsf{Str})] \leq 1/H_n$, and the upper bound is achieved by $\mathbf{q}^{(1)}$, i.e. picking the first number.

3 Main Results

Our goal is to use the available information about the distribution \mathbf{p} to find a strategy $\mathsf{Str} \in \overline{S}$ which approximately maximizes $\mathbb{P}[\mathsf{Success}(\mathsf{Str})]$ when $N \sim \mathbf{p}$. Thanks to the results in Lemma 2.1 and 2.2, instead of considering all possible strategies $\mathsf{Str} \in \overline{S}$, it suffices to consider the subfamily of strategies $\{\mathsf{Str}(\mathbf{q}) | \mathbf{q} \in S\}$, which has performance $\mathbb{P}[\mathsf{Success}(\mathbf{q})] = A(\mathbf{p}, \mathbf{q})$ when $N \sim \mathbf{p}$. Below, we first formally state the three types of prior information and the results in Section 3.1. Then, we provide more discussions and sketch of proofs in the following subsections. The full proofs are deferred to the Appendix.

3.1 Overview of main results

Full information: **p** is known. We focus on strategy simplicity, and compare the performance of the optimal single-threshold type of strategies with the optimal one in \overline{S} . This is motivated by [38] which shows that for many distributions (such as uniform, Poisson, and geometric) there exists $l \in \mathbb{N}$ (which depends on the parameters of the distribution) where the simple strategy $\mathbf{q}^{(l)}$ is in fact optimal. We show that this is approximately true for general **p**, in the sense that restricting **q** from $\mathcal{S} = [0, 1]^{\mathbb{N}}$ to the family of single-threshold strategies $\{\mathbf{q}^{(l)} | l \in \mathbb{N}\}$ gives a constant factor approximation to the optimal performance:

Theorem 3.1. For any $\mathbf{p} \in \Delta$, let $\lambda_i(\mathbf{p}) := \sum_{l=i}^{\infty} p_l / l$ for $i \in \mathbb{N}$, and $\theta(\mathbf{p}) := \max_{i \in \mathbb{N}} i \lambda_i(\mathbf{p}) = K^*(\mathbf{p}) \lambda_{K^*(\mathbf{p})}(\mathbf{p})$ for some $K^*(\mathbf{p}) \in \mathbb{N}$, then

$$\theta(\mathbf{p})/e \le A(\mathbf{p}, \mathbf{q}^{(K^*(\mathbf{p}))}) \le \sup_{\mathbf{q} \in \mathcal{S}} A(\mathbf{p}, \mathbf{q}) = \sup_{\mathsf{Str} \in \overline{\mathcal{S}}} \mathbb{P}[\mathsf{Success}(\mathsf{Str})] \le \theta(\mathbf{p}).$$

On a high level, when $N \sim \mathbf{p}$, there exists a classical secretary problem scaled by $\theta(\mathbf{p})$ hidden inside the problem. Since for the classical problem, there exists a single-threshold strategy which succeeds with probability at least 1/e, the same strategy applied to the variant succeeds with probability at least $\theta(\mathbf{p})/e$.

Known upper bound: $N \leq \overline{n}$ (or $\mathbb{E}[N] \leq \overline{\mu}$) and \overline{n} (or $\overline{\mu}$) is known. We focus on performance guarantees against a worst or average **p** subject to the available information. When $N \leq \overline{n}$, Corollary 2.1 and the randomized algorithm proposed in [36]⁴ shows

$$1/(1+H_{\overline{n}-1}) \leq \sup_{\mathsf{Str}\in\overline{\mathcal{S}}} \inf_{\mathbf{p}\in\Delta^{(\overline{n})}} \mathbb{P}[\mathsf{Success}(\mathsf{Str})] \leq 1/H_{\overline{n}}.$$

where $\Delta^{(n)} = \{ \mathbf{x} \in \Delta | x_i = 0, \forall i \ge n+1 \}$. Since $N \le \overline{n}$ implies $\mathbb{E}[N] \le \overline{n}$, the above upper bound implies that when $\mathbb{E}[N] \le \overline{\mu}$, no algorithm can achieve a guarantee better than $1/H_{\lfloor \overline{\mu} \rfloor}$ in the worst case.

We extend these existing results in two directions. First, we show that the $1/(1 + H_{\overline{n}-1})$ lower bound can in fact be achieved by a randomization over single-threshold strategies, as compared to the strategies $Str(\mathbf{q})$ proposed in [36] and [28]. Similar results hold when $\mathbb{E}[N] \leq \overline{\mu}$ and $\overline{\mu}$ is known.

Notice that according to Lemma 2.2, $\mathbb{P}[\mathsf{Success}(\mathbf{q})] = A(\mathbf{p}, \mathbf{q})$ is not linear in the strategy parameter \mathbf{q} , and so for a random strategy $\hat{\mathbf{q}}$ with $\overline{\mathbf{q}} = \mathbb{E}[\hat{\mathbf{q}}]$, the performance of $\hat{\mathbf{q}}$, $\mathbb{P}[\mathsf{Success}(\hat{\mathbf{q}})]$, does not equal to $\mathbb{P}[\mathsf{Success}(\overline{\mathbf{q}})]$ in general.

Theorem 3.2. For each $n \in \mathbb{N}$, define the following distribution $\mathbf{x}^{(n)} \in \Delta^{(n)}$: $x_1^{(n)} = \frac{1}{1+H_{n-1}}$, and $x_i^{(n)} = \frac{1/(1+H_{n-1})}{i-1}$ for $i = 2, \ldots, \overline{n}$. Then for any $\mathbf{p} \in \Delta^{(\overline{n})}$, let $L \sim \mathbf{x}^{(\overline{n})}$,

$$\mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)})] = \frac{1}{1 + H_{\overline{n}-1}}$$

For any $\mathbf{p} \in \Delta$, such that if $N \sim \mathbf{p}$, $\mathbb{E}[N] \leq \overline{\mu}$, let $L \sim \mathbf{x}^{(\lceil \overline{\mu} \log(\overline{\mu}) \rceil)}$,

$$\mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)})] \geq \frac{1}{\log(\overline{\mu})}(1 - O(\frac{\log(\log(\overline{\mu}))}{\log(\overline{\mu})}))$$

The above theorem shows that in the *worst case* when **p** is supported on $[\overline{n}]$, no algorithm can succeed with probability more than $1/H_{\overline{n}}$, which converges to 0 as $\overline{n} \to \infty$. This motivates us to study instead the *most likely cases*: for most **p** supported on $[\overline{n}]$, is $\sup_{\mathsf{Str}\in\overline{S}} \mathbb{P}[\mathsf{Success}(\mathsf{Str})]$ lower bounded by a constant, or converging to 0 as $\overline{n} \to \infty$? Surprisingly, we find that there is a single-threshold strategy (depending on \overline{n}) that guarantees a constant probability of success for most distributions supported on $[\overline{n}]$:

Theorem 3.3. For each $n \in \mathbb{N}$, define $l_n^* := \lceil n/e^2 \rceil$. Then for any $0 < \epsilon < 2/e^2$, there exists $0 < \rho < 1$, such that if $\hat{\mathbf{p}}^{(n)} \sim Uniform(\Delta^{(n)})$ for each $n \in \mathbb{N}$, then $\mathbb{P}[A(\hat{\mathbf{p}}^{(n)}, \mathbf{q}^{(l_n^*)}) \leq \epsilon] = O(\rho^n)$ as $n \to \infty$. For instance, for $\epsilon \leq 0.03$, $\mathbb{P}[A(\hat{\mathbf{p}}^{(n)}, \mathbf{q}^{(l_n^*)}) \leq \epsilon] = O(2^{-n})$. The probability is over the randomness in $\hat{\mathbf{p}}^{(n)}$ $(A(\cdot, \mathbf{q}^{(l_n^*)}) : \Delta \to [0, 1]$ is a deterministic function of \mathbf{p}).

⁴Precisely, the algorithm proposed in [36] succeeds with probability $1/(1 + H_{n-1})$ against a non-adaptive adversary who picks $N \leq n$ in advance, i.e. not based on the relative ranks or the algorithm's decisions as compared to the adversary we consider in Appendix A.3. And it can easily be shown that this algorithm achieves the same guarantee against a random **p** supported on [n].

The above theorem shows that for all but an exponentially small fraction of distributions supported on [n], the strategy $\mathbf{q}^{(l_n^*)}$ achieves a constant probability of success. Thus, for most distributions with bounded support, $\sup_{\mathsf{Str}\in\overline{\mathcal{S}}} \mathbb{P}[\mathsf{Success}(\mathsf{Str})]$ is lower bounded by a positive constant that does not depend on the upper bound of the support.

Samples: $N \sim \mathbf{p}$ for some unknown distribution $\mathbf{p} \in \Delta$, and *m* samples $N^{(1)}, \ldots, N^{(m)} \sim_{iid} \mathbf{p}$ are given. We investigate learning algorithms and the sampling complexity of finding an ϵ suboptimal (as compared to knowing **p**) strategy $\hat{\mathbf{q}} \in \mathcal{S}$, such that with high probability,

$$A(\mathbf{p}, \hat{\mathbf{q}}) \ge \sup_{\mathbf{q} \in \mathcal{S}} A(\mathbf{p}, \mathbf{q}) - \epsilon.$$
⁽²⁾

We propose a block based approach stated in Algorithm 1, where "block" refers to the following mapping, called Block:

Definition 3.1. For each $\rho > 1$, let $I(\rho) := \{ \lceil \rho^l \rceil, l \in \{0\} \cup \mathbb{N} \}$ be an index set, with elements $I(\rho) = \{ 1 = 1 \}$ $I(\rho,1) < I(\rho,2) < \cdots \}^5$. For convenience, denote $I(\rho,0) = 0$. Let Block : $\Delta \times (1,\infty) \to \Delta$ be a mapping such that for $\mathbf{p} \in \Delta$, $\rho > 1$, Block (\mathbf{p}, ρ) is a distribution supported only on $I(\rho)$, where

$$\mathsf{Block}(\mathbf{p},\rho)_i = \begin{cases} 0 & i \notin I(\rho) \\ \sum_{j=I(\rho,l-1)+1}^{I(\rho,l)} p_j & i = I(\rho,l), l \in \mathbb{N} \end{cases}$$

On a high level, our Algorithm 1 uses samples to construct an approximation to $A(\mathsf{Block}(\mathbf{p},\rho),\cdot)$ for $\rho = 1 + \epsilon/4$. We show that the bias due to the use of Block, i.e. the difference between $A(\mathsf{Block}(\mathbf{p},\rho),\cdot)$ and $A(\mathbf{p}, \cdot)$, is small for ρ close to 1 (Corollary 3.1). In addition, a good estimation of $i\lambda_i(\mathsf{Block}(\mathbf{p}, \rho))$ for i in the support of $\mathsf{Block}(\mathbf{p},\rho)$ is sufficient for a good function approximation (Lemma 3.2 and D.5). Together, we get the following guarantee:

Theorem 3.4. Let $0 < \epsilon, \delta < 1$. Let $T \in \mathbb{N}$ be such that $\sum_{i=T+1}^{\infty} p_i \le \epsilon/12$. If T = 1, and $m \ge \frac{18}{\epsilon} \log(\frac{2}{\delta})$, or if $T \ge 2$ and $m \ge \max(\frac{18}{\epsilon}\log(\frac{50\log(T)}{\epsilon\delta}), \frac{1}{2\epsilon^2}\log(\frac{1200}{\epsilon^2\delta}))$, with probability at least $1 - \delta$, Algorithm 1 outputs $\hat{\mathbf{q}}$ satisfying Equation (2).

Algorithm 1: Secretary Problem with Samples: Block based Algorithm

Input: *m* samples $N^{(1)}, N^{(2)}, ..., N^{(m)} \sim_{iid} \mathbf{p}, 0 < \epsilon \leq 1$. **Output:** $\hat{\mathbf{q}} \in \mathcal{S}$. $\rho \leftarrow 1 + \epsilon/4;$ $\hat{p}_{i} \leftarrow \inf\{i \in \mathbb{N}, \ I(\rho, i) \ge \max_{j \in [m]} N^{(j)}\}; \\ \hat{p}_{i} \leftarrow \begin{cases} \frac{1}{m} \sum_{j=1}^{m} \mathbf{1}[I(\rho, l-1) + 1 \le N^{(j)} \le I(\rho, l)] & i = I(\rho, l), \ l \in [l_{\max}] \\ 0 & o.w. \end{cases};$ $G_{i} \leftarrow \hat{i}\lambda_{I(\rho,l)}(\hat{\mathbf{p}}), \quad I(\rho,l-1) < i \le I(\rho,l), \ l \in [l_{\max}];$ $\hat{\mathbf{q}} \leftarrow \sup_{\mathbf{q} \in \mathcal{S}} \sum_{i=1}^{I(\rho,l_{\max})} U_{i-1}(\mathbf{q}) \frac{q_{i}}{i} \cdot G_{i}$

We make three additional comments about Algorithm 1. First, due to the use of Block, Algorithm 1 only requires a much weaker sampling oracle, where iid samples from the less refined distribution $\mathsf{Block}(\mathbf{p}, \rho)$ are needed. Second, if T is not known in advance, we can sample $N^{(1)}, \ldots, N^{(m_1)} \sim_{iid} \mathbf{p}$ where $m_1 = \lceil \frac{12}{\epsilon} \log(\frac{2}{\delta}) \rceil$, and then, with probability at least $1-\delta/2$, setting $T = \max_{j \in [m_1]} N^{(j)}$ will satisfy the requirement in Theorem 3.4. Third, in the last step, an optimization problem needs to be solved. Due to its special structure, it can be solved exactly in a backward fashion in $O(I(\rho, l_{\text{max}}))$ steps⁶. If an ϵ' suboptimal solution is obtained instead of an exact one, then $\hat{\mathbf{q}}$ is at least $\epsilon + \epsilon'$ suboptimal for the original problem.

In addition, we show that $\Omega(\frac{1}{\epsilon^2})$ samples are in fact necessary:

⁵The notation $I(\rho, \cdot)$ denotes distinctive terms in $I(\rho)$. This is to deal with cases where ρ is very small and $\lceil \rho^l \rceil = \lceil \rho^{l+1} \rceil$ for *l* small. For instance, when $\rho < 2$, for all $l < \frac{\log(2)}{\log(\rho)}$, $\lceil \rho^l \rceil = 2$. ⁶Notice that each q_i only appears in the terms $U_{j-1}(\mathbf{q})\frac{q_j}{j} \cdot G_j$ for $j \ge i$, and once q_j 's are determined for all j > i, the

objective function is linear in q_i .

Theorem 3.5. For any $n \ge 3$, there exists a threshold c_n , such that for $0 < \epsilon < c_n$, at least $m \ge \frac{c_n - \epsilon}{9\epsilon^2}$ samples are needed to find an $\epsilon/3$ suboptimal strategy $\hat{\mathbf{q}}$ with probability at least 2/3. In addition, $\lim_{n\to\infty} c_n = \frac{1}{1+\epsilon}$.

In fact, the hard cases are supported only on $\{1, n\}$, with the probability $p_1 = c_n \pm \epsilon$. Here, c_n is chosen such that the optimal strategy at time 1, i.e. q_1 , switches between 0 and 1. Thus to output an $O(\epsilon)$ suboptimal strategy, the algorithm needs to know p_1 .

3.2 Full information

Theorem 3.1 shows that the optimal single-threshold strategy gives a 1/e approximation to the optimal algorithm in \overline{S} , that is,

$$\sup_{l \in \mathbb{N}} \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(l)})] \geq \frac{1}{e} \sup_{\mathsf{Str} \in \overline{\mathcal{S}}} \mathbb{P}[\mathsf{Success}(\mathsf{Str})].$$

In addition, it shows that $\theta(\mathbf{p})$ is a good approximation to the optimal performance when $N \sim \mathbf{p}$, in the sense that

$$\frac{1}{e}\theta(\mathbf{p}) \leq \sup_{\mathsf{Str}\in\overline{\mathcal{S}}} \mathbb{P}[\mathsf{Success}(\mathsf{Str})] \leq \theta(\mathbf{p}).$$

The above inequality is in fact tight. The lower bound is achieved by the classical secretary problem with $\mathbf{p} = \boldsymbol{\delta}^{(n)}$ for any $n \in \mathbb{N}$, since $\lambda_i(\boldsymbol{\delta}^{(n)}) = \frac{1}{n}\mathbf{1}[i \leq n]$, and so $\theta(\boldsymbol{\delta}^{(n)}) = 1$, and it's known that as $n \to \infty$, the optimal success probability of the classical secretary problem with T numbers converges to $\frac{1}{e} = \frac{1}{e}\theta(\mathbf{p})$. For the upper bound, notice that the distribution $\mathbf{p}_*^{(n)}$ defined in Equation (1) satisfies $i\lambda_i(\mathbf{p}_*^{(n)}) = 1/H_n\mathbf{1}[i \leq n]$ and so $\theta(\mathbf{p}_*^{(n)}) = 1/H_n$. In addition, Corollary 2.1 shows that the optimal performance is in fact $1/H_n$, showing the tightness of the upper bound.

Last, we provide the sketch of proof of Theorem 3.1. For the upper bound, recall that

$$A(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{\infty} (U_{i-1}(\mathbf{q})q_i/i) \cdot (i\lambda_i(\mathbf{p})), \quad U_0(\mathbf{q}) = 1, \ U_i(\mathbf{q}) := \prod_{l=1}^{i} (1 - q_l/l)$$

is linear in $i\lambda_i$, and the sum of the coefficients does not exceed 1; for the lower bound, we compare $A(\mathbf{p}, \mathbf{q})$ with $T\lambda_T \cdot A(\boldsymbol{\delta}^{(T)}, \mathbf{q})$ where $\boldsymbol{\delta}^{(T)}$ is a delta distribution supported at T. Since when $\mathbf{p} = \boldsymbol{\delta}^{(T)}$ is a delta distribution, the problem is just the classical secretary where the time horizon is T, and the optimal success probability $\approx 1/e$. Taking supremum over T gives the desired result.

3.3 Known upper bound

For Theorem 3.2, the proof of the case when $N \leq \overline{n}$ is by calculating the probability of success when L is sampled according to $\mathbf{x} \in \Delta$, then $\mathbf{q}^{(L)}$ is used. The proof of the case when $\mathbb{E}[N] \leq \overline{\mu}$ is by truncating the tail using Markov inequality, and using the result of the bounded case.

For Theorem 3.3, the proof uses two main ideas: for fixed \mathbf{q} , $A(\mathbf{p}, \mathbf{q})$ is linear in \mathbf{p} , so $\mathbb{E}[A(\mathbf{p}, \mathbf{q})] = A(\mathbb{E}[\mathbf{p}], \mathbf{q})$. In particular, when \mathbf{p} is uniformly sampled in $\Delta^{(n)}$, $\mathbb{E}[\mathbf{p}]$ is the uniform distribution over [n], and so in expectation, $A(\mathbf{p}, \mathbf{q})$ is the same as the performance when N is uniformly randomly selected from [n] – this suggests considering the optimal strategy for the case when N follows the uniform distribution on [n]; second, we use concentration inequalities to show that with high probability, the performance is close to its expectation.

[38] shows that when **p** is the uniform distribution over [n], the optimal strategy is $\mathbf{q}^{(\tilde{l}_n^*)}$ for some $\tilde{l}_n^* \in \mathbb{N}$ satisfying $\frac{1}{e^2}(n+1) - 0.5 < \tilde{l}_n^* < \frac{1}{e^2}(n+1) + 1.5$, and the probability of success for this strategy converges to $\frac{2}{e^2}$ as $n \to \infty$. With $l_n^* := \lceil n/e^2 \rceil$, the strategy $\mathbf{q}^{(l_n^*)}$ in Theorem 3.3 is an approximately optimal strategy when N follows the uniform distribution on [n]. Thus, Theorem 3.3 shows:

- for small ϵ , the fraction of distributions $\mathbf{p} \in \Delta^{(n)}$ where no algorithm can achieve better than ϵ probability of success decays exponentially in n, despite the $O(\log(n)^{-1})$ worst case \mathbf{p} ;
- single-threshold strategies can achieve constant probability of success for most bounded distributions.

3.4 Samples

We take a learning theoretical approach: the samples are used to construct an approximation to $A(\mathbf{p}, \cdot)$: $S \rightarrow [0, 1]$ viewed as a function of the strategy parameter $\mathbf{q} \in S$. If the approximation error is at most $\epsilon/2$ for all $\mathbf{q} \in S$, then any optimal policy for the approximation function satisfies the goal Equation (2) (Lemma D.3). Recall that from Lemma 2.2,

$$A(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{\infty} p_i(\frac{1}{i} \sum_{l=1}^{i} U_{l-1}(\mathbf{q})q_l) = \sum_{i=1}^{\infty} U_{i-1}(\mathbf{q})q_i\lambda_i(\mathbf{p})$$

where $U_0(\mathbf{q}) = 1$, $U_i(\mathbf{q}) := \prod_{l=1}^{i} (1 - q_l/l)$ and $\lambda_i(\mathbf{p}) := \sum_{l=i}^{\infty} p_l/l$ for $i \in \mathbb{N}$. Since $\frac{1}{i} \sum_{l=1}^{i} U_{l-1}(\mathbf{q})q_l \leq 1$ for all $\mathbf{q} \in \mathcal{S}$, $A(\cdot, \mathbf{q})$ is 1-Lipschitz in \mathbf{p} with respect to the l_1 norm. Thus, estimating \mathbf{p} up to accuracy of $\epsilon/2$ in l_1 norm, which usually requires $O(\frac{n}{\epsilon^2})$ samples [25], is enough to find a ϵ suboptimal strategy. However, due to the specific structure of the secretary problem, the function $A(\mathbf{p}, \cdot)$ has two nice properties which we state in Section 3.4.1. These properties allow much more efficient sampling algorithms.

Indeed, our Algorithm 1 achieves the optimal sampling complexity of $O(\frac{1}{\epsilon^2})$ (up to a $\log(\frac{1}{\epsilon})$ factor) in the regime $\epsilon = O(\frac{1}{\log \log(T)})$, and remains sublinear in T for all $\epsilon \in (0, 1)$. It's still open whether this dimension dependence in the large ϵ regime is necessary. However, we conjecture that the overall sample complexity is at least $\Omega(\log \log(T))$, since it requires $\Omega(\log(\frac{\log(n)}{\epsilon}))$ bits to just write down an ϵ suboptimal strategy for distributions supported on [n] for $0 < \epsilon < \frac{1}{2\epsilon}$ (Theorem D.1).

3.4.1 Two properties of $A(\cdot, \cdot)$

First, for any strategy $\mathbf{q} \in \mathcal{S}$, $A(\cdot, \mathbf{q})$ is insensitive to *local* variation in \mathbf{p} : if some probability mass that belongs to p_i is assigned to $p_{i'}$, as long as i' is close to i, the resulting change in the function value is small. To be more precise, notice that by linearity,

$$A(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{\infty} p_i \cdot A(\boldsymbol{\delta}^{(i)}, \mathbf{q}).$$

We show that for any $\mathbf{q} \in \mathcal{S}$, the coefficients of p_n and $p_{n'}$ are approximately the same if $n \approx n'$:

Lemma 3.1. For any $\mathbf{q} \in S$ and any $n, n' \in \mathbb{N}$, $n \leq n'$,

$$\frac{n}{n'}A(\boldsymbol{\delta}^{(n)}, \mathbf{q}) \le A(\boldsymbol{\delta}^{(n')}, \mathbf{q}) \le \frac{n}{n'}A(\boldsymbol{\delta}^{(n)}, \mathbf{q}) + (1 - \frac{n}{n'})$$

Intuitively, for the classical secretary problem with n' candidates, with probability $\frac{n}{n'}$ the best candidate is among the first n arrivals, and the probability of success under this event is $A(\boldsymbol{\delta}^{(n)}, \mathbf{q})$. For the event that the best candidate arrives after n, the probability of success is upper bounded by 1.

Due to this insensitivity, it suffices to consider the "block" version for \mathbf{p} , $\mathsf{Block}(\mathbf{p}, \rho)$, where the probability mass in each block (interval) is assigned to the end of the block. The following guarantee is achieved:

Corollary 3.1. For any $\mathbf{p} \in \Delta$, any $\mathbf{q} \in S$ and $\rho > 1$, denote $\tilde{\mathbf{p}} = \mathsf{Block}(\mathbf{p}, \rho)$, then

$$|A(\mathbf{p}, \mathbf{q}) - A(\tilde{\mathbf{p}}, \mathbf{q})| \le \rho - 1.$$

The second property we take advantage of is that $A(\cdot, \cdot)$ depends on **p** only through the sequence $(i\lambda_i(\mathbf{p}))_{i\in\mathbb{N}}$, and is 1-Lipschitz in $(i\lambda_i)_{i\in\mathbb{N}}$ with respect to the l_{∞} norm. Thus, a good estimation of all the λ_i 's suffices for a good function approximation:

Lemma 3.2. Let $\mathbf{p} \in \Delta$ be a fixed distribution, $\epsilon > 0$. Let $n \in \mathbb{N}$, be such that for $i \ge n+1$, $i\lambda_i(\mathbf{p}) \le \epsilon$. Let $\mathbf{G} \in [0,1]^n$ be a sequence of length n, and for $i \in [n]$, $|G_i - i\lambda_i(\mathbf{p})| \le \epsilon$. Then for any $\mathbf{q} \in S$,

$$|A(\mathbf{p},\mathbf{q}) - \sum_{i=1}^{n} U_{i-1}(\mathbf{q}) \frac{q_i}{i} \cdot G_i| \le \epsilon.$$

In fact, if the sequence **G** above is defined as $G_i = i\lambda_i(\hat{\mathbf{p}})$ for some distribution $\hat{\mathbf{p}}$ whose support is a subset of the support of \mathbf{p} (for example, $\hat{\mathbf{p}}$ is the empirical distribution of samples from \mathbf{p}), then the conditions in Lemma 3.2 can be relaxed to $|G_i - i\lambda_i(\mathbf{p})| \leq \epsilon$ for $i \in [n]$ and $p_i > 0$ (Lemma D.5). In other words, the number of parameters that needs to be estimated depends only on the number of non-zero terms in the (truncated) distribution, which makes the block-based technique extremely efficient.

3.4.2 Proof sketch of Theorem 3.4

The proof of Theorem 3.4 consists of two parts. First, we use concentration bounds to show that with high probability, the tail of **p** starting from $N_{\max} := I(\rho, l_{\max})$ is $O(\epsilon)$, and for each $i = I(\rho, l)$ for some $l \in [l_{\max}]$, G_i is a good estimate of $i\lambda_i(\mathsf{Block}(\mathbf{p}, \rho))$. Precisely, we lower bound the probability of the following good event (\mathcal{E}^*) :

$$\sum_{i=N_{\max}+1}^{\infty} p_i \le \epsilon/12, \text{ and } |G_i - i\lambda_i(\mathsf{Block}(\mathbf{p}, \rho))| \le \epsilon/4 \ i = I(\rho, l), \ l \in [l_{\max}].$$
 (\$\mathcal{E}^*\$)

There are two technicalities. First, the number of G_i 's to be bounded is l_{\max} , a random variable which depends on how "long" the tail of **p** is, and which can potentially be much larger than $\log(T)^7$ We fix this by showing that $\frac{1}{m} \sum_{j=1}^{m} \mathbf{1}[N^{(j)} \ge \rho T] = O(\epsilon)$ with high probability, which implies $|G_i| = O(\epsilon)$ for all $i \ge T$ (jointly). Another issue is that if all of $|G_i - i\lambda_i|$'s in (\mathcal{E}^*) are bounded using Hoeffding's inequality, the sample complexity will be $O(\frac{\log \log(T)}{\epsilon^2})$. To further improve it to $O(\frac{1}{\epsilon^2} \max(\epsilon \log \log(T), 1))$, we upper bound the sum of variances of G_i 's, thereby showing that at most $O(\frac{1}{\epsilon^2})$ of them can have large variances, and the rest of the terms can be bounded using the tighter Bernstein's inequality.

For the second part of the proof, we first use Lemma D.5 to show that under Event (\mathcal{E}^*) , all $|G_i - i\lambda_i|$'s are small for all $i \in [N_{\max}]$, which together with Event (\mathcal{E}^*) , implies that the conditions in Lemma 3.2 holds with **p** replaced by Block(**p**, ρ). With the approximation bound in Corollary 3.1, we show that $A(\mathbf{p}, \cdot)$ is approximated up to accuracy $\epsilon/2$, and so output $\hat{\mathbf{q}}$ is at least ϵ -suboptimal.

4 Beyond secretary: matroid, prophet, online LP

We apply the idea of Theorem 3.2 – randomizing over algorithms (strategies) designed for problems with known time horizon – to three other classical online problems: matroid secretary problem, iid prophet inequality, and online packing LP. In the classical setting, the performance of algorithms for these problems are often evaluated using the competitive ratio, defined as the expectation of the ratio between the output of the algorithm and the optimal. When the time horizon N for these problems is random, we can look at a modified performance metric: the expectation (over the randomness in N) of the classical metric restricted to the first N arrivals. Detailed setup and comparison with previous results are deferred to Appendix B.

We assume here that N is a random variable with known upper and lower bound. Interestingly, for these three problems, all algorithms that achieve the optimal or best known performance require the time horizon as an input. Let $\{\mathcal{A}^{(s)}\}_{s\in\mathbb{N}}$ be a family of algorithms designed for problem instances when the time horizon is s. Just like in the random horizon secretary problem, the key quantity here is how sensitive the performance of $\{\mathcal{A}^{(s)}\}_{s\in\mathbb{N}}$ is to "misspeficiation" of the time horizon. More concretely, let $M : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be the performance of $\mathcal{A}^{(s)}$ applied to a problem whose time horizon is n in the metrics of interest, and assume that $\underline{n} \leq N \leq \overline{n}$ almost surely, where $\underline{n}, \overline{n} \in \mathbb{N}$ are known. We have the following result:

Theorem 4.1. Assume that there exists a constant $c_0 > 0$ and a nondecreasing function $f : \mathbb{N} \to \mathbb{R}_{>0}$, such that for each $s, n \in \mathbb{N}$, $M(s, n) \ge c_0 \frac{f(s)}{f(n)} \mathbf{1}[s \le n]$. Then for any $\underline{n} \le \overline{n} \in \mathbb{N}$, there exists a distribution $\mathbf{x}^{(\underline{n},\overline{n})} \in \Delta$ (depending on $f, \underline{n}, \overline{n}$) supported on $\{\underline{n}, \ldots, \overline{n}\}$, such that if $S \sim \mathbf{x}^{(\underline{n},\overline{n})}$, and $\mathcal{A}^{(S)}$ is used, then the expected performance of this randomized algorithm is at least $\frac{c_0}{1 + \log(f(\overline{n})/f(\underline{n}))}$ for any distribution of Nthat is supported on $\{\underline{n}, \ldots, \overline{n}\}$.

Surprisingly, independent of the actual algorithms $\{\mathcal{A}^{(s)}\}_{s\in\mathbb{N}}$, just due to the random permutation or iid assumption, the specific form $M(s,n) \ge c_0 \frac{f(s)}{f(n)} \mathbf{1}[s \le n]$ holds for all three problems we consider:

⁷For example, if **p** is such that $p_{T'} = \epsilon/20$ for some T' >> T, then if $m = \Omega(\frac{1}{\epsilon})$, with constant probability, $N^{(j)} = T'$ for some $j \in [m]$, and $l_{\max} \gtrsim \log(T') >> \log(T)$.

- matroid secretary problem: c₀ = Ω(1/log log(k)) for rank-k matroid, and f(i) = i;
 iid prophet inequality: c₀ = 0.745, and f(i) = E[max_{j∈[i]} X_j] where X_j ~_{iid} F is the observed sequence, for instance
 - if $X_i \sim Uniform([0,1]), \mathbb{E}[\max_{i \in [l]} X_i] = \frac{l}{l+1},$
- if $X_i \sim Exp(1)$, $\mathbb{E}[\max_{i \in [l]} X_i] = \sum_{i=1}^{l} 1/i \approx \log(l)$; online packing LP: $c_0 = 1 O(\epsilon)$ where $\epsilon = \Omega(\frac{m}{\sqrt{B}})$ with B the budget-to-bid ratio and m the number of constraints in the LP, and f(i) = i.

The constant c_0 represents the algorithm's performance when the time horizon is fixed and known, and the additional factor of $\frac{1}{1+\log(f(\overline{n})/f(\underline{n}))}$ represents the extra difficulty due to the uncertainty in N.

In fact, for the iid prophet inequality, when there exists $x \in (0, 1]$ such that all $X_i \in [x, 1]$ almost surely, our randomization algorithm is at least $\frac{c_0}{1 + \log(1/x)}$ -competitive, and we show an upper bound of $\frac{1}{1 + \log(1/x)}$ (Theorem B.1), meaning that our algorithm is optimal up to a constant factor.

References

- A. R. Abdel-Hamid, J. A. Bather, and G. B. Trustrum. "The Secretary Problem with an Unknown Number of Candidates". In: *Journal of Applied Probability* 19.3 (1982), pp. 619–630.
- [2] Shipra Agrawal, Zizhuo Wang, and Yinyu Ye. "A Dynamic Near-Optimal Algorithm for Online Linear Programming". In: Oper. Res. 62.4 (Aug. 2014), pp. 876–890.
- [3] Nima Anari et al. "Nearly Optimal Pricing Algorithms for Production Constrained and Laminar Bayesian Selection". In: Proceedings of the 2019 ACM Conference on Economics and Computation. EC '19. Phoenix, AZ, USA: Association for Computing Machinery, 2019, pp. 91–92.
- [4] Pablo D. Azar, Robert Kleinberg, and S. Matthew Weinberg. "Prophet Inequalities with Limited Information". In: Proceedings of the 2014 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pp. 1358–1377.
- [5] Moshe Babaioff, Nicole Immorlica, and Robert Kleinberg. "Matroids, Secretary Problems, and Online Mechanisms". In: Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms. SODA '07. New Orleans, Louisiana: Society for Industrial and Applied Mathematics, 2007, pp. 434–443.
- [6] Moshe Babaioff et al. "Online Auctions and Generalized Secretary Problems". In: *SIGecom Exch.* 7.2 (June 2008).
- [7] Santiago Balseiro, Christian Kroer, and Rachitesh Kumar. "Online Resource Allocation under Horizon Uncertainty". In: *SIGMETRICS Perform. Eval. Rev.* 51.1 (June 2023), pp. 63–64.
- [8] Mark Braverman, Mahsa Derakhshan, and Antonio Molina Lovett. "Max-Weight Online Stochastic Matching: Improved Approximations Against the Online Benchmark". In: Proceedings of the 23rd ACM Conference on Economics and Computation. EC '22. Boulder, CO, USA: Association for Computing Machinery, 2022, pp. 967–985.
- [9] Niv Buchbinder, Kamal Jain, and Mohit Singh. "Secretary Problems via Linear Programming". In: Mathematics of Operations Research 39.1 (2014), pp. 190–206.
- [10] Niv Buchbinder and Joseph (Seffi) Naor. "The Design of Competitive Online Algorithms via a Primal–Dual Approach". In: Foundations and Trends® in Theoretical Computer Science 3.2–3 (2009), pp. 93–263.
- [11] Sourav Chakraborty and Oded Lachish. "Improved Competitive Ratio for the Matroid Secretary Problem". In: Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms. SODA '12. Kyoto, Japan: Society for Industrial and Applied Mathematics, 2012, pp. 1702–1712.
- [12] T-H. Hubert Chan, Fei Chen, and Shaofeng H.-C. Jiang. "Revealing Optimal Thresholds for Generalized Secretary Problem via Continuous LP: Impacts on Online K-Item Auction and Bipartite K-Matching with Random Arrival Order". In: *Proceedings of the Twenty-Sixth Annual ACM-SIAM* Symposium on Discrete Algorithms. SODA '15. San Diego, California: Society for Industrial and Applied Mathematics, 2015, pp. 1169–1188.
- [13] Yuan Shih Chow, Herbert Robbins, and David Siegmund. The theory of optimal stopping. eng. New York: Dover, 1971 - 1991.
- [14] Jose Correa et al. "Recent Developments in Prophet Inequalities". In: SIGecom Exch. 17.1 (May 2019), pp. 61–70.
- [15] José Correa et al. "Posted Price Mechanisms for a Random Stream of Customers". In: Proceedings of the 2017 ACM Conference on Economics and Computation. EC '17. Cambridge, Massachusetts, USA: Association for Computing Machinery, 2017, pp. 169–186.
- [16] José Correa et al. "Prophet Inequalities for I.I.D. Random Variables from an Unknown Distribution". In: Proceedings of the 2019 ACM Conference on Economics and Computation. EC '19. Phoenix, AZ, USA: Association for Computing Machinery, 2019, pp. 3–17.
- [17] José Correa et al. "Sample-Driven Optimal Stopping: From the Secretary Problem to the i.i.d. Prophet Inequality". In: *Mathematics of Operations Research* 0.0 (0), null.

- [18] José Correa et al. "The Secretary Problem with Independent Sampling". In: Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10 - 13, 2021. Virtual, United States: Society for Industrial and Applied Mathematics, Jan. 2021, pp. 2047– 2058.
- [19] José Correa et al. "Unknown I.I.D. Prophets: Better Bounds, Streaming Algorithms, and a New Impossibility". In: 12th Innovations in Theoretical Computer Science Conference (ITCS 2021). Ed. by James R. Lee. Vol. 185. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021, 86:1–86:1.
- [20] Nikhil R. Devanur et al. "Near Optimal Online Algorithms and Fast Approximation Algorithms for Resource Allocation Problems". In: J. ACM 66.1 (Jan. 2019).
- [21] Wedad Elmaghraby and Pinar Keskinocak. "Dynamic Pricing in the Presence of Inventory Considerations: Research Overview, Current Practices, and Future Directions". In: *Management Science* 49.10 (2003), pp. 1287–1309.
- [22] Moran Feldman, Ola Svensson, and Rico Zenklusen. "A Simple O(log log(rank))-Competitive Algorithm for the Matroid Secretary Problem". In: *Mathematics of Operations Research* 43.2 (2018), pp. 638–650.
- [23] Thomas S. Ferguson. "Who Solved the Secretary Problem?" In: Statistical Science 4.3 (1989), pp. 282–289.
- [24] John P. Gilbert and Frederick Mosteller. "Recognizing the Maximum of a Sequence". In: Journal of the American Statistical Association 61.313 (1966), pp. 35–73.
- [25] Yanjun Han, Jiantao Jiao, and Tsachy Weissman. "Minimax estimation of discrete distributions". In: 2015 IEEE International Symposium on Information Theory (ISIT). 2015, pp. 2291–2295.
- [26] T. P. Hill and D. P. Kennedy. "Minimax-Optimal Strategies for the Best-Choice Problem When a Bound is Known for the Expected Number of Objects". In: SIAM Journal on Control and Optimization 32.4 (1994), pp. 937–951.
- [27] Theodore P. Hill and Robert P. Kertz. "Stop Rule Inequalities for Uniformly Bounded Sequences of Random Variables". In: Transactions of the American Mathematical Society 278.1 (1983), pp. 197–207.
- [28] Theodore P. Hill and Ulrich Krengel. "Minimax-Optimal Stop Rules and Distributions in Secretary Problems". In: *The Annals of Probability* 19.1 (1991), pp. 342–353.
- [29] Robert P Kertz. "Stop rule and supremum expectations of i.i.d. random variables: A complete comparison by conjugate duality". In: *Journal of Multivariate Analysis* 19.1 (1986), pp. 88–112.
- [30] Thomas Kesselheim et al. "Primal Beats Dual on Online Packing LPs in the Random-Order Model". In: Proceedings of the Forty-Sixth Annual ACM Symposium on Theory of Computing. STOC '14. New York, New York: Association for Computing Machinery, 2014, pp. 303–312.
- [31] Robert Kleinberg. "A Multiple-Choice Secretary Algorithm with Applications to Online Auctions". In: Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms. SODA '05. Vancouver, British Columbia: Society for Industrial and Applied Mathematics, 2005, pp. 630–631.
- [32] Oded Lachish. "O(log log Rank) Competitive Ratio for the Matroid Secretary Problem". In: 2014 IEEE 55th Annual Symposium on Foundations of Computer Science. 2014, pp. 326–335.
- [33] Jeffrey I. McGill and Garrett J. van Ryzin. "Revenue Management: Research Overview and Prospects". In: Transportation Science 33.2 (1999), pp. 233–256.
- [34] Aranyak Mehta et al. "AdWords and Generalized Online Matching". In: J. ACM 54.5 (Oct. 2007), 22–es.
- [35] Marco Molinaro and R. Ravi. "Geometry of Online Packing Linear Programs". In: Automata, Languages, and Programming. Ed. by Artur Czumaj et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 701–713.
- [36] Shayan Oveis Gharan and Jan Vondrák. "On Variants of the Matroid Secretary Problem". In: Algorithms – ESA 2011. Ed. by Camil Demetrescu and Magnús M. Halldórsson. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 335–346.

- [37] Christos Papadimitriou et al. "Online Stochastic Max-Weight Bipartite Matching: Beyond Prophet Inequalities". In: Proceedings of the 22nd ACM Conference on Economics and Computation. EC '21. Budapest, Hungary: Association for Computing Machinery, 2021, pp. 763–764.
- [38] E. L. Presman and I. M. Sonin. "The Best Choice Problem for a Random Number of Objects". In: Theory of Probability & Its Applications 17.4 (1973), pp. 657–668.
- [39] Aviad Rubinstein, Jack Z. Wang, and S. Matthew Weinberg. "Optimal Single-Choice Prophet Inequalities from Samples". In: 11th Innovations in Theoretical Computer Science Conference (ITCS 2020).
 Ed. by Thomas Vidick. Vol. 151. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2020, 60:1–60:10.
- [40] Igal Sason. On Reverse Pinsker Inequalities. 2015.
- [41] Martin J. Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019.

Α Additional comments

A.1Notations

We use the following common mathematical notations:

- $|\cdot|$ and $|\cdot|$ denote the floor function and the ceiling function, i.e. the largest integer no greater than and the smallest integer no less than the variable;
- $\mathbf{1}[\mathcal{E}]$ denotes the indicator function of the event \mathcal{E} which is 1 when \mathcal{E} happens and 0 otherwise;

- $[n] = \{1, 2, ..., n\}$ for each $n \in \mathbb{N}$; $H_n := \sum_{i=1}^n 1/i$ is the *n*-th Harmonic number; $\Delta := \{\mathbf{x} = (x_1, x_2, ...) | \sum_{i=1}^\infty x_i = 1, x_i \ge 0 \ \forall i \in \mathbb{N}\}$ and $\Delta^{(n)} = \{\mathbf{x} \in \Delta | x_i = 0, \forall i \ge n+1\}$ represents distributions supported on \mathbb{N} and [n] respectively;
- for $n \in \mathbb{N}$, $\boldsymbol{\delta}^{(n)} \in \Delta^{(n)}$ is the delta distribution supported at n, i.e. $\delta_i^{(n)} = \mathbf{1}[i=n];$
- $X \sim \mathbf{p}$ means X is a random variable sampled from distribution \mathbf{p} ;
- all the $\log(\cdot)$ have base e, i.e. $\log(e) = 1$.

For quick reference, for $n \in \mathbb{N}$, the distribution $\mathbf{p}_*^{(n)} \in \Delta^{(n)}$ is defined in Equation (1), and the functions $A: \Delta \times S \to [0,1], U_{n-1}: S \to [0,1], \text{ and } \lambda_n: \Delta \to [0,1] \text{ are defined in Lemma 2.2.}$

A.2Why the problem becomes harder?

In this section, we provide some intuition on how the randomness in the time horizon increases the uncertainty in the reward of the decisions, thereby making the problem harder.

We start with the following optimal stopping problem: there is a sequence of random variables $\{X_i\}_{i \in [n]}$ defined on the same probability space. $\{\mathcal{F}_i\}_{i\in[n]}$ is a filtration on this probability space. (Different from the classical optimal stopping problem, we don't assume that X_j is measurable with respect to \mathcal{F}_i for $j \leq i \leq n$.)

For convenience, let " $\tau \in \{\mathcal{F}_i\}_{i \in [n]}$ " denote a stopping time τ which is adapted to $\{\mathcal{F}_i\}_{i \in [n]}$ and satisfies $\tau < n$ almost surely. In addition, we define

$$V_{\mathcal{F}}^* := \sup_{\tau \in \{\mathcal{F}_i\}_{i \in [n]}} \mathbb{E}[X_{\tau}].$$

By a simple conditioning argument, we can reduce the problem to the classical setting:

$$\mathbb{E}[X_{\tau}] = \mathbb{E}[\sum_{i=1}^{n} X_{i} \mathbf{1}[\tau = i]] = \sum_{i=1}^{n} \mathbb{E}[X_{i} \mathbf{1}[\tau = i]] = \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[X_{i} \mathbf{1}[\tau = i]|\mathcal{F}_{i}]]$$
$$= \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[X_{i}|\mathcal{F}_{i}] \cdot \mathbf{1}[\tau = i]] = \mathbb{E}[\sum_{i=1}^{n} \mathbb{E}[X_{i}|\mathcal{F}_{i}] \cdot \mathbf{1}[\tau = i]] = \mathbb{E}[\tilde{X}_{\tau}]$$

where $\tilde{X}_i := \mathbb{E}[X_i | \mathcal{F}_i]$. Thus, under mild conditions, classical results about optimal stopping (e.g. in [13]) applied to the sequence $\{X_i\}_{i \in [n]}$ gives that

$$V_{\mathcal{F}}^{(n)} = \tilde{X}_n, \quad V_{\mathcal{F}}^{(i)} = \max(\tilde{X}_i, \mathbb{E}[V_{\mathcal{F}}^{(i+1)} | \mathcal{F}_i]) \ i \in [n-1], \quad V_{\mathcal{F}}^* = \mathbb{E}[V_{\mathcal{F}}^{(1)}].$$

Now let $\{\mathcal{G}_i\}_{i\in[n]}$ be another filtration such that $\mathcal{G}_i \subset \mathcal{F}_i$ for all $i \in [n]$. That is, \mathcal{G}_i contains no more information than \mathcal{F}_i for all *i*. Similarly, let $\tilde{Y}_i := \mathbb{E}[X_i | \mathcal{G}_i]$, and

$$V_{\mathcal{G}}^{(n)} = \tilde{Y}_n, \quad V_{\mathcal{G}}^{(i)} = \max(\tilde{Y}_i, \mathbb{E}[V_{\mathcal{G}}^{(i+1)}|\mathcal{G}_i]) \ i \in [n-1], \quad V_{\mathcal{G}}^* = \mathbb{E}[V_{\mathcal{G}}^{(1)}].$$

Intuitively, the performance of a stopping time should not be hurt by having more information. The lemma below formalizes this idea.

Lemma A.1. $V_G^{(i)} \leq \mathbb{E}[V_F^{(i)}|\mathcal{G}_i]$ for all $i \in [n]$. Thus, $V_G^* \leq V_F^*$.

Proof. We prove it by induction. For i = n, since $\mathcal{G}_n \subset \mathcal{F}_n$,

$$\mathbb{E}[V_{\mathcal{F}}^{(n)}|\mathcal{G}_n] = \mathbb{E}[\mathbb{E}[X_n|\mathcal{F}_n]|\mathcal{G}_n] = \mathbb{E}[X_n|\mathcal{G}_n] = V_{\mathcal{G}}^{(n)}.$$

Now suppose the statement is true for i + 1, then for i,

$$\begin{split} \mathbb{E}[V_{\mathcal{F}}^{(i)}|\mathcal{G}_i] &= \mathbb{E}[\max(\tilde{X}_i, \mathbb{E}[V_{\mathcal{F}}^{(i+1)}|\mathcal{F}_i])|\mathcal{G}_i]\\ &\geq \max(\mathbb{E}[\tilde{X}_i|\mathcal{G}_i], \mathbb{E}[\mathbb{E}[V_{\mathcal{F}}^{(i+1)}|\mathcal{F}_i]|\mathcal{G}_i])\\ &= \max(\tilde{Y}_i, \mathbb{E}[V_{\mathcal{F}}^{(i+1)}|\mathcal{G}_i])\\ &= \max(\tilde{Y}_i, \mathbb{E}[\mathbb{E}[V_{\mathcal{F}}^{(i+1)}|\mathcal{G}_{i+1}]|\mathcal{G}_i])\\ &\geq \max(\tilde{Y}_i, \mathbb{E}[V_{\mathcal{G}}^{(i+1)}|\mathcal{G}_i]) = V_{\mathcal{G}}^{(i)}. \end{split}$$

This finishes the induction. The second statement is by noticing

$$V_{\mathcal{F}}^* = \mathbb{E}[V_{\mathcal{F}}^{(1)}] = \mathbb{E}[\mathbb{E}[V_{\mathcal{F}}^{(1)}|\mathcal{G}_1]] \ge \mathbb{E}[V_{\mathcal{G}}^{(1)}] = V_{\mathcal{G}}^*.$$

The key step in the proof above is the inequality

$$\mathbb{E}[\max(\tilde{X}_i, \mathbb{E}[V_{\mathcal{F}}^{(i+1)} | \mathcal{F}_i]) | \mathcal{G}_i] \ge \max(\mathbb{E}[\tilde{X}_i | \mathcal{G}_i], \mathbb{E}[\mathbb{E}[V_{\mathcal{F}}^{(i+1)} | \mathcal{F}_i] | \mathcal{G}_i])$$

which captures the idea that making decision based on $\{\mathcal{F}_i\}_{i\in[n]}$ (taking maximum over \mathcal{F}_i -measurable random variables, then taking expectation over $\mathcal{G}_i \subset \mathcal{F}_i$) gives higher return than making decision based on $\{\mathcal{G}_i\}_{i\in[n]}$ only (averaging the variable over $\mathcal{G}_i \subset \mathcal{F}_i$ first, then taking maximum over \mathcal{G}_i -measurable random variables).

Going back to the problem when the time horizon is a random variable, assume that there is an underlying process $\{Z_i\}_{i\in[n]}$ which the algorithm observes, a random time horizon $N \leq n$, and the reward $X_i = X_i(Z_i, N)$ of picking the *i*-th number, which depends on Z_i and possibly also on N. The information we have at time *i* is $\mathcal{G}_i = \sigma(Z_1, \ldots, Z_i, \mathbf{1}[N \geq 1], \ldots, \mathbf{1}[N \geq i])$ the σ -algebra generated by the observations Z_j and $\mathbf{1}[N \geq j]$ for $j \leq i$.

As a comparison, define $\mathcal{F}_i = \sigma(N, Z_1, \ldots, Z_i)$, which consists of all information about N as well as the observations up till time *i*, thus $\mathcal{G}_i \subset \mathcal{F}_i$. Under this setting, an algorithm which doesn't know (the realization of) the time horizon needs to choose $\tau \in {\mathcal{G}_i}_{i \in [n]}$, while an algorithm that knows (the realization of) the time horizon picks $\tau \in {\mathcal{F}_i}_{i \in [n]}$. Then by Lemma A.1, the algorithm that knows N can achieve better return.

More concretely, in the setting of the secretary problem, $Z_i \sim Bernoulli(\frac{1}{i})$ denotes whether the *i*-th number is the best among the first *i* numbers, and

$$X_i(z_i,k) = \mathbb{P}[i\text{-th number is the best among all } N \text{ numbers}|Z_i = z_i, N = k] = \begin{cases} \frac{i}{k} & z_i = 1, \ i \le k \\ 0 & o.w. \end{cases}$$

For the prophet inequality problem with the sequence of random variable $\{Z_i\}_{i \in [n]}$, let $X_i(z_i, k) = \frac{z_i \mathbf{1}[i \leq k]}{\mathbb{E}[\max_{i \in [k]} Z_i]}$, then the metric of interest becomes

$$\mathbb{E}\left[\frac{\mathbb{E}[Z_{\tau}\mathbf{1}[\tau \leq N]|N]}{\mathbb{E}[\max_{i \in [N]} Z_i|N]}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{Z_{\tau}\mathbf{1}[\tau \leq N]}{\mathbb{E}[\max_{i \in [N]} Z_i|N]}|N]\right] = \mathbb{E}[X_{\tau}]$$

where $\tau \in {\mathcal{G}_i}_{i \in [n]}$.

A.3 Adversarially picked N

Assume that $N \leq n$ has an upper bound, and N is picked by an adversary. At time t, the adversary is given the relative ranking of the t-th arriving number together with the decision maker's decision at time t - 1, and needs to decide if N = t.

Consider the following strategy for the adversary: for $t = 1, 2, ..., \lfloor \sqrt{n} \rfloor$, the adversary chooses N > t. At time $t = \lfloor \sqrt{n} \rfloor + 1$, if the decision maker has picked a number (among the first $\lfloor \sqrt{n} \rfloor$ arrivals), then the adversary chooses N = n, otherwise the adversary chooses $N = \lfloor \sqrt{n} \rfloor + 1$.

Lemma A.2. Under the above strategy of the adversary, for any decision maker, the probability that he picks the largest number among the N numbers that arrive is no more than $\frac{1}{\sqrt{n}}$.

Proof. We consider the two cases in the strategy. If the decision maker picks some number among the first $|\sqrt{n}|$ arrivals, then the probability that he picks the largest one among N = n numbers is upper bounded by the probability that the largest one of n numbers is among the first $|\sqrt{n}|$ arrivals, which due to the random permutation has probability upper bounded by $\frac{|\sqrt{n}|}{n} \leq \frac{1}{\sqrt{n}}$.

If no number is picked among the first $|\sqrt{n}|$ arrivals, if the decision maker does not pick the $|\sqrt{n}| + 1$ -th number, he automatically fails, and if he picks, due to the random permutation, the probability that the $\lfloor \sqrt{n} \rfloor$ + 1-th number is the largest one among the first $\lfloor \sqrt{n} \rfloor$ + 1 arrivals is $\frac{1}{\lfloor \sqrt{n} \rfloor + 1} \leq \frac{1}{\sqrt{n}}$. In both cases, the decision maker cannot succeed with probability more than $\frac{1}{\sqrt{n}}$.

Β Details for Section 4

Matroid Secretary **B.1**

Since the matroid secretary problem was first proposed in [5], the focus has been on improving the competitive ratio: [11] improves the ratio to $\Omega(\sqrt{\log(k)^{-1}})$ and [32, 22] to $\Omega(\log(\log(k))^{-1})$. Nevertheless, all algorithms proposed in these papers assume that n, the cardinality of the ground set, is known. Related to our problem is the variant of matroid secretary problem studied in [36], where no information about the size of the ground set of the matroid is available at all. The authors propose, for any $\epsilon > 0$, an algorithm which achieves a competitive ratio $\Omega(\frac{\epsilon}{\log(\log(k))\log^{1+\epsilon}(n)})$. This algorithm randomly samples an n' according to a distribution with polynomial tail, and uses any matroid secretary algorithm for n' as a black box. In addition, in this setting, they present an instance of a rank-1 matroid and a distribution of weight such that when the weights are sampled iid according to this distribution, no algorithm can achieve competitive ratio better than $O(\frac{\log(\log(n))}{\log(n)})$. The algorithm we propose also samples n' and then use existing algorithms as black boxes, but we sample n' with a different distribution to take into account the upper and lower bound information about N.

Next, we describe the setup for the classical and the random time horizon variant of the matroid secretary problem. In the classical setting introduced in [5], there is a rank-k weighted matroid $\mathcal{M} = (E, \mathcal{I})$ with ground set $E = \{e_1, \ldots, e_n\}$, a family of independent sets \mathcal{I} , and a weight function $w : E \to \mathbb{R}_{\geq 0}$ that is injective (i.e. there is no tie for the weights). The elements and their weights are revealed in an order chosen uniformly at random, and an independence oracle which gives answers to if any subset of elements revealed so far is independent is provided. The algorithm needs to decide irrevocably whether to accept the current element before the next element arrives, with the goal to select an independent set of maximum weight, where the weight of any set $S \subset E$ is $w(S) := \sum_{e \in S} w(e)$. The performance is measured in terms of the competitive ratio, i.e. $\mathbb{E}\left[\frac{w(B^{(\mathcal{A})})}{w(\mathsf{OPT}(\mathcal{M},w))}\right]$ where $\mathsf{OPT}(\mathcal{M},w) = \arg\max_{S \in \mathcal{I}} w(S)$ and $B^{(\mathcal{A})}$ is the set chosen by the algorithm \mathcal{A} . The best known algorithm for the matroid secretary problem achieves a competitive ratio of $c_k = \Omega(\log(\log(k))^{-1})$ [32, 22] but requires the size of the ground set E as an input. Let's denote this algorithm as $\mathcal{A}^{(s)}$ when s = |E|.

In this work, we consider the variant where the above process stops after revealing the first N < |E|elements (and their weights), where N is a random variable independent of the random permutation. Denoting $B_N^{(\mathcal{A})}$ as the set chosen by the algorithm \mathcal{A} by time N, and $\mathcal{M}_N = (E_N, \mathcal{I}_N)$ the matroid restricted to the first N elements revealed, the performance we are interested in is a modified competitive ratio where we compare $w(B_N^{(\mathcal{A})})$ with the offline-optimal (i.e. max-weight basis) for the matroid \mathcal{M}_N :

$$\mathbb{E}[\frac{w(B_N^{(\mathcal{A})})}{w(\mathsf{OPT}(\mathcal{M}_N, w))}].$$

Thus, when \mathcal{A} is independently randomly chosen in $\{\mathcal{A}^{(s)}\}_{s=1}^{\infty}$, with $\mathbb{P}[\mathcal{A} = \mathcal{A}^{(s)}] = x_s$, we have

$$\mathbb{E}\left[\frac{w(B_{N}^{(\mathcal{A})})}{w(\mathsf{OPT}(\mathcal{M}_{N},w))}\right] = \sum_{n=1}^{|E|} \sum_{s=1}^{\infty} x_{s} p_{n} \cdot \mathbb{E}\left[\frac{w(B_{n}^{(\mathcal{A}_{s})})}{w(\mathsf{OPT}(\mathcal{M}_{n},w))}\right]$$
$$\geq \sum_{n=1}^{|E|} \sum_{s=1}^{n} x_{s} p_{n} \cdot \mathbb{E}\left[\mathbb{E}\left[\frac{w(B_{n}^{(\mathcal{A}_{s})})}{w(\mathsf{OPT}(\mathcal{M}_{n},w))}|\mathcal{M}_{n}\right]\right]$$
$$\geq c_{k} \sum_{n=1}^{|E|} \sum_{s=1}^{n} x_{s} p_{n} \cdot \mathbb{E}\left[\mathbb{E}\left[\frac{w(\mathsf{OPT}(\mathcal{M}_{s},w))}{w(\mathsf{OPT}(\mathcal{M}_{n},w))}|\mathcal{M}_{n}\right]\right]$$
$$\geq c_{k} \sum_{n=1}^{|E|} \sum_{s=1}^{n} x_{s} p_{n} \cdot \frac{s}{n}.$$

where the last inequality is because for any $e \in \mathsf{OPT}(\mathcal{M}_n, w), e \in \mathcal{M}_s$ with probability $\frac{s}{n}$, and $w(\mathsf{OPT}(\mathcal{M}_s, w)) \ge w(\mathsf{OPT}(\mathcal{M}_n, w) \cap E_s)$. Thus, the corresponding $M^{(mat)}(s, n) \ge c_k \frac{s}{n} \mathbf{1}[s \le n]$

By Theorem 4.1, if $\underline{n} \leq N \leq \overline{n}$ almost surely, then the sampling algorithm described above is at least $\frac{c_k}{1+\log(\overline{n}/n)} = \Omega(\log(\overline{n}/\underline{n})^{-1}\log(\log(k))^{-1})$ competitive.

B.2 IID prophet inequality

In the iid prophet inequality problem, there are n numbers X_1, \ldots, X_n drawn independently according to some known distribution F supported on $[0, \infty)$. A decision maker wants to pick one number as large as possible out of these n numbers, but only sees these numbers sequentially, and upon seeing X_t , must decide irrevocably whether to accept X_t or not. Let X_{τ} be the number being picked. Then the performance is measured in terms of $\frac{\mathbb{E}[X_{\tau}]}{\mathbb{E}[\max_{i \in [n]} X_i]}$.

[15] proposes a threshold rule algorithm, where the thresholds depend only on the distribution F and the number of numbers n. The algorithm achieves a $\frac{\mathbb{E}[X_{\tau}]}{\mathbb{E}[\max_{i \in [n]} X_i]} > 0.745 = c$, which matches the known upper bound [14, 29]. We denote this threshold rule algorithm for problems with n numbers as $\mathcal{A}^{(n)}$.

In the random time horizon setting, we take N to be a random variable with distribution **p**, independent of X_i 's. The performance is measured in terms of $\mathbb{E}\left[\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \leq N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}\right]$. Notice that for $\mathcal{A}^{(s)}$, we have $\tau \leq s$, and so with an algorithm that uses $\mathcal{A}^{(s)}$ with probability x_s , we have

$$\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \leq N]|N=k]}{\mathbb{E}[\max_{i \in [N]} X_i|N=k]} \geq c \cdot \sum_{s=1}^k x_s \frac{\mathbb{E}[\max_{i \in [s]} X_i]}{\mathbb{E}[\max_{i \in [k]} X_i]}.$$

Thus taking the expectation of the above equation with respect to N, we can take $M^{(prophet)}(s,k) \geq c \cdot \frac{\mathbb{E}[\max_{i \in [s]} X_i]}{\mathbb{E}[\max_{i \in [k]} X_i]} \mathbf{1}[s \leq k].$

Now for X_i supported on [x, 1] where $x \in (0, 1]$ and any random horizon bounded by $N \leq \overline{n}$, by Theorem 4.1, the sampling based algorithm guarantees an expected performance

$$\mathbb{E}\left[\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \le N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}\right] \ge \frac{c}{1 + \log(\mathbb{E}[\max_{i \in [\overline{n}]} X_i)]/\mathbb{E}[X_1])} \ge \frac{c}{1 + \log(1/x)}$$

We provide the following upper bound for any stopping time τ adapted to the process $\{X_i\}$, thereby showing that the sampling based method is in fact optimal up to a factor of c = 0.745.

Theorem B.1. For any $x \in (0,1]$, there exists a family of pairs of distributions $\{(G_{x,n}, P_n)\}_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$, $G_{x,n}$ is supported on [x,1] and P_n is supported on $\{1, 2, \ldots, k_n\}$ for some $k_n \in \mathbb{N}$. Then for $X_i \sim_{iid} G_{x,n}$ and independently $N \sim P_n$, defining $\mathcal{M}_n := \{\tau' | \tau' \text{ a stopping time adapted to } \{X_i\}_{i \in [k_n]}\}$

$$\limsup_{n \to \infty} \sup_{\tau \in \mathcal{M}_n} \mathbb{E}_{N \sim P_n, X_i \sim G_{x,n}} \left[\frac{\mathbb{E}[X_{\tau} \mathbf{1}[\tau \le N] | N]}{\mathbb{E}[\max_{i \in [N]} X_i | N]} \right] \le \frac{1}{1 + \log(1/x)}$$

The proof of Theorem B.1 is provided in Appendix E.

B.3 Online Packing LP

For the online (packing) linear programming problem, the early lines of research (e.g. [10, 9]) usually don't assume that the time horizon is known (or even finite), and the arrival order can be arbitrary. However, it has been shown that the optimal competitive ratio is $O(\log(m\frac{a_{\max}}{a_{\min}})^{-1})$ where m is the number of constraints, and a_{\max}, a_{\min} are the upper and lower(non-zero) bound on the constraint coefficients. As a result, the competitive ratio is usually bounded away from 1. In contrast, recent lines of research have been focusing on achieving high (close to 1) competitive ratio under the random permutation, large budget-to-bid ratio model [2, 30, 35]. However, the proposed algorithms all require the time horizon, or at least an approximation of the time horizon within $1 \pm \epsilon$ factor [2], as part of the input in order to achieve $1 - O(\epsilon)$ competitiveness for small ϵ . The sampling based algorithm we propose in Section 4, when given such approximation, also achieves similar performance. Recently, [7] studies the online LP problem with known upper bound \overline{n} and lower bound \underline{n} on the time horizon. The dual mirror descent method they propose achieves a competitive ratio of $1/(1 + \log(\overline{n}/\underline{n}))$ against the worst case horizon $n \in \{\underline{n}, \ldots, \overline{n}\}$, which is optimal up to a $\log \log(\overline{n}/\underline{n})$ factor.

Below, we first state the setup for the classical online packing LP:

$$\max_{x_1,\dots,x_n} \sum_{t=1}^n \pi_t x_t \quad s.t. \ \sum_{t=1}^n \mathbf{a}_t x_t \le \mathbf{b}, \ x_t \in [0,1]$$
(3)

where $\mathbf{a}_t, \mathbf{b} \in \mathbb{R}_{\geq 0}^m$, and $\pi_t \geq 0$ for all t. The algorithm knows the budget \mathbf{b} in the beginning, but not the coefficients $(\pi_1, \mathbf{a}_1), \ldots, (\pi_n, \mathbf{a}_n)$. Instead, a permutation $\sigma : [n] \to [n]$ is chosen uniformly at random, and $(\pi_{\sigma(t)}, \mathbf{a}_{\sigma(t)})$ is revealed at time t. The algorithm needs to decide irrevocably $x_{\sigma(t)}$ before time t + 1 while satisfying the constraints. The performance is measured using competitive ratio defined as $\mathbb{E}[\frac{\sum_{t=1}^n \pi_t \hat{x}_t}{\mathsf{OPT}}]$ where OPT is the optimal value to the problem (3), and \hat{x}_t is the algorithm's decision for x_t . [35] has proposed a dual-resolving algorithm for $\epsilon \in (0, 1/100)$ which, under the assumption that $\max_{i \in [m], t \in [n]} \frac{b_i}{a_{t,i}} = \Omega(\frac{m^2}{\epsilon^2} \log(\frac{m}{\epsilon}))$, achieves the competitive ratio $1 - 50\epsilon$ when n is known, and we denote this algorithm as $\mathcal{A}^{(n)}$.

Now consider the variant where the time horizon is $N \leq n$ a random variable, and for $n' \leq n, \sigma : [n] \rightarrow [n]$, $\mathsf{OPT}(\sigma, n')$ denotes the solution to the following problem:

$$\max_{x_{\sigma(1)},\dots,x_{\sigma(n')}} \sum_{t=1}^{n'} \pi_{\sigma(t)} x_{\sigma(t)} \quad s.t. \ \sum_{t=1}^{n'} \mathbf{a}_{\sigma(t)} x_{\sigma(t)} \le \mathbf{b}, \ x_{\sigma(t)} \in [0,1].$$

Then the performance measure of interest is $\mathbb{E}\left[\frac{\sum_{t=1}^{N} \pi_{\sigma}(t) \hat{x}_{\sigma(t)}}{\mathsf{OPT}(\sigma,N)}\right]$. And similar to the argument for the matroid secretary problem, we can take $M^{(LP)}(s,n) \geq (1-50\epsilon)\frac{s}{n}\mathbf{1}[s \leq n]$, leading to a performance of $\mathbb{E}\left[\frac{\sum_{t=1}^{N} \pi_{\sigma}(t) \hat{x}_{\sigma(t)}}{\mathsf{OPT}(\sigma,N)}\right] \geq (1-50\epsilon)/(1+\log(\overline{n}/\underline{n}))$, matching the result in [7].

C Proof for results in Section 2

Proof of Lemma 2.1. Let $\mathbf{p} \in \Delta$ denote the distribution for N. For any algorithm $\mathsf{Str} \in \overline{\mathcal{S}}$, we can consider the following $\mathbf{q} \in \mathcal{S}$:

 $q_i = \mathbb{P}[\text{Str picks } x_{\sigma(i)} | N \ge i, R_i = 1, x_{\sigma(j)} \text{ not picked for } 1 \le j \le i-1].$

For Str, for any $1 \le i \le l$, it's easy to show by induction (on *i*) that

$$\mathbb{P}[x_{\sigma(j)} \text{ not picked for } 1 \leq j \leq i | N \geq l] \leq U_i(\mathbf{q}).$$

Thus,

$$\mathbb{P}[\text{Str picks } x_{\sigma(i)} | N \ge i, R_i = 1] \le q_i U_{i-1}(\mathbf{q}).$$

And so

 Sin

$$\begin{split} \mathbb{P}[\text{Str picks } x_{\sigma(i)}, \ x_{\sigma(i)} &= \max_{1 \le j \le N} x_{\sigma(j)} | N \ge i, \ R_i = 1] \\ &= \mathbb{P}[\text{Str picks } x_{\sigma(i)} | N \ge i, \ R_i = 1] \cdot \mathbb{P}[\arg\max_{1 \le j \le N} x_{\sigma(j)} \le i | N \ge i, \ R_i = 1] \\ &\leq q_i U_{i-1}(\mathbf{q}) \cdot \sum_{j=i}^{n'} (\frac{i}{j} \cdot \frac{p_j}{\sum_{j'=i}^{n'} p_{j'}}) = \frac{i}{\sum_{j'=i}^{\infty} p_{j'}} \cdot q_i U_{i-1}(\mathbf{q}) \lambda_i(\mathbf{p}). \\ & \text{ce } \mathbb{P}[N \ge i, R_i = 1] = \mathbb{P}[R_i = 1 | N \ge i] \cdot \mathbb{P}[N \ge i] = \frac{\sum_{j'=i}^{\infty} p_{j'}}{i}, \\ \mathbb{P}[\text{Success}(\text{Str})] &= \sum_{i=1}^{\infty} \mathbb{P}[N \ge i, \ \text{Str picks } x_{\sigma(i)}, \ x_{\sigma(i)} = \max_{1 \le j \le N} x_{\sigma(j)}] \le \sum_{i=1}^{\infty} q_i U_{i-1}(\mathbf{q}) \lambda_i(\mathbf{p}) = A(\mathbf{p}, \mathbf{q}). \\ & \Box \end{split}$$

D Proof for results in Section 3

D.1 Full information

Proof of Theorem 3.1. For convenience, denote $\theta(\mathbf{p}) = \theta$ and $\lambda_i(\mathbf{p}) = \lambda_i$ for $i \in \mathbb{N}$. First, notice that $0 \leq i\lambda_i \leq \sum_{j=i}^{\infty} p_j \to 0$ as $i \to \infty$, thus by Lemma F.3, there exists an index $K^* \in \mathbb{N}$ such that

$$\theta = \max_{i \in \mathbb{N}} i\lambda_i = K^* \lambda_{K^*}.$$

Due to Lemma 2.1 and 2.2,

$$\sup_{\mathbf{q}\in\mathcal{S}} A(\mathbf{p},\mathbf{q}) = \sup_{\mathsf{Str}\in\overline{\mathcal{S}}} \mathbb{P}[\mathsf{Success}(\mathsf{Str})]$$

For the upper bound, since $k\lambda_k \leq \theta$ for all k

$$A(\mathbf{p}, \mathbf{q}) = \sum_{k=1}^{\infty} (\prod_{l=1}^{k-1} (1 - q_l/l)) (q_k/k) \cdot k\lambda_k \le \theta \cdot \sum_{k=1}^{\infty} (\prod_{l=1}^{k-1} (1 - q_l/l)) (q_k/k).$$

Since for any finite $K \in \mathbb{N}$

$$\sum_{k=1}^{K} \left(\prod_{l=1}^{k-1} (1-q_l/l)\right) (q_k/k) = 1 - \prod_{l=1}^{K} (1-q_l/l) \le 1.$$

Taking $K \to \infty$, we get

$$A(\mathbf{p},\mathbf{q}) \le \theta \cdot \lim_{K \to \infty} \sum_{k=1}^{K} (\prod_{l=1}^{k-1} (1-q_l/l))(q_k/k) \le \theta.$$

For the lower bound, the idea is to compare the sequence of λ_k with the sequence $\lambda_k^{(T)} = \frac{1}{T} \mathbf{1}[k \leq T]$ generated by a delta distribution at T, i.e. $p_i = \mathbf{1}[i = T]$, since we know that for the delta distribution, the problem is just the classical secretary problem with known time horizon, and achieves a probability of success $\approx 1/e$.

We first truncate the sum at $K\in\mathbb{N}$

$$A(\mathbf{p}, \mathbf{q}) = \sum_{k=1}^{\infty} (\prod_{l=1}^{k-1} (1 - q_l/l)) q_k \lambda_k \ge \sum_{k=1}^{K} (\prod_{l=1}^{k-1} (1 - q_l/l)) q_k \lambda_k.$$

Since λ_k is a nonincreasing sequence, we have $\lambda_k \geq \lambda_K$ for all $k \leq K$, and so

$$A(\mathbf{p},\mathbf{q}) \ge K\lambda_K \cdot \sum_{k=1}^K (\prod_{l=1}^{k-1} (1-q_l/l))q_k \cdot \frac{1}{K}.$$

For K = 1, taking $q_1 = 1$, we have $\sup_{\mathbf{q}\in\mathcal{S}} A(\mathbf{p},\mathbf{q}) \ge \lambda_1$. For K = 2, taking $q_1 = 0, q_2 = 1$, we have $\sup_{\mathbf{q}\in\mathcal{S}} A(\mathbf{p},\mathbf{q}) \ge 2\lambda_2 \cdot \frac{1}{2} \ge 2\lambda_2 \cdot \frac{1}{e}$. For $K \ge 3$, let $s_K^* \in \mathbb{N}$ be such that $\sum_{i=s_K^*}^{K-1} \frac{1}{i} \le 1$ but $\sum_{i=s_K^*-1}^{K-1} \frac{1}{i} > 1$, take $q_l = \mathbf{1}[l \ge s_K^*]$ then by Lemma F.1

$$\frac{1}{K}\sum_{k=1}^{K} (\prod_{l=1}^{k-1} (1-q_l/l))q_k = \frac{1}{K}\sum_{k=s_K^*}^{K} (\prod_{l=s_K^*}^{k-1} (1-1/l)) = \frac{s_K^* - 1}{K} (\sum_{i=s_K^* - 1}^{K-1} \frac{1}{i}) \ge \frac{1}{e}$$

which implies that for all $K \in \mathbb{N}$ (and in particular for K^*),

$$\sup_{\mathbf{q}\in\mathcal{S}} A(\mathbf{p},\mathbf{q}) \ge K\lambda_K \cdot \frac{1}{e} \implies \sup_{\mathbf{q}\in\mathcal{S}} A(\mathbf{p},\mathbf{q}) \ge K^*\lambda_{K^*} \cdot \frac{1}{e} = \frac{1}{e} \cdot \theta.$$

D.2 Upper bounds

Proof of Theorem 3.2. For any $\mathbf{p} \in \Delta^{(\overline{n})}$, if $L \sim \mathbf{x} \in \Delta$, $N \sim \mathbf{p}$, since if $l \ge \overline{n} + 1$, the algorithm rejects all numbers, and can never succeed,

$$\mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)})] = \sum_{l=1}^{\infty} \sum_{i=1}^{\overline{n}} x_l p_i \cdot \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(l)}) | N = i] = \sum_{l=1}^{\overline{n}} \sum_{i=1}^{\overline{n}} x_l p_i \cdot \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(l)}) | N = i].$$

In addition, conditioning on $\{N = i\}$, the largest number is uniformly distributed in [i], and conditioning on $\{N = i, \text{ the } j\text{-th number is the largest number}\}$, an algorithm succeeds if and only if it rejects all first j - 1 numbers, and accept the j-th number. Thus

$$\mathbb{P}[\mathsf{Success}(\mathbf{q}^{(l)})|N=i] = \frac{1}{i} \sum_{j=1}^{i} \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(l)})|N=i, \text{ the } j\text{-th number is the largest number}]$$
$$= \frac{1}{i} \sum_{j=1}^{i} \prod_{k=1}^{j-1} (1 - \frac{q_k^{(l)}}{k}) q_j^{(l)} = \begin{cases} \frac{1}{i} & l=1\\ 0 & l\ge 2, \ 1\le i\le l-1 \\ \frac{l-1}{i} \sum_{j=l}^{i} \frac{1}{j-1} & l\ge 2, \ l\le i\le \overline{n} \end{cases}$$

By design, $x_l^{(\overline{n})} = \frac{1}{1+H_{\overline{n}-1}} \begin{cases} 1 & l=1\\ \frac{1}{l-1} & l=2,\ldots,\overline{n} \end{cases}$, thus for any $i \in [\overline{n}]$,

$$\sum_{l=1}^{\overline{n}} x_l \cdot \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(l)}) | N = i] = \frac{1}{1 + H_{\overline{n}-1}} \cdot (\frac{1}{i} + \sum_{l=2}^{i} \frac{1}{i} \cdot (\sum_{j=l}^{i} \frac{1}{j-1})) = \frac{1}{1 + H_{\overline{n}-1}}.$$

Thus when $N \sim \mathbf{p} \in \Delta^{(\overline{n})}, L \sim \mathbf{x}^{(\overline{n})}$,

$$\mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)})] = \sum_{l=1}^{\overline{n}} \sum_{i=1}^{\overline{n}} x_l p_i \cdot \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(l)}) | N = i] = \frac{1}{1 + H_{\overline{n}-1}} \cdot \sum_{i=1}^{\overline{n}} p_i = \frac{1}{1 + H_{\overline{n}-1}}$$

This shows the first part. For the second part, if $\mathbb{E}[N] \leq \overline{\mu}$, by Markov inequality, for any $n \in \mathbb{N}$, $\mathbb{P}[N \geq n] \leq \frac{\overline{\mu}}{n}$. Thus if $L \sim \mathbf{x}^{(n)}$,

$$\mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)})] \ge \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)}), \ N \le n] \ge \mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)})|N \le n] \cdot (1 - \frac{\overline{\mu}}{n}).$$

Conditioning on $N \leq n$, N follows a distribution supported on [n], and so by the above result,

$$\mathbb{P}[\operatorname{Success}(\mathbf{q}^{(L)})] \geq \frac{1}{1+H_{n-1}} \cdot (1-\frac{\overline{\mu}}{n}) \geq \frac{1}{2+\log(n-1)} \cdot (1-\frac{\overline{\mu}}{n}).$$

Taking $n = \lceil \overline{\mu} \log(\overline{\mu}) \rceil$,

$$\mathbb{P}[\mathsf{Success}(\mathbf{q}^{(L)})] \geq \frac{1}{2 + \log(\overline{\mu}\log(\overline{\mu}))} \cdot (1 - \frac{1}{\log(\overline{\mu})}) = \frac{1}{\log(\overline{\mu})}(1 - O(\frac{\log(\log(\overline{\mu}))}{\log(\overline{\mu})})).$$

Proof of Theorem 3.3. For convenience, denote $\mathbf{a}^{(n)} = (a_1^{(n)}, \ldots, a_n^{(n)})$ where $a_i^{(n)} = A(\boldsymbol{\delta}^{(i)}, \mathbf{q}^{(l_n^*)})$, then by linearity $A(\mathbf{p}, \mathbf{q}^{(l_n^*)}) = \langle \mathbf{p}, \mathbf{a}^{(n)} \rangle$. Denoting $\overline{a}^{(n)} = \langle \mathbf{1}/n, \mathbf{a}^{(n)} \rangle = \frac{1}{n} \sum_{i=1}^n a_i^{(n)}$ and $a^{*,(n)} = \max_{i \in [n]} a_i^{(n)}$, then by Lemma F.2, we have for any $0 \leq \epsilon \leq \overline{a}^{(n)}$, if $\mathbf{p} \sim Uniform(\Delta^{(n)})$,

$$\mathbb{P}[A(\mathbf{p}, \mathbf{q}^{(l_n^*)}) \le \epsilon] \le h_{\overline{a}^{(n)}/a^{*,(n)}} (\epsilon/\overline{a}^{(n)})^n,$$

where $h_{\alpha}: [0,1] \to \mathbb{R}$ is defined for $0 < \alpha \leq 1$, $h_{\alpha}(x) := x^{\alpha} (\frac{1/\alpha + 1}{1/\alpha + x})^{\alpha + 1}$.

By Lemma D.2, $\liminf \overline{a}^{(n)} \ge 2/e^2$, $\lim a^{*,(n)} = 1/e$. Thus for any $\epsilon < 2/e^2$, there exists $n_{\epsilon} \in \mathbb{N}$ such that for all $n \ge n_{\epsilon}$, $\overline{a}^{(n)} \ge \frac{2/e^2 + \epsilon}{2}$ and $a^{*,(n)} \le 1.001/e$, which implies that $\overline{a}^{(n)}/a^{*,(n)} \ge \frac{(2/e^2 + \epsilon)/2}{1.001/e}$ and $\epsilon/\overline{a}^{(n)} \le \frac{2\epsilon}{2/e^2 + \epsilon}$.

By Lemma D.1, h(x) is non-increasing in α , and non-decreasing in x, so

$$h_{\overline{a}^{(n)}/a^{*,(n)}}(\epsilon/\overline{a}^{(n)}) \le h_{\frac{(2/e^2 + \epsilon)/2}{1.001/e}}(\frac{2\epsilon}{2/e^2 + \epsilon}) := \rho_{\epsilon} < 1.$$

Putting the above together, we get that for any $\epsilon < 2/e^2$, there exists $n_{\epsilon} \in \mathbb{N}$ and $\rho_{\epsilon} < 1$, such that for all $n \ge n_{\epsilon}$,

$$\mathbb{P}[A(\mathbf{p}, \mathbf{q}^{(l_n^*)}) \le \epsilon] \le \rho_{\epsilon}^n.$$

Since $h_{0.72}(x) < 1/2$ for $x \le 0.12$, we can take $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we have $\overline{a}^{(n)} \ge \frac{2/e^2}{1.01}$ and $a^{*,(n)} \le \frac{2/e^2}{1.01}/0.72$ (notice that $\frac{2/e^2}{1.01}/0.72 > 1/e$), which implies that $\overline{a}^{(n)}/a^{*,(n)} \ge 0.72$. Thus for $\epsilon \le 0.03 < \frac{0.24/e^2}{1.01}$,

$$\mathbb{P}[A(\mathbf{p}, \mathbf{q}^{(l_n^*)}) \le \epsilon] = O(2^{-l}).$$

Lemma D.1. For $0 < \alpha \leq 1$, define $h_{\alpha} : [0,1] \to \mathbb{R}$, $h_{\alpha}(x) := x^{\alpha} (\frac{1/\alpha+1}{1/\alpha+x})^{\alpha+1}$. Then $h_{\alpha}(x)$ is non-increasing in α , and non-decreasing in x. In addition, for any $0 < \alpha \leq 1$, $x \in [0,1)$, $h_{\alpha}(x) < 1$.

Proof of Lemma D.1. For the first part, let function $g_{\beta}(x) : (0,1] \to \mathbb{R}$ where $0 \le \beta \le 1$ and $g_{\beta}(x) = \log(\beta)x + (x+1)\log(\frac{1+x}{1+\beta x})$, i.e. $g_{\beta}(x) = \log(h_x(\beta))$. Then

$$\frac{d}{dx}g_{\beta}(x) = \log(\beta) + \log(\frac{1+x}{1+\beta x}) + 1 - \frac{\beta(1+x)}{1+\beta x} = \log(1 - \frac{1-\beta}{1+\beta x}) + \frac{1-\beta}{1+\beta x} \le 0.$$

Thus $g_{\beta}(x)$ is non-increasing in x, and so $h_{\alpha}(x) \leq h_{\alpha'}(x)$ for all $x \in [0, 1], \alpha \geq \alpha'$.

For the second part, $h_{\alpha}(0) = 0 \le h_{\alpha}(x)$ for all $x \in [0, 1]$, and for $x \in (0, 1]$,

$$\frac{d}{dx}\log(h_{\alpha}(x)) = \frac{\alpha}{x} - \frac{\alpha+1}{1/\alpha+x} = \frac{1-x}{x(1/\alpha+x)} \ge 0.$$

For the third part, suppose there exists $\xi \in (0, 1)$ such that $h_{\alpha}(\xi) = 1 = h_{\alpha}(1)$, since h_{α} is non-decreasing in x, we have that for all $x \in [\xi, 1]$, $\log(h_{\alpha}(x)) = \log(1) = 0$, i.e. $\log(h_{\alpha}(x))$ is constant on $[\xi, \frac{\xi+1}{2}]$. However, $\frac{d}{dx}\log(h_{\alpha}(x)) > 0$ strictly on $[\xi, \frac{\xi+1}{2}]$, contradiction, so $h_{\alpha}(x) < 1$ for all $x \in (0, 1)$. And $h_{\alpha}(0) = 0 \neq 1$.

Lemma D.2. For each $n \in \mathbb{N}$, denote $l_n^* = \lceil n/e^2 \rceil$, $\mathbf{a}^{(n)} = (a_1^{(n)}, \dots, a_n^{(n)})$ where $a_i^{(n)} = A(\boldsymbol{\delta}^{(i)}, \mathbf{q}^{(l_n^*)})$, $\overline{a}^{(n)} = \frac{1}{n} \sum_{i=1}^n a_i^{(n)}$ and $a^{*,(n)} = \max_{i \in [n]} a_i^{(n)}$. Then we have $\liminf \overline{a}^{(n)} \ge 2/e^2$, $\lim a^{*,(n)} = 1/e$.

Proof of Lemma D.2. Since the theorem's statement is for the limit as $n \to \infty$, below we consider the case when $n \ge 15$, which implies that $l_n^* \ge 3$. By direct computation,

$$a_i^{(n)} = \begin{cases} 0 & 1 \le i \le l_n^* - 1\\ \frac{l_n^* - 1}{i} \sum_{j=l_n^*}^i \frac{1}{j-1} \ge \frac{l_n^* - 1}{i} \log(\frac{i}{l_n^* - 1}) & l_n^* \le i \le n \end{cases}$$

where we use the fact that for $1 \leq k_1 \leq k_2 \in \mathbb{N}$,

$$\sum_{i=k_1}^{k_2} \frac{1}{i} \ge \int_{k_1}^{k_2+1} \frac{1}{x} dx = \log(\frac{k_2+1}{k_1}).$$

Notice that $x \to \log(x)/x$ is increasing on (0, e] and decreasing on $[e, \infty)$. Since $l_n^* - 1 \le n/e^2$, we have $n/(l_n^* - 1) > e$, thus there exists $i_n^* \in \{l_n^*, \ldots, n\}$ such that $i_n^*/(l_n^* - 1) \le e < (i_n^* + 1)/(l_n^* - 1)$

$$\begin{split} \overline{a}^{(n)} &\geq \frac{1}{n} \sum_{i=l_n^n}^n \frac{l_n^* - 1}{i} \log(\frac{i}{l_n^* - 1}) \\ &\geq \frac{1}{n} \sum_{i=l_n^*}^{i_n^*} \frac{l_n^* - 1}{i} \log(\frac{i}{l_n^* - 1}) + \frac{1}{n} \sum_{i=l_n^* + 2}^n \frac{l_n^* - 1}{i} \log(\frac{i}{l_n^* - 1}) \\ &\geq \frac{l_n^* - 1}{n} \left(\int_1^{\frac{i_n^*}{l_n^* - 1}} \frac{\log(x)}{x} dx + \int_{\frac{i_n^* + 1}{n - 1}}^{\frac{n^*}{l_n^* - 1}} \frac{\log(x)}{x} dx \right) \\ &= \frac{l_n^* - 1}{2n} \cdot \left(\log^2(\frac{i_n^*}{l_n^* - 1}) + \left(\log^2(\frac{n}{l_n^* - 1}) - \log^2(\frac{i_n^* + 1}{l_n^* - 1}) \right) \right) \\ &\geq \frac{l_n^* - 1}{2n} \cdot \left(\log^2(\frac{n}{l_n^* - 1}) + \log^2(e - \frac{1}{l_n^* - 1}) - \log^2(e + \frac{1}{l_n^* - 1}) \right). \end{split}$$

where the last step uses $i_n^*/(l_n^*-1) \le e < (i_n^*+1)/(l_n^*-1)$. Thus for $l_n^* := \lceil n/e^2 \rceil$,

$$\liminf_{n \to \infty} \overline{a}^{(n)} \ge \liminf_{n \to \infty} \frac{l_n^*}{2n} \cdot \log^2(\frac{n}{l_n^*}) = 2/e^2.$$

For the second part, notice that for $2 \leq k_1 \leq k_2 \in \mathbb{N}$, we have

$$\sum_{i=k_1}^{k_2} \frac{1}{i} \le \int_{k_1-1}^{k_2} \frac{1}{x} dx = \log(\frac{k_2}{k_1-1}).$$

For $3 \leq l_n^* \leq i \leq n$, using $\log(x)/x \leq 1/e$ for $x \geq 1$, we have

$$a_i^{(n)} = \frac{l_n^* - 1}{i} \sum_{j=l_n^*}^i \frac{1}{j-1} \le \frac{l_n^* - 1}{i} \log(\frac{i-1}{l_n^* - 2}) \le \frac{l_n^* - 1}{l_n^* - 2} \cdot \frac{i-1}{i} \cdot \frac{1}{e} \le \frac{l_n^* - 1}{l_n^* - 2} \cdot \frac{1}{e}$$

Thus we have $a^{*,(n)} \leq \frac{l_n^* - 1}{l_n^* - 2} \cdot \frac{1}{e}$, and so $\limsup a^{*,(n)} \leq \frac{1}{e}$. On the other hand, for $k_n^* = \lceil n/e \rceil$, we have $l_n^* \leq k_n^* \leq n$,

$$a^{*,(n)} \ge a_{k_n^*}^{(n)} \ge \frac{l_n^* - 1}{k_n^*} \log(\frac{k_n^*}{l_n^* - 1}).$$

Since $\frac{k_n^*}{l_n^*} \to e$, we have $\liminf a^{*,(n)} \ge \frac{1}{e}$.

D.3 Samples

Lemma D.3. Let $\mathbf{p} \in \Delta$ be a fixed distribution, $\epsilon > 0$. Let $f_{\mathbf{p}} : S \to \mathbb{R}$ be a function such that $|A(\mathbf{p}, \mathbf{q}) - f_{\mathbf{p}}(\mathbf{q})| \le \epsilon$ for all $\mathbf{q} \in S$, and assume that there exists $\mathbf{q}^* \in S$ such that $f_{\mathbf{p}}(\mathbf{q}^*) = \sup_{\mathbf{q} \in S} f(\mathbf{q})$ then

$$A(\mathbf{p}, \mathbf{q}^*) \ge \sup_{\mathbf{q} \in S} A(\mathbf{p}, \mathbf{q}) - 2\epsilon$$

Proof of Lemma D.3.

$$A(\mathbf{p}, \mathbf{q}^*) \ge f_{\mathbf{p}}(\mathbf{q}^*) - \epsilon = \sup_{\mathbf{q} \in \mathcal{S}} f_{\mathbf{p}}(\mathbf{q}) - \epsilon \ge \sup_{\mathbf{q} \in \mathcal{S}} A(\mathbf{p}, \mathbf{q}) - 2\epsilon.$$

Lemma D.4. For any $\rho > 1$, we have $I(\rho, l) \leq \rho \cdot (I(\rho, l-1)+1)$ for all $l \in \mathbb{N}$, and $I(\rho, l') \geq \frac{\rho^{l'-l}}{2} \cdot I(\rho, l)$ for all $l' \geq l \geq 1$.

Proof of Lemma D.4. Recall that by definition, $I(\rho) := \{ \lceil \rho^l \rceil, l \in \{0\} \cup \mathbb{N} \}$, and has elements $I(\rho) = \{ 1 = I(\rho, 1) < I(\rho, 2) < \cdots \}$, with $I(\rho, 0) = 0$.

For l = 1, $I(\rho, l) = 1$ and $I(\rho, l-1) + 1 = 1$, the statement holds since $\rho > 1$. For any $l \in \mathbb{N}$, $l \ge 2$, let $k \in \mathbb{N}$ be the largest integer, such that $I(\rho, l-1) = \lceil \rho^k \rceil$, then

$$I(\rho, l) = \lceil \rho^{k+1} \rceil \le \rho^{k+1} + 1 \le \rho^{k+1} + \rho = \rho(\rho^k + 1) \le \rho(\lceil \rho^k \rceil + 1) = \rho \cdot (I(\rho, l-1) + 1).$$

For the second part, let $I(\rho, l) = \lceil \rho^k \rceil$ for some $k \in \mathbb{N}$, then $I(\rho, l') \ge \lceil \rho^{k+l'-l} \rceil$.

$$\frac{I(\rho,l)}{I(\rho,l')} \leq \frac{\lceil \rho^k \rceil}{\lceil \rho^{k+l'-l} \rceil} \leq \frac{\rho^k+1}{\rho^{k+l'-l}} \leq \frac{2\rho^k}{\rho^{k+l'-l}} = \frac{2}{\rho^{l'-l}}.$$

Proof of Lemma 3.1. For convenience, denote $s_j := A(\boldsymbol{\delta}^{(j)}, \mathbf{q}) = \frac{1}{j} \sum_{l=1}^{j} U_{l-1}(\mathbf{q}) q_l$ for $j \in \mathbb{N}$. For the lower bound, for any $j' \geq j$, we have

$$j's_{j'} = \sum_{l=1}^{j'} U_{l-1}(\mathbf{q})q_l \ge \sum_{l=1}^{j} U_{l-1}(\mathbf{q})q_l = js_j.$$

For the other direction,

$$s_{j'} = \frac{1}{j'} \sum_{l=1}^{j'} U_{l-1}(\mathbf{q}) q_l = \frac{j}{j'} s_j + \frac{1}{j'} \sum_{l=j+1}^{j'} U_{l-1}(\mathbf{q}) q_l \le \frac{j}{j'} s_j + \sum_{l=j+1}^{j'} U_{l-1}(\mathbf{q}) \frac{q_l}{l}.$$

Plugging in the definition $U_i(\mathbf{q}) := \prod_{l=1}^i (1 - q_l/l)$, the second term above becomes

$$\sum_{l=j+1}^{j'} U_{l-1}(\mathbf{q}) \frac{q_l}{l} = U_j(\mathbf{q}) \cdot (1 - \prod_{l=j+1}^{j'} (1 - \frac{q_l}{l})) \le 1 \cdot (1 - \frac{j}{j+1} \cdot \frac{j+1}{j+2} \cdots \frac{j'-1}{j'}) = 1 - \frac{j}{j'}$$

thus we get

$$s_{j'} \le \frac{j}{j'} s_j + (1 - \frac{j}{j'}).$$

Proof of Corollary 3.1. For convenience, denote $s_i := \frac{1}{i} \sum_{l=1}^{i} U_{l-1}(\mathbf{q}) q_l$ and $k_i := I(\rho, i)$ for $i \in \mathbb{N}$, $k_0 = 0$. With these notations,

$$A(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{\infty} p_i s_i = \sum_{l=1}^{\infty} \sum_{j=k_{l-1}+1}^{k_l} p_j s_j.$$

By Lemma D.4, $\rho(k_{l-1}+1) \ge k_l$ for all $l \in \mathbb{N}$, thus by Lemma 3.1 we get

$$A(\mathbf{p}, \mathbf{q}) \le \rho \cdot \sum_{l=1}^{\infty} \sum_{j=k_{l-1}+1}^{k_l} p_j s_{k_l} = \rho \cdot \sum_{l=1}^{\infty} s_{k_l} \cdot (\sum_{j=k_{l-1}+1}^{k_l} p_j) = \rho \cdot \sum_{l=1}^{\infty} s_{k_l} \tilde{p}_{k_l} = \rho \cdot A(\tilde{\mathbf{p}}, \mathbf{q}).$$

For the upper bound, by Lemma 3.1 again,

$$A(\mathbf{p},\mathbf{q}) \ge \sum_{l=1}^{\infty} \sum_{j=k_{l-1}+1}^{k_l} p_j(s_{k_l} - (1 - \frac{1}{\rho})) = \sum_{l=1}^{\infty} \sum_{j=k_{l-1}+1}^{k_l} p_j s_{k_l} - (1 - \frac{1}{\rho}) = A(\tilde{\mathbf{p}},\mathbf{q}) - (1 - \frac{1}{\rho}).$$

Thus we get the following bounds

$$A(\mathbf{p},\mathbf{q}) \ge A(\tilde{\mathbf{p}},\mathbf{q}) - (1 - \frac{1}{\rho}) \ge A(\tilde{\mathbf{p}},\mathbf{q}) - (\rho - 1), \quad A(\mathbf{p},\mathbf{q}) \le \rho A(\tilde{\mathbf{p}},\mathbf{q}) \le A(\tilde{\mathbf{p}},\mathbf{q}) + (\rho - 1)$$

which together implies that for all \mathbf{q} ,

$$|A(\mathbf{p}, \mathbf{q}) - A(\tilde{\mathbf{p}}, \mathbf{q})| \le \rho - 1.$$

Lemma D.5. Let $\mathbf{p} \in \Delta$ be a distribution. Let $k_1 \leq k_2 \in \mathbb{N}$ be two indices such that $p_i = 0$ for $i = k_1, \ldots, k_2 - 1$. If $|G - k_2 \lambda_{k_2}(\mathbf{p})| \leq \epsilon$ for some $G \in \mathbb{R}$, then $|\frac{i}{k_2}G - i\lambda_i(\mathbf{p})| \leq \epsilon$ for all $i = k_1, \ldots, k_2 - 1$.

Proof of Lemma D.5. Since $p_i = 0$ for $i = k_1, \ldots, k_2 - 1$, $\lambda_i(\mathbf{p}) = \lambda_{k_2}(\mathbf{p})$ for $i = k_1, \ldots, k_2 - 1$. Thus

$$\left|\frac{i}{k_2}G - i\lambda_i(\mathbf{p})\right| = \frac{i}{k_2} \cdot |G - k_2\lambda_i(\mathbf{p})| = \frac{i}{k_2} \cdot |G - k_2\lambda_{k_2}(\mathbf{p})| \le \frac{i}{k_2} \cdot \epsilon \le \epsilon.$$

Proof of Lemma 3.2. For any $\mathbf{q} \in \mathcal{S}$, notice that

$$\forall k \in \mathbb{N}, \ \sum_{i=1}^{k} (U_{i-1}(\mathbf{q})q_i/i) = 1 - \prod_{i=1}^{k} (1 - \frac{q_i}{i}) \le 1 \implies \sum_{i=1}^{\infty} (U_{i-1}(\mathbf{q})q_i/i) \le 1$$
(4)

Thus for any $\mathbf{q} \in \mathcal{S}$,

$$\begin{aligned} |A(\mathbf{p},\mathbf{q}) - \sum_{i=1}^{n} U_{i-1}(\mathbf{q}) \frac{q_i}{i} \cdot G_i| &= |\sum_{i=1}^{n} (U_{i-1}(\mathbf{q})q_i/i) \cdot (i\lambda_i(\mathbf{p}) - G_i) + \sum_{i=n+1}^{\infty} (U_{i-1}(\mathbf{q})q_i/i) \cdot i\lambda_i(\mathbf{p})| \\ &\leq \sum_{i=1}^{n} (U_{i-1}(\mathbf{q})q_i/i) \cdot |i\lambda_i(\mathbf{p}) - G_i| + \sum_{i=n+1}^{\infty} (U_{i-1}(\mathbf{q})q_i/i) \cdot |i\lambda_i(\mathbf{p})| \\ &\leq \epsilon \cdot \sum_{i=1}^{\infty} (U_{i-1}(\mathbf{q})q_i/i) \leq \epsilon \end{aligned}$$

Proof of Theorem 3.4. The proof consists of two parts. First, we show that the good event \mathcal{E}^* holds with high probability. Then, we show that under this event, the output $\hat{\mathbf{q}}$ is at least ϵ -suboptimal.

For convenience, denote $\overline{\mathbf{p}} = \mathsf{Block}(\mathbf{p}, \rho)$, and $k_i = I(\rho, i)$ for $i \in \mathbb{N}$, $k_0 = 0$, and $N_{\max} = I(\rho, l_{\max})$. Step 1: Recall that we define the following event as good (denoted as \mathcal{E}^*),

$$\sum_{i=N_{\max}+1}^{\infty} p_i \le \epsilon/12, \text{ and } |G_i - i\lambda_i(\overline{\mathbf{p}})| \le \epsilon/4 \ i = I(\rho, l), \ l \in [l_{\max}].$$

For the first part of the event,

$$\mathbb{P}\left[\sum_{i=N_{\max}+1}^{\infty} p_i > \epsilon/12\right] \le \mathbb{P}\left[\sum_{i=\max_{i\in[m]}N^{(i)}+1}^{\infty} p_i > \epsilon/12\right] \text{ since } N_{\max} \ge \max_{j\in[m]} N^{(j)}$$
$$= \mathbb{P}\left[\sum_{i=N^{(j)}+1}^{\infty} p_i > \epsilon/12, \ \forall \ j \in [m]\right]$$
$$= \mathbb{P}\left[\sum_{i=N+1}^{\infty} p_i > \epsilon/12\right]^m \text{ by assumption } N^{(j)} \sim_{iid} \mathbf{p}, \ \forall j \in [m]$$
$$\le (1 - \epsilon/12)^m \le e^{-\epsilon m/12}.$$

For the second part, we consider two cases: $i \leq \rho T$, which we apply the standard Hoeffding or Bernstein concentration bound, and $i > \rho T$ (if $N_{\text{max}} > \rho T$), which we show by bounding the tail of $\hat{\mathbf{p}}$.

For $i \leq \rho T$: first, notice that we can rewrite $\hat{\mathbf{p}}$ in the following way:

$$\hat{\mathbf{p}} = \frac{1}{m} \sum_{j=1}^{m} \sum_{l=1}^{l_{\max}} \mathbf{1}[k_{l-1} + 1 \le N^{(j)} \le k_l] \cdot \boldsymbol{\delta}^{(k_l)} = \frac{1}{m} \sum_{j=1}^{m} \mathsf{Block}(\boldsymbol{\delta}^{N^{(j)}}, \rho).$$

Then, for any $i = I(\rho, l)$ for some $l \in [l_{\max}]$, by linearity of λ_i

$$G_i = i\lambda_i(\hat{\mathbf{p}}) = \frac{1}{m} \sum_{j=1}^m i\lambda_i(\mathsf{Block}(\boldsymbol{\delta}^{(N^{(j)})}, \rho)).$$

By linearity of the mapping Block, if $N^{(j)} \sim \mathbf{p}$ for all $j \in [m]$,

$$\mathbb{E}[G_i] = \mathbb{E}[i\lambda_i(\mathsf{Block}(\boldsymbol{\delta}^{(N)}, \rho))] = i\lambda_i(\mathsf{Block}(\mathbb{E}[\boldsymbol{\delta}^{(N)}], \rho)) = i\lambda_i(\mathsf{Block}(\mathbf{p}, \rho)) = i\lambda_i(\overline{\mathbf{p}}).$$

Thus, G_i is an unbiased estimator for $i\lambda_i(\mathbf{\overline{p}})$ for $i = I(\rho, l), l \in [l_{\max}]$. Next, we show that G_i 's are concentrated around their means. To do so, we first notice that for any distribution $\mathbf{p}' \in \Delta$ and any $i \in \mathbb{N}$,

$$i\lambda_i(\mathbf{p}') = \sum_{j=i}^{\infty} \frac{i}{j} p'_j \le \sum_{j=i}^{\infty} p'_j \le 1$$

and since $G_i = i\lambda_i(\hat{\mathbf{p}})$, implying that $0 \leq G_i \leq 1$ for all $i = I(\rho, l), l \in [l_{\max}]$. By Hoeffding's inequality (Lemma F.4)

$$\mathbb{P}[|G_i - i\lambda_i(\overline{\mathbf{p}})| > \epsilon/4] \le 2e^{-2m\epsilon^2}.$$

Next, we will bound $\sum_{l=1}^{l_{\max}} Var[G_i]$, thereby showing that most of the G_i 's have small variance, which allows tighter concentration bound such as Bernstein's inequality.

To be precise, for $i = k_l$ for some $l \in [l_{\max}]$, and $j \in \mathbb{N}$ where $k_{s-1} < j \le k_s$ for some $s \in \mathbb{N}$,

$$i\lambda_i(\mathsf{Block}(\pmb{\delta}^{(j)},\rho)) = \frac{k_l}{k_s}\mathbf{1}[s \geq l].$$

Since for $N \sim \mathbf{p}$, by our definition that $\overline{\mathbf{p}} = \mathsf{Block}(\mathbf{p}, \rho)$, we have $\mathbb{P}[k_{l'-1} + 1 \leq N \leq k_{l'}] = \overline{p}_{k_{l'}}$, thus

$$\mathbb{E}[(i\lambda_i(\mathsf{Block}(\boldsymbol{\delta}^{(N)},\rho)))^2] = \sum_{s=l}^{\infty} \overline{p}_{k_s}(\frac{k_l}{k_s})^2 \leq \sum_{s=l}^{\infty} \overline{p}_{k_s} \cdot \frac{4}{\rho^{2(s-l)}}.$$

where the last step is by Lemma D.4. In particular, we get

$$\mathbb{E}[(k_1\lambda_{k_1}(\mathsf{Block}(\boldsymbol{\delta}^{(N)},\rho)))^2] + (1-\frac{1}{\rho^2})\sum_{l=2}^{\infty}\mathbb{E}[(k_l\lambda_{k_l}(\mathsf{Block}(\boldsymbol{\delta}^{(N)},\rho)))^2] \le \sum_{s=1}^{\infty}4\overline{p}_{k_s} = 4$$

which implies that

$$\sum_{l=1}^{l_{\max}} Var[k_l \lambda_{k_l}(\mathsf{Block}(\boldsymbol{\delta}^{(N)}, \rho))] \leq \sum_{l=1}^{l_{\max}} \mathbb{E}[(k_l \lambda_{k_l}(\mathsf{Block}(\boldsymbol{\delta}^{(N)}, \rho)))^2] \leq \frac{4}{1 - 1/\rho^2} \leq \frac{25}{\epsilon}.$$

In particular, at most $\min(300/\epsilon^2, l_{\max})$ of them can have variance larger than $\epsilon/12$. For the rest of these terms, we use Bernstein's inequality (Lemma F.5)

$$\mathbb{P}[|G_i - i\lambda_i(\overline{\mathbf{p}})| > \epsilon/4] \le 2e^{-m\epsilon/8}.$$

Next, we consider the second case where $i > \rho T$. Notice that

$$\sum_{j=1}^{m} \mathbf{1}[N^{(j)} \ge T+1] \sim Bino(m, \sum_{j=T+1}^{\infty} p_j),$$

Under the assumption, $\sum_{j=T+1}^{\infty} p_j \leq \epsilon/12$, thus by Chernoff bound (Lemma F.6) we have

$$\mathbb{P}[\frac{1}{m}\sum_{j=1}^{m}\mathbf{1}[N^{(j)} \ge T+1] \ge \epsilon/6] \le e^{-m\epsilon/18}.$$

In addition, for any $i = k_l$ for some $l \in [l_{\max}]$, by triangle inequality,

$$\sum_{j=i}^{\infty} |\hat{p}_j - \overline{p}_j| \le \sum_{j=i}^{\infty} \hat{p}_j + \sum_{j=i}^{\infty} \overline{p}_j \le \frac{1}{m} \sum_{j=1}^m \mathbf{1}[N^{(j)} \ge k_{l-1} + 1] + \sum_{j=k_{l-1}+1}^{\infty} \overline{p}_j.$$

However, for any $k_l > \rho T$, we have $k_{l-1} \ge T$. This is because $k_l > \rho T$ implies $l \ge 2$, in particular $l-1 \ge 1$, and so $k_{l-1} \in I(\rho)$. Now let l_0 be the largest integer such that $k_{l-1} = \lceil \rho^{l_0} \rceil$, then $k_l = \lceil \rho^{l_0+1} \rceil$. By assumption, $\lceil \rho^{l_0+1} \rceil > \rho T$. If $\lceil \rho^{l_0} \rceil < T$, then $\rho^{l_0} \le T-1$, which implies $\lceil \rho^{l_0+1} \rceil \le \lceil \rho T-\rho \rceil \le \rho T-\rho+1 < \rho T$ contradiction. Thus, under the event $\{\frac{1}{m} \sum_{j=1}^m \mathbf{1}[N^{(j)} \ge T+1] < \frac{\epsilon}{6}\}$, which has probability at least $1 - e^{-m\epsilon/18}$, for all $i \in I(\rho)$ such that $i > \rho T$,

$$\sum_{j=i}^{\infty} |\hat{p}_j - \overline{p}_j| \le \frac{1}{m} \sum_{j=1}^m \mathbf{1}[N^{(j)} \ge T+1] + \sum_{j=T+1}^{\infty} \overline{p}_j \le \frac{\epsilon}{6} + \frac{\epsilon}{12} \le \frac{\epsilon}{4}.$$

Thus

$$|G_i - i\lambda_i(\overline{\mathbf{p}})| = |\sum_{j=i}^{\infty} \frac{i}{j}(\hat{p}_j - \overline{p}_j)| \le \sum_{j=i}^{\infty} \frac{i}{j}|\hat{p}_j - \overline{p}_j| \le \sum_{j=i}^{\infty} |\hat{p}_j - \overline{p}_j| \le \epsilon/4.$$

Since the indices in the first case satisfies $i \leq \rho T$ and $i \in I(\rho)$, i.e. $i = \lceil \rho^l \rceil$ for some $l \in \mathbb{N}$. If $T \geq 2$, the number of terms in the first case is upper bounded by

$$1 + \frac{\log(T)}{\log(\rho)} \le 1 + \frac{5\log(T)}{\epsilon} \le \frac{10\log(T)}{\epsilon}$$

where we use $\log(\rho) = \log(1 + \epsilon/4) \ge 0.2\epsilon$ for $0 < \epsilon < 1^8$. Thus, if $m \ge \max(\frac{18}{\epsilon}\log(\frac{50\log(T)}{\epsilon\delta}), \frac{1}{2\epsilon^2}\log(\frac{1200}{\epsilon^2\delta}))$,

$$\mathbb{P}[\mathcal{E}^*] \ge 1 - e^{-\epsilon m/12} - \frac{600}{\epsilon^2} \cdot e^{-2m\epsilon^2} - \frac{20\log(T)}{\epsilon} \cdot e^{-\epsilon m/8} - e^{-\epsilon m/18} \ge 1 - \delta.$$

If T = 1, since $\lceil \rho \rceil > \rho \cdot T$, there is no term in the first case, with $m \ge \frac{18}{\epsilon} \log(\frac{2}{\delta})$

$$\mathbb{P}[\mathcal{E}^*] \ge 1 - e^{-\epsilon m/12} - e^{-\epsilon m/18} \ge 1 - \delta.$$

Step 2: If the event \mathcal{E}^* holds, the following series of approximation results holds.

 $^{{}^{8}\}log(1+x) \ge 0.8x$ for $x \in [0, 0.5]$.

• The sequence **G** approximates the sequence $(i\lambda_i(\overline{\mathbf{p}}))_{i\in\mathbb{N}}$ well. Notice that under \mathcal{E}^* , this is already true for $i \in I(\rho) \cap [N_{\max}]$, and here we show that this is true for all $i \in \mathbb{N}$. First, notice that $\hat{\mathbf{p}}$ is supported only on $I(\rho) \cap [N_{\max}]$. Thus, for $i \in [N_{\max}]$ but $i \notin I(\rho)$, let i' be the smallest index such that $i' \geq i$ and $i' \in I(\rho)$, then $i' \leq N_{\max}$, and by the second part in \mathcal{E}^* , $|G_{i'} - i'\lambda_{i'}(\overline{\mathbf{p}})| \leq \epsilon/4$. By Lemma D.5,

$$|G_i - i\lambda_i(\mathbf{\overline{p}})| \le \epsilon/4.$$

• $\sum_{i=1}^{N_{\max}} U_{i-1}(\mathbf{q}) \frac{q_i}{i} \cdot G_i$ approximates $A(\mathbf{\overline{p}}, \mathbf{q})$ well. This follows by observing that the sequence **G** satisfies the conditions in Lemma 3.2 with $n = N_{\max}$, which implies that for any $\mathbf{q} \in \mathcal{S}$,

$$|A(\overline{\mathbf{p}}, \mathbf{q}) - \sum_{i=1}^{N_{\max}} U_{i-1}(\mathbf{q}) \frac{q_i}{i} \cdot G_i| \le \epsilon/4.$$

• $A(\overline{\mathbf{p}}, \mathbf{q})$ approximates $A(\mathbf{p}, \mathbf{q})$ well since $\overline{\mathbf{p}}$ is the blocked version of \mathbf{p} : by Corollary 3.1,

$$|A(\mathbf{p}, \mathbf{q}) - A(\overline{\mathbf{p}}, \mathbf{q})| \le \rho - 1 = \epsilon/4.$$

Given these approximations, by triangle inequality,

$$|A(\mathbf{p},\mathbf{q}) - \sum_{i=1}^{N_{\max}} U_{i-1}(\mathbf{q}) \frac{q_i}{i} \cdot G_i| \le |A(\mathbf{p},\mathbf{q}) - A(\overline{\mathbf{p}},\mathbf{q})| + |A(\overline{\mathbf{p}},\mathbf{q}) - \sum_{i=1}^{N_{\max}} U_{i-1}(\mathbf{q}) \frac{q_i}{i} \cdot G_i| \le \epsilon/2.$$

By Lemma D.3, $\hat{\mathbf{q}}$ is at least ϵ -suboptimal.

Proof of Theorem 3.5. For convenience, define $A^*: \Delta \to [0,1]$ as

$$A^*(\mathbf{p}) = \sup_{\mathbf{q}\in\mathcal{S}} A(\mathbf{p},\mathbf{q}).$$

For $n \ge 3$ and $s \in [0, 1]$, let $\mathbf{w}^{(n,s)}$ be the distribution supported on $\{1, n\}$ with $w_1^{(n,s)} = s$ and $w_n^{(n,s)} = 1-s$. For any $\mathbf{q} \in \mathcal{S}$, by linearity,

$$A(\mathbf{w}^{(n,s)},\mathbf{q}) = q_1 s + (1-s)A(\boldsymbol{\delta}^{(n)},\mathbf{q}) = q_1 s + \frac{1-s}{n}(q_1 + (1-q_1)\sum_{i=2}^n \prod_{j=2}^{i-1} (1-q_j/j)q_i)$$

Let $\mathbf{q}_*^{(n)}$ be an optimal strategy for $\boldsymbol{\delta}^{(n)}$, with $A_n := A(\boldsymbol{\delta}^{(n)}, \mathbf{q}_*^{(n)})$. From [24], for $n \ge 3$, we have $q_{*,1}^{(n)} = 0$, and so

$$\sup_{q_2,\dots,q_n \in [0,1]} \sum_{i=2}^n \prod_{j=2}^{i-1} (1 - q_j/j) q_i = \sum_{i=2}^n \prod_{j=2}^{i-1} (1 - q_{*,j}^{(n)}/j) q_{*,i}^{(n)} = nA_n.$$

Thus

$$\sup_{q_2,\dots,q_n \in [0,1]} A(\mathbf{w}^{(n,s)}, \mathbf{q}) = q_1 s + (1-s)(q_1/n + (1-q_1)A_n).$$

To maximize the above expression over $q_1 \in [0, 1]$, it's easy to see that for $s \ge s_n^* := \frac{A_n - \frac{1}{n}}{1 + A_n - \frac{1}{n}} \approx \frac{1}{1 + e}$, taking $q_1 = 1$ is optimal, otherwise $q_1 = 0$ is optimal. Take $0 < \epsilon < \min(s_n^*, 1 - s_n^*)$, using $A_n \ge 1/e$,

$$A^{*}(\mathbf{w}^{(n,s_{n}^{*}+\epsilon)}) = (1-\frac{1}{n})(s_{n}^{*}+\epsilon) + \frac{1}{n} = A^{*}(\mathbf{w}^{(n,s_{n}^{*})}) + (1-\frac{1}{n})\epsilon \ge A^{*}(\mathbf{w}^{(n,s_{n}^{*})}) + \epsilon/e,$$
$$A^{*}(\mathbf{w}^{(n,s_{n}^{*}-\epsilon)}) = (1-s_{n}^{*}+\epsilon)A_{n} = A^{*}(\mathbf{w}^{(n,s_{n}^{*})}) + \epsilon A_{n} \ge A^{*}(\mathbf{w}^{(n,s_{n}^{*})}) + \epsilon/e.$$

Thus for any $\mathbf{q} \in \mathcal{S}$,

$$\frac{1}{2}(A(\mathbf{w}^{(n,s_n^*+\epsilon)},\mathbf{q}) + A(\mathbf{w}^{(n,s_n^*-\epsilon)},\mathbf{q})) = A(\mathbf{w}^{(n,s_n^*)},\mathbf{q}) \le A^*(\mathbf{w}^{(n,s_n^*)}) \le \frac{1}{2}(A^*(\mathbf{w}^{(n,s_n^*+\epsilon)}) + A^*(\mathbf{w}^{(n,s_n^*-\epsilon)})) - \epsilon/e.$$

		-

That is, there is no $\mathbf{q} \in \mathcal{S}$, such that at the same time

$$A(\mathbf{w}^{(n,s_n^*+\epsilon)}, \mathbf{q}) \ge A^*(\mathbf{w}^{(n,s_n^*+\epsilon)}) - \epsilon/3, \quad A(\mathbf{w}^{(n,s_n^*-\epsilon)}, \mathbf{q}) \ge A^*(\mathbf{w}^{(n,s_n^*-\epsilon)}) - \epsilon/3$$

As a result, consider $S \sim Uniform\{s_n^* + \epsilon, s_n^* - \epsilon\}$, the algorithm is given m iid samples from $\mathbf{w}^{(n,S)}$. If an algorithm can find a $\mathbf{q} \in S$ which is at least $\epsilon/3$ suboptimal for $\mathbf{w}^{(n,S)}$, with probability at least 2/3, using m samples, then this algorithm can correctly predict which value S takes with probability at least 2/3 using m samples.

For simplicity, for any distribution \mathbf{p} , use $\otimes^m \mathbf{p}$ to denote $\mathbf{p} \times \cdots \times \mathbf{p}$ the *m*-product distribution of \mathbf{p} . Then from classical results (textbook reference [41]), the optimal test for S makes mistake with probability p_e ,

$$p_e = \frac{1}{2} (1 - D_{TV}(\otimes^m \mathbf{w}^{(n,s_n^* + \epsilon)}, \otimes^m \mathbf{w}^{(n,s_n^* - \epsilon)})).$$

To achieve $p_e \leq 1/3$, *m* needs to be large enough such that $D_{TV}(\otimes^m \mathbf{w}^{(n,s_n^*+\epsilon)}, \otimes^m \mathbf{w}^{(n,s_n^*-\epsilon)}) \geq 1/3$.

$$2D_{TV}^{2}(\otimes^{m} \mathbf{w}^{(n,s_{n}^{*}+\epsilon)}, \otimes^{m} \mathbf{w}^{(n,s_{n}^{*}-\epsilon)})$$

$$\leq D_{KL}(\otimes^{m} \mathbf{w}^{(n,s_{n}^{*}+\epsilon)} || \otimes^{m} \mathbf{w}^{(n,s_{n}^{*}-\epsilon)}) \quad \text{Pinsker's inequality}$$

$$= mD_{KL}(\mathbf{w}^{(n,s_{n}^{*}+\epsilon)} || \mathbf{w}^{(n,s_{n}^{*}-\epsilon)}) \quad \text{tensorization for KL-divergence}$$

By reverse Pinsker inequality [40], we have for $\epsilon < s_n^*$,

$$D_{KL}(\mathbf{w}^{(n,s_n^*+\epsilon)}||\mathbf{w}^{(n,s_n^*-\epsilon)}) \le \frac{2}{\min(s_n^*-\epsilon, 1-s_n^*+\epsilon)} D_{TV}^2(\mathbf{w}^{(n,s_n^*+\epsilon)}, \mathbf{w}^{(n,s_n^*-\epsilon)}) = \frac{2\epsilon^2}{\min(s_n^*-\epsilon, 1-s_n^*+\epsilon)}$$

However, for any $n \ge 3$, we have $A_n - \frac{1}{n} < A_n < 1$, and so $s_n^* = \frac{A_n - \frac{1}{n}}{1 + A_n - \frac{1}{n}} < \frac{1}{2}$, simplifying the above to

$$D_{KL}(\mathbf{w}^{(n,s_n^*+\epsilon)}||\mathbf{w}^{(n,s_n^*-\epsilon)}) \le \frac{2\epsilon^2}{s_n^*-\epsilon}$$

Thus we have the following bound:

$$m \ge \frac{s_n^* - \epsilon}{9\epsilon^2}$$

Taking $c_n = s_n^* = \frac{A_n - \frac{1}{n}}{1 + A_n - \frac{1}{n}}$, and using $\lim_{n \to \infty} A_n = 1/e$, we get $\lim_{n \to \infty} c_n = \frac{1}{e+1}$.

Theorem D.1. Let $0 < \epsilon < \frac{1}{2e}$, then for $n \in \mathbb{N}$ large enough, it requires at least $\Omega(\log(\frac{\log(n)}{\epsilon}))$ bits to write down an ϵ suboptimal strategy $\mathbf{q} \in S$ for distributions supported on [n].

Proof of Theorem D.1. For convenience, define $A^*: \Delta \to [0,1]$ as

$$A^*(\mathbf{p}) = \sup_{\mathbf{q}\in\mathcal{S}} A(\mathbf{p},\mathbf{q}).$$

First, notice that for any $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$, we have

$$i\lambda_i(\frac{1}{2}\boldsymbol{\delta}^{(k_1)} + \frac{1}{2}\boldsymbol{\delta}^{(k_2)}) = \begin{cases} \frac{1/k_1 + 1/k_2}{2} \cdot i & i = 1, \dots, k_1\\ \frac{1/k_2}{2} \cdot i & i = k_1 + 1, \dots, k_2\\ 0 & i > k_2 \end{cases}$$

which implies that

$$\theta(\frac{1}{2}\boldsymbol{\delta}^{(k_1)} + \frac{1}{2}\boldsymbol{\delta}^{(k_2)}) = \sup_{i \in \mathbb{N}} i\lambda_i(\frac{1}{2}\boldsymbol{\delta}^{(k_1)} + \frac{1}{2}\boldsymbol{\delta}^{(k_2)}) = k_1\lambda_{k_1}(\frac{1}{2}\boldsymbol{\delta}^{(k_1)} + \frac{1}{2}\boldsymbol{\delta}^{(k_2)}) = \frac{1 + k_1/k_2}{2}.$$

In particular, by Theorem 3.1, this means that for any strategy $\mathbf{q} \in \mathcal{S}$,

$$A(\boldsymbol{\delta}^{(k_1)}, \mathbf{q}) + A(\boldsymbol{\delta}^{(k_2)}, \mathbf{q}) = 2A(\frac{1}{2}\boldsymbol{\delta}^{(k_1)} + \frac{1}{2}\boldsymbol{\delta}^{(k_2)}), \mathbf{q}) \le \frac{1 + k_1/k_2}{e}$$

Since $A^*(\boldsymbol{\delta}^{(n)}) \geq \frac{1}{e}$ for all $n \in \mathbb{N}$, we conclude that there is no strategy $\mathbf{q} \in \mathcal{S}$ which is at least ϵ suboptimal for both $\boldsymbol{\delta}^{(k_1)}$ and $\boldsymbol{\delta}^{(k_2)}$ for any $\epsilon > \frac{1-k_1/k_2}{2e}$. Now for any $0 < \epsilon < \frac{1}{2e}$, consider the sequence $(k_i)_{i \in \mathbb{N}}$ where $k_1 = 1$ and $k_{i+1} = \lceil \frac{1}{(1-2e)}k_i \rceil + 1$ for all

Now for any $0 < \epsilon < \frac{1}{2\epsilon}$, consider the sequence $(k_i)_{i \in \mathbb{N}}$ where $k_1 = 1$ and $k_{i+1} = \lceil \frac{1}{(1-2\epsilon\epsilon)}k_i \rceil + 1$ for all $i \in \mathbb{N}$, then $\frac{k_i}{k_{i+1}} < 1 - 2\epsilon\epsilon$ for all $i \in \mathbb{N}$. From the argument above, there is no strategy that is ϵ suboptimal for both $\delta^{(k_i)}$ and $\delta^{(k_j)}$ at the same time if $i \neq j$. Thus let $n \in \mathbb{N}$ and $l \in \mathbb{N}$ be such that $k_l \leq n < k_{l+1}$, then it requires at least $\frac{\log(l)}{\log(2)}$ bits to write down an ϵ suboptimal strategy for $\{\delta^{(k_i)}, i \in [l]\}$. Since for n large enough, $l = \Omega(\frac{\log(n)}{\log(1/(1-2\epsilon\epsilon))}) = \Omega(\frac{\log(n)}{\epsilon})$, and so the number of bits required is $\Omega(\log(\frac{\log(n)}{\epsilon}))$.

E Proof for results in Section 4 and B

 $Proof of Theorem 4.1. \text{ Consider } \mathbf{x} \in \Delta \text{ where } x_s = 0 \text{ if } s < \underline{n} \text{ or } s > \overline{n}, x_s = \begin{cases} \frac{1}{\overline{n} - \underline{n} + 1 - \sum_{i=\underline{n}}^{\overline{n} - \underline{n}} f(i)/f(i+1)} & s = \underline{n} \\ \frac{1 - f(s-1)/f(s)}{\overline{n} - \underline{n} + 1 - \sum_{i=\underline{n}}^{\overline{n} - \underline{n}} f(i)/f(i+1)} & s = \underline{n} + 1, \dots, \overline{n} \end{cases}$

It's easy to check that for each $n \in \{\underline{n}, \underline{n} + 1, \dots, \overline{n}\},\$

$$\sum_{s=\underline{n}}^{n} x_s M(s,n) = \frac{c_0}{f(n)} \sum_{s=\underline{n}}^{n} x_s f(s) = \frac{c_0}{f(n)} \cdot \frac{f(n)}{\overline{n} - \underline{n} + 1 - \sum_{i=\underline{n}}^{\overline{n} - 1} f(i) / f(i+1)}$$
$$= \frac{c_0}{\overline{n} - \underline{n} + 1 - \sum_{i=\underline{n}}^{\overline{n} - 1} f(i) / f(i+1)}.$$

Thus, no matter what distribution N takes, as long as $\underline{n} \leq N \leq \overline{n}$ almost surely, the sampling algorithm described above achieves a performance of $\frac{c_0}{\overline{n}-\underline{n}+1-\sum_{i=n}^{\overline{n}-1}f(i)/f(i+1)}$ in expectation.

Next, consider the function $h(x) := 1 - x + \log(x)$ defined for $x \in (0, 1]$. h(1) = 0, and h'(x) = -1 - 1/x < 0, and so $h(x) \le 0$ on (0, 1], and so $1 - x \le \log(1/x)$ for $0 < x \le 1$. In particular,

$$\overline{n} - \underline{n} + 1 - \sum_{i=\underline{n}}^{\overline{n}-1} f(i) / f(i+1) \leq 1 + \sum_{i=\underline{n}}^{\overline{n}-1} \log(f(i+1) / f(i)) = 1 + \log(f(\overline{n}) / f(\underline{n})).$$

This gives the desired lower bound.

To prove Theorem B.1, we construct an explicit family of distributions for X_i and N based on the following 2 ideas:

- Section E.1 provides a simple and intuitive proof for the upper bound (which is also $\frac{1}{1+\log(1/x)}$) when X_i 's are not restricted to identically distributed random variable but $X_i \in [x, 1]$ almost surely.
- Section E.2 constructs distributions (say G) such that when $X_i \sim_{iid} G$, $\max_{j \in [n]} X_j$ increases in a "controlled way" as n increases, with high probability.

E.1 non iid case

Theorem E.1. Let $\{X_l\}_{l\in[n]}$ be a sequence of random variables such that $X_l = a_l$ almost surely, where $0 < a_1 \le a_2 \le \cdots \le a_n$. Let N be a random variable independent of $\{X_l\}_{l\in[n]}$ and supported on [n], with $\mathbb{P}[N=l] = p_l = c \cdot \begin{cases} 1 - \frac{a_l}{a_{l+1}} & l \in [n-1] \\ 1 & l = n \end{cases}$, where $c = (n - \sum_{i=1}^{n-1} \frac{a_i}{a_{i+1}})^{-1}$. Then for any stopping time τ adapted to the filtration generated by $\{X_l\}_{l\in[n]}$,

$$\mathbb{E}\left[\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \le N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}\right] = (n - \sum_{i=1}^{n-1} \frac{a_i}{a_{i+1}})^{-1}.$$

If
$$a_i = x^{\frac{n-i}{n-1}}$$
 for some $x \in (0,1]$,

$$\mathbb{E}[\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \le N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}] = (1 + (n-1)(1 - x^{1/(n-1)}))^{-1} \to \frac{1}{1 + \log(1/x)} \quad as \ n \to \infty.$$

Proof. Clearly $\mathbb{E}[\max_{i \in [N]} X_i | N] = a_N$, and the numerator satisfies

$$\mathbb{E}[X_{\tau}\mathbf{1}[\tau \le N]|N] = \mathbb{E}[\sum_{i=1}^{n} a_{i}\mathbf{1}[\tau = i] \cdot \mathbf{1}[i \le N]|N] = \sum_{i=1}^{n} a_{i}\mathbf{1}[i \le N] \cdot \mathbb{E}[\mathbf{1}[\tau = i]|N] = \sum_{i=1}^{n} a_{i}q_{i}\mathbf{1}[i \le N]$$

where $q_i := \mathbb{P}[\tau = i] = \mathbb{E}[\mathbf{1}[\tau = i]|N]$ and the equality is because N is independent of $\{X_l\}_{l \in [n]}$ and τ is adapted to $\{X_l\}_{l \in [n]}$. Thus the ratio of interest is

$$\mathbb{E}\left[\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \leq N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}\right] = \mathbb{E}\left[\sum_{i=1}^n \frac{a_i}{a_N} \cdot q_i\mathbf{1}[i \leq N]\right] = \sum_{i=1}^n \sum_{j=1}^n \frac{a_i}{a_j} \cdot q_i\mathbb{E}\left[\mathbf{1}[i \leq N = j]\right] = \sum_{i=1}^n \sum_{j=i}^n \frac{a_i}{a_j} \cdot q_ip_j.$$
With $p_l = c \cdot \begin{cases} 1 - \frac{a_l}{a_{l+1}} & l \in [n-1]\\ 1 & l = n \end{cases}$, where $c = (n - \sum_{i=1}^{n-1} \frac{a_i}{a_{i+1}})^{-1}$, it's easy to check that
$$\sum_{i=1}^n a_i = n = \sum_{i=1}^n \sum_{j=i}^n \frac{a_i}{a_i} = \sum_{i=1}^n \sum_{j=i}^n \frac{a_i}{a_i} \cdot q_ip_j.$$

 $\sum_{j=i}^{n} \frac{a_i}{a_j} \cdot p_j = c \; \forall i \in [n] \implies \mathbb{E}\left[\frac{\mathbb{E}[X_\tau \mathbf{1}[\tau \le N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}\right] = c \sum_{i=1}^{n} q_i = c.$

When $a_i = x^{\frac{n-i}{n-1}}$,

$$(n - \sum_{i=1}^{n-1} \frac{a_i}{a_{i+1}})^{-1} = (n - (n-1)x^{1/(n-1)})^{-1}$$

and the limit is by noticing that

$$\lim_{h \to 0^+} (1 + \frac{1 - x^h}{h})^{-1} = \frac{1}{1 + \log(1/x)}.$$

In fact, [27] proves that for any non-negative sequence of random variables $\{X_i\}_{i \in [n]}$ such that $X_i \in [0, 1]$ for all i and $\sup_{\tau} \mathbb{E}[X_{\tau}] = x$,

$$\mathbb{E}[\max_{i \in [n]} X_i] \le x(1 + (n-1)(1 - x^{1/(n-1)}))$$

and the upper bound is achieved by the sequence $X_i = x^{\frac{n-i}{n-1}} \prod_{j=1}^{i-1} Z_j$ where $Z_i \sim_{iid} Bernoulli(x^{1/(n-1)})$, which is exactly our construction above. It's an interesting open question whether there is connection between the two problems.

E.2 iid case

Definition E.1. Let $G : \mathbb{R} \to [0,1]$ be a right continuous, non-decreasing function (i.e. it's the cdf for a random variable). Then we say that G satisfies the

• $(\theta_0, \theta_1, \ldots, \theta_n, K, \xi)$ property for some $n, K \in \mathbb{N}$, $\theta_0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_n$, $\xi \in [0, 1]$ if there exist $k_1 \leq \ldots \leq k_n \leq K$, $k_i \in \mathbb{N}$

$$\sum_{i=1}^{n} 1 - (G(\theta_i)^{k_i} - G(\theta_{i-1})^{k_i}) \le \xi$$

- (n, K, ξ) property if $(\theta_0, \theta_1, \dots, \theta_n, K, \xi)$ property with $\theta_i = i/n$
- approximate (n, K, ξ) property if (n, K, ξ) property but k_i 's don't need to be integers.

Lemma E.1. If G satisfies the $(\theta_0, \theta_1, \ldots, \theta_n, K, \xi)$ property, and let $X_i \sim_{iid} G$, then there exist $k_1 \leq \ldots \leq k_n \leq K$, $k_i \in \mathbb{N}$ such that

$$\mathbb{P}[\max_{j \in [k_i]} X_j \in (\theta_{i-1}, \theta_i], \ \forall i \in [n]] \ge 1 - \xi.$$

Proof. By definition there exist $k_1 \leq \ldots \leq k_n \leq K$, $k_i \in \mathbb{N}$, such that

$$\sum_{i=1}^{n} 1 - (G(\theta_i)^{k_i} - G(\theta_{i-1})^{k_i}) \le \xi$$

For i = 1, ..., n, let \mathcal{E}_i denote the event $\max_{i \in [k_i]} X_i \in (\theta_{i-1}, \theta_i]$. Then

$$\mathbb{P}[\mathcal{E}_i] = G(\theta_i)^{k_i} - G(\theta_{i-1})^{k_i}$$

Thus, by union bound,

$$\mathbb{P}[\bigcap_{i \in [n]} \mathcal{E}_i] \ge 1 - \sum_{i=1}^n 1 - (G(\theta_i)^{k_i} - G(\theta_{i-1})^{k_i}) \ge 1 - \xi.$$

Lemma E.2. If G satisfies (n, K, ξ) property, and let $\theta_0 \leq \cdots \leq \theta_n$, then $G \circ h$ satisfies $(\theta_0, \theta_1, \ldots, \theta_n, K, \xi)$ property, where $h : \mathbb{R} \to [0,1]$ is a piecewise linear linear interpolation defined as

$$h(x) := \begin{cases} 0 & x \le \theta_0 \\ 1 & x \ge \theta_n \\ \frac{1}{n} \cdot \frac{x - \theta_i}{\theta_{i+1} - \theta_i} + \frac{i}{n} & x \in [\theta_i, \theta_{i+1}), \ i = 0, 1, \dots, n-1 \end{cases}$$

In particular, if there exists G satisfying (n, K, ξ) property, then there exists G' satisfying $(\theta_0, \theta_1, \ldots, \theta_n, K, \xi)$ property.

Proof. Follows from $h(\theta_i) = \frac{i}{n}$ for i = 0, 1, ..., n.

Lemma E.3. Let $\tilde{h}(z) := z^{-\frac{z}{z-1}} - z^{-\frac{1}{z-1}}$ and $\overline{h}(z) := z^{-\frac{z}{2(z-1)}} - z^{-\frac{1}{z-1}}$ be defined on $(1,\infty)$, then

- For any $\xi \ge (n-1)(1+\tilde{h}(K^{\frac{1}{n-1}}))$, there exists a G satisfying the approximate (n, K, ξ) property, with G(0) = 0, G(1) = 1.
- Assume that $K^{\frac{1}{n-1}} \ge 2$, for any $\xi \ge (n-1)(1+\overline{h}(K^{\frac{1}{n-1}}))$, there exists a G satisfying the (n, K, ξ) property, with G(0) = 0, G(1) = 1.

Proof. For the approximate (n, K, ξ) property, consider the following choice:

$$k_i^* = K^{\frac{i-1}{n-1}}, \quad i = 1, 2, \dots, n$$

 $G(i/n) = \alpha^{1/k_i^*}, \quad i = 1, 2, \dots, n-1$

where $\alpha = K^{-\frac{1}{(n-1)(K^{\frac{1}{n-1}}-1)}}$, and G(0) = 0, G(1) = 1. Then

$$\sum_{i=1}^{n} 1 - (G(i/n)^{k_i^*} - G((i-1)/n)^{k_i^*}) = (n-1)(1 - \alpha + \alpha^{K^{1/(n-1)}}) = (n-1)(1 + \tilde{h}(K^{\frac{1}{n-1}})).$$

Thus for $\xi \ge (n-1)(1+\tilde{h}(K^{\frac{1}{n-1}}))$, there exists a G satisfying the approximate (n, K, ξ) property, with G(0) = 0, G(1) = 1.

For the second part, we consider $\tilde{k}_i^* = \lceil k_i^* \rceil = \lceil K^{\frac{i-1}{n-1}} \rceil$ where $\lceil \cdot \rceil$ is the ceiling function, with G(0) =

 $0, G(1) = 1 \text{ and } G(i/n) = \alpha^{1/\tilde{k}_i^*} \text{ for } i = 1, \dots, n-1.$ For i = 1, we have $\frac{\tilde{k}_{i+1}^*}{\tilde{k}_i^*} = \frac{[K^{\frac{1}{n-1}}]}{1} \ge K^{\frac{1}{n-1}}$. Under the assumption that $K \ge 2^{n-1}$, we have $k_i^* \ge 2$ for all $2 \le i \le n$, and so for all $2 \le i \le n-1$, we have

$$\frac{k_{i+1}^*}{\tilde{k}_i^*} \ge \frac{k_{i+1}^*}{k_i^*+1} = \frac{k_{i+1}^*}{k_i^*} (1 - \frac{1}{k_i^*+1}) \ge \frac{k_{i+1}^*}{2k_i^*} = \frac{1}{2} K^{\frac{1}{n-1}}.$$

Thus for i = 1, 2, ..., n - 1,

$$G(i/n)^{k_{i+1}} = \alpha^{k_{i+1}/k_i} \le \alpha^{2^{2K}} .$$

$$\sum_{i=1}^n 1 - (G(i/n)^{k_i^*} - G((i-1)/n)^{k_i^*}) \le (n-1)(1-\alpha + \alpha^{\frac{1}{2}K^{1/(n-1)}}) = (n-1)(1+\overline{h}(K^{\frac{1}{n-1}})).$$

 $C(\cdot, \cdot, \cdot) \tilde{k}^* = \tilde{k}^* / \tilde{k}^* = \frac{1}{k} K \frac{1}{n-1}$

Thus for $\xi \ge (n-1)(1+\overline{h}(K^{\frac{1}{n-1}}))$, there exists a G satisfying the (n, K, ξ) property, with G(0) = 0, G(1) = 1.

Lemma E.4. Let $\overline{h}(z) := z^{-\frac{z}{2(z-1)}} - z^{-\frac{1}{z-1}}$ be defined on $(1,\infty)$. Assume that $K^{\frac{1}{n-1}} \ge 2$ and $\theta_0 \le \theta_1 \le \theta_2 \le \ldots \le \theta_n$, for any $\xi \ge (n-1)(1+\overline{h}(K^{\frac{1}{n-1}}))$, there exists a G and $k_1 \le \cdots \le k_n \le K$, $k_i \in \mathbb{N}$ such that if $X_i \sim_{iid} G$

$$\mathbb{P}[\max_{j\in[k_i]} X_j \in (\theta_{i-1}, \theta_i], \ \forall i \in [n]] \ge 1-\xi.$$

Proof. Follows from Lemma E.1, E.2, and E.3.

E.3 main proof

Theorem E.2. Let G be a distribution satisfying the condition in Lemma E.4 for $\theta_l = x^{1-l/n}$ for some $x \in (0,1]$ for l = 0, 1, ..., n, and $X_i \sim_{iid} G$. Let N be a random variable supported on $\{k_1, k_2, ..., k_n\}$, independent of the sequence $\{X_i\}$, and $\mathbb{P}[N = k_l] = p_l = c \cdot \begin{cases} 1 - x^{1/n} & l \in [n-1] \\ 1 & l = n \end{cases}$, where $c = (1 + (n - 1)(1 - x^{1/n}))^{-1}$, then for any τ a stopping time adapted to $\{X_i\}$,

$$\mathbb{E}\left[\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \le N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}\right] \le x^{-1/n}c + \xi \to \frac{1}{1 + \log(1/x)} + \xi \quad as \ n \to \infty.$$

Proof. The proof is similar to the proof of Theorem E.1. To deal with iid samples, now we

- condition on the event that $\max_{j \in [k_i]} X_j \in (\theta_{i-1}, \theta_i]$ holds for all i = 1, 2, ..., n which holds with high probability
- on the above event, bound $\{\max_{j \in [k_i]} X_i\}_{i \in [n]}$ using $\{\theta_i\}_{i \in [n]}$
- First, denote $A := \{\max_{j \in [k_i]} X_j \in (\theta_{i-1}, \theta_i] \ \forall i \in [n]\}, \text{ then } \mathbb{P}[A^c] \leq \xi.$

For convenience, abbreviate $\mathbb{E}[\cdot|A] = \mathbb{E}_A[\cdot]$, then $\mathbb{E}_A[\max_{i \in [N]} X_i | N = k_l] \ge \theta_{l-1}$, denote $k_0 = 0$, and let $I_i = \{k_{i-1} + 1, \dots, k_i\}$. For the numerator,

$$\mathbb{E}_A[X_\tau \mathbf{1}[\tau \le N] | N = k_l] = \mathbb{E}_A[\sum_{i=1}^l X_\tau \mathbf{1}[\tau \in I_i] | N = k_l] \le \sum_{i=1}^l \theta_i \mathbb{P}[\tau \in I_i | N = k_l, A].$$

For the conditional probability, since the sequence $\{X_i\}$ (and thus the event A, and the stopping time τ which is adapted to $\{X_i\}$) is independent of N, we have

$$\mathbb{P}[\tau \in I_i | N = k_l, A] = \frac{\mathbb{P}[\tau \in I_i, N = k_l, A]}{\mathbb{P}[N = k_l, A]} = \frac{\mathbb{P}[\tau \in I_i, A] \mathbb{P}[N = k_l]}{\mathbb{P}[A] \mathbb{P}[N = k_l]} = \mathbb{P}[\tau \in I_i | A] := q_i.$$

And so

$$\mathbb{E}_A[X_\tau \mathbf{1}[\tau \le N] | N = k_l] \le \sum_{i=1}^l \theta_i q_i \implies \frac{\mathbb{E}_A[X_\tau \mathbf{1}[\tau \le N] | N = k_l]}{\mathbb{E}_A[\max_{i \in [N]} X_i | N = k_l]} \le \sum_{i=1}^l \frac{\theta_i}{\theta_{l-1}} q_i.$$

The ratio of interest can be upper bounded by

$$\mathbb{E}_{A}\left[\frac{\mathbb{E}[X_{\tau}\mathbf{1}[\tau \leq N]|N]}{\mathbb{E}[\max_{i \in [N]} X_{i}|N]}\right] \leq \sum_{l=1}^{n} \sum_{i=1}^{l} \frac{\theta_{i}}{\theta_{l-1}} q_{i} p_{l} = x^{-1/n} \sum_{l=1}^{n} \sum_{i=1}^{l} \frac{\theta_{i}}{\theta_{l}} q_{i} p_{l}.$$

With
$$p_l = c \cdot \begin{cases} 1 - x^{1/n} & l \in [n-1] \\ 1 & l = n \end{cases}$$
, where $c = (1 + (n-1)(1 - x^{1/n}))^{-1}$, thus
$$\sum_{j=i}^n \frac{\theta_i}{\theta_j} \cdot p_j = c \ \forall i \in [n] \implies \mathbb{E}_A[\frac{\mathbb{E}[X_{\tau} \mathbf{1}[\tau \le N]|N]}{\mathbb{E}[\max_{i \in [N]} X_i|N]}] \le x^{-1/n}c.$$

Since on A^c the above ratio is bounded by 1, we get the desired result.

Proof of Theorem B.1. Take $G_{x,n}$ to be the distribution in Theorem E.2, with $\xi_n = 1/n$ (or any sequence such that $\xi_n \to 0$), and P_n correspondingly, then by Theorem E.2

$$\sup_{\tau \in \mathcal{M}_n} \mathbb{E}_{N \sim P_n, X_i \sim G_{x,n}} \left[\frac{\mathbb{E}[X_{\tau} \mathbf{1}[\tau \le N] | N]}{\mathbb{E}[\max_{i \in [N]} X_i | N]} \right] \le (x^{1/n} (1 + (n-1)(1 - x^{1/n})))^{-1} + \xi_n.$$

Taking $n \to \infty$ proves the result.

F Auxiliary results

Lemma F.1. For $T \ge 3$, let $s_T^* \in \mathbb{N}$ be such that $\sum_{i=s_T^*}^{T-1} \frac{1}{i} \le 1$ but $\sum_{i=s_T^*-1}^{T-1} \frac{1}{i} > 1$, then $\frac{s_T^*-1}{T} (\sum_{i=s_T^*-1}^{T-1} \frac{1}{i}) \ge 1/e$.

Proof. For $T \leq 9$, [24] lists the values of $\frac{s_T^* - 1}{T} (\sum_{i=s_T^* - 1}^{T-1} \frac{1}{i})$, which are all $\geq 1/e$. Below we assume that $T \geq 10$. Since $\sum_{i=s_T^* - 1}^{T-1} \frac{1}{i} \geq 1$, if $\frac{s_T^* - 1}{T} \geq 1/e$, the result holds. Below, we consider the case when $\frac{s_T^* - 1}{T} < 1/e$. Next, we give a lower bound for $\frac{s_T^* - 1}{T}$.

$$1 \ge \sum_{i=s_T*}^{T-1} \frac{1}{i} \ge \int_{s_T^*}^T \frac{1}{x} dx = \log(\frac{T}{s_T^*}) \implies \frac{s_T^* - 1}{T} \ge \frac{1}{e} - \frac{1}{T}$$

Since for any $a \ge 1$, we have $\frac{1}{a} \ge \int_a^{a+1} \frac{1}{z} dz + \frac{1}{2} (\frac{1}{a} - \frac{1}{a+1})$

$$\sum_{i=s_T^*-1}^{T-1} \frac{1}{i} \ge \int_{s_T^*-1}^T \frac{1}{x} dx + \frac{1}{2} \sum_{i=s_T^*-1}^{T-1} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \log\left(\frac{T}{s_T^*-1}\right) + \frac{1}{2T} \left(\frac{T}{s_T^*-1} - 1\right).$$

Thus, defining $g_T(x) = -x \log(x) + \frac{1}{2T}(1-x)$, we have

$$\frac{s_T^* - 1}{T} \left(\sum_{i=s_T^* - 1}^{T-1} \frac{1}{i}\right) \ge g_T\left(\frac{s_T^* - 1}{T}\right).$$

Since $g_T(\frac{1}{e}) = \frac{1}{e} + (1 - \frac{1}{e})/2T > \frac{1}{e}$, and $g_T(x)$ is concave in x, as long as $g_T(\frac{1}{e} - \frac{1}{T}) \ge \frac{1}{e}$, we have that $g_T(x) \ge \frac{1}{e}$ for all $x \in [\frac{1}{e} - \frac{1}{T}, \frac{1}{e}]$. So it remains to show $g_T(\frac{1}{e} - \frac{1}{T}) \ge \frac{1}{e}$. Let's define $h(y) = -(\frac{1}{e} - y)\log(\frac{1}{e} - y) + \frac{1}{2}y(1 - \frac{1}{e} + y)$ for $y \in [0, \frac{1}{10}]$, then $g_T(\frac{1}{e} - \frac{1}{T}) = h(\frac{1}{T})$. Notice that

$$h'(y) = \log(\frac{1}{e} - y) + 1 + y + \frac{1}{2} \cdot (1 - \frac{1}{e}), \quad h''(y) = -\frac{1}{1/e - y} + 1 \le 0.$$

And so for $y \in [0, \frac{1}{10}]$,

$$h'(y) \ge h'(\frac{1}{10}) = \log(\frac{1}{e} - \frac{1}{10}) + 1 + \frac{1}{10} + \frac{1}{2} \cdot (1 - \frac{1}{e}) > 0.09 > 0.$$

Thus h is non-decreasing on $[0, \frac{1}{10}]$, and so for all $y \in [0, \frac{1}{10}]$,

$$h(y) \ge h(0) = \frac{1}{e}$$

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Lemma F.2. Let $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_l)$ be a random vector uniformly sampled on the *l*-dimensional probability simplex, i.e. $\mathbf{Y} \sim Dir(1, 1, \ldots, 1)$. For any fixed $\mathbf{a} \in \mathbb{R}^l_{\geq 0}$, let $\overline{a} = \frac{1}{l} \langle \mathbf{a}, \mathbf{1} \rangle$ and $a^* = \max_{i \in [l]} a_i$, then $\forall 0 \leq \epsilon \leq \overline{a}$,

$$\mathbb{P}[\langle \mathbf{a}, \mathbf{Y} \rangle \le \epsilon] \le (h_{\overline{a}/a^*}(\epsilon/\overline{a}))^l$$

where $h_{\alpha}: [0,1] \to \mathbb{R}$ defined for $0 < \alpha \leq 1$, $h_{\alpha}(x) := x^{\alpha} \left(\frac{1/\alpha+1}{1/\alpha+x}\right)^{\alpha+1}$, and $h_{\alpha}(x) < 1$ for x < 1.

Proof of Lemma F.2. It's well known that for a random vector $\mathbf{X} = (X_1, X_2, \dots, X_l)$ where $\{X_i\}_{i \in [l]}$ are iid $Exp(1), \mathbf{Y} = (Y_1, Y_2, \dots, Y_l) = \frac{1}{\sum_{i=1}^l X_i} \mathbf{X}$ follows the uniform distribution on $\Delta^{(l)}$. For any $\mathbf{a} \in \mathbb{R}^l_{\geq 0}$,

$$\langle \mathbf{a}, \mathbf{Y} \rangle \leq \epsilon \iff \langle \epsilon \mathbf{1} - \mathbf{a}, \mathbf{X} \rangle \geq 0.$$

Consider the moment generating function of $\langle \epsilon \mathbf{1} - \mathbf{a}, \mathbf{X} \rangle$

$$\phi(\xi) = \mathbb{E}[e^{\xi \langle \epsilon \mathbf{1} - \mathbf{a}, \mathbf{X} \rangle}] = \prod_{i=1}^{l} \mathbb{E}[e^{\xi \langle \epsilon - a_i \rangle X_i}] = \prod_{i=1}^{l} \frac{1}{1 - \xi(\epsilon - a_i)}$$

defined for ξ such that $\xi(\epsilon - a_i) < 1$ for all $i \in [l]$. Since $(1 - \xi\epsilon)(1 + \xi a_i) = 1 - \xi(\epsilon - a_i) - \xi^2 \epsilon a_i \le 1 - \xi(\epsilon - a_i)$, we have

$$\frac{1}{1 - \xi(\epsilon - a_i)} \le \frac{1}{(1 - \xi\epsilon)(1 + \xi a_i)}$$

By convexity of the exponential function, for any $\overline{x} > 0$ fixed, $\forall x \in [0, \overline{x}]$, we have $1 + x \ge e^{\frac{\log(1+\overline{x})}{\overline{x}}x}$. Taking $a^* = \max_{i \in [l]} a_i, \overline{x} = \xi a^*$, and $\overline{a} = \sum_{i \in [l]} a_i/l$, then

$$\phi(\xi) \le (1 - \xi\epsilon)^{-l} \cdot e^{-\frac{\log(1 + \overline{x})}{\overline{x}}\xi \sum_{i \in [l]} a_i} = (\frac{(1 + \xi a^*)^{-\frac{\overline{a}}{a^*}}}{1 - \xi\epsilon})^l.$$

For $\epsilon \leq \overline{a}$, take $\xi^* = \frac{1/\epsilon - 1/\overline{a}}{a^*/\overline{a} + 1} < 1/\epsilon$, denote $\alpha_0 = \overline{a}/a^*$,

$$\frac{(1+\xi^*a^*)^{-\frac{\overline{\alpha}}{a^*}}}{1-\xi^*\epsilon} = \frac{(\epsilon/\overline{a})^{\overline{a}/a^*}}{((a^*+\epsilon)/(a^*+\overline{a}))^{\overline{a}/a^*+1}} = (\epsilon/\overline{a})^{\alpha_0} \cdot (\frac{1/\alpha_0+1}{1/\alpha_0+\epsilon/\overline{a}})^{\alpha_0+1} = h_{\alpha_0}(\epsilon/\overline{a}).$$

Thus, for $\epsilon \leq \overline{a}$, we have

$$\mathbb{P}[\langle \mathbf{a}, \mathbf{Y} \rangle \leq \epsilon] = \mathbb{P}[\langle \epsilon \mathbf{1} - \mathbf{a}, \mathbf{X} \rangle \geq 0] \leq \phi(\xi^*) \leq (h_{\alpha_0}(\epsilon/\overline{a}))^l.$$

Since $h_{\alpha}(1) = 1$,

$$\frac{d}{dx}h_{\alpha}(x) = \left(\frac{\alpha}{x} - \frac{\alpha+1}{1/\alpha+x}\right)h_{\alpha}(x) = \frac{1-x}{x(1/\alpha+x)}h_{\alpha}(x)$$

Thus h_{α} is increasing on (0, 1), and so $\forall x \in [0, 1), h_{\alpha}(x) < 1$.

Lemma F.3. Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be two sequences in \mathbb{R} such that for each $i \in \mathbb{N}$, $0 \le a_i \le b_i$. In addition, $\lim_{n\to\infty} b_n = 0$. Then there exists an index $i^* \in \mathbb{N}$ such that $a_{i^*} \ge a_j$ for all $j \in \mathbb{N}$.

Proof. If $a_i = 0$ for all $i \in \mathbb{N}$, then taking $i^* = 1$ works. Otherwise, let $l \in \mathbb{N}$ be the first index such that $a_l > 0$. Since $\lim_{n\to\infty} b_n = 0$, there exists $l' \in \mathbb{N}$ such that for all $n \ge l'$, $a_n \le b_n \le a_l$. Thus taking $i^* \in \arg \max_{j \in [l']} a_j$ works.

Lemma F.4 (Hoeffding inequality). Let X_1, \ldots, X_n be independent random variable such that $0 \le X_i \le 1$ for all $i \in [n]$ almost surely. Then for all t > 0,

$$\mathbb{P}[\sum_{i=1}^{n} X_i - \mathbb{E}[\sum_{i=1}^{n} X_i] \ge t] \le e^{-2t^2/n}.$$

Lemma F.5 (Bernstein's inequality). Let X_1, \ldots, X_n be independent random variable. Suppose that there exists $M \in \mathbb{R}$ such that $|X_i - \mathbb{E}[X_i]| \leq M$ for all $i \in [n]$ almost surely. Then for all t > 0,

$$\mathbb{P}[\sum_{i=1}^{n} X_i - \mathbb{E}[\sum_{i=1}^{n} X_i] \ge t] \le e^{-\frac{t^2/2}{V+Mt/3}}$$

where $V := \sum_{i=1}^{n} Var[X_i]$.

Lemma F.6 (Chernoff bound). Let X_1, \ldots, X_n be independent Bernoulli random variables. Denote $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then for all $t \ge 0$,

$$\mathbb{P}[X \ge (1+t)\mu] \le e^{-t^2\mu/(2+t)}.$$

For all 0 < t < 1,

$$\mathbb{P}[X \le (1-t)\mu] \le e^{-t^2\mu/2}.$$