Adversarial Rewards in Universal Learning for Contextual Bandits

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Abstract

We study the fundamental limits of learning in contextual bandits, where a learner's rewards depend on their actions and a known context, which extends the canonical multi-armed bandit to the case where side-information is available. We are interested in *universally consistent* algorithms, which achieve sublinear regret compared to any measurable fixed policy, without any function class restriction. For stationary contextual bandits, when the underlying reward mechanism is time-invariant, [1] characterized *learnable* context processes for which universal consistency is achievable; and further gave algorithms ensuring universal consistency whenever this is achievable, a property known as *optimistic universal consistency*. It is well understood, however, that reward mechanisms can evolve over time, possibly adversarially, and depending on the learner's actions. We show that optimistic universal learning for contextual bandits with adversarial rewards is impossible in general, contrary to all previously studied settings in online learning-including standard supervised learning. We also give necessary and sufficient conditions for universal learning under various adversarial reward models, and an exact characterization for online rewards. In particular, the set of learnable processes for these reward models is still extremely general—larger than i.i.d., stationary or ergodic—but in general strictly smaller than that for supervised learning or stationary contextual bandits, shedding light on new adversarial phenomena.

Keywords. Contextual bandits, Universal consistency, Optimistically universal learning, Online learning, Adversarial rewards, Statistical learning theory

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1 Introduction

The contextual bandit setting is a central problem in statistical decision-making. This setting models the interaction between a learner or decision maker, and a reward mechanism. At each iteration of the learning process, the learner observes a *context* $x \in \mathcal{X}$ (also known as covariate in the statistical learning literature), then selects an *action* $a \in A$ to perform. The decision maker then receives a reward based on the context and selected action, which can then be used to perform informed future actions. As a classical example, this framework can model the problem of online personalized recommendations. For any new customer, an online store provides a list of product recommendations. Based on the reward obtained from actions of the customer, e.g., if they purchase an item, the store can then update its recommendations for future customers. The major difference with the standard supervised learning framework is that the learner can only observe the reward of the selected action, referred to as partial feedback, instead of the full-feedback case of supervised learning in which a learner can directly compute the reward (or loss) of non-selected actions. Further, instead of estimating the reward mechanism, the goal in contextual bandits is to achieve low regret compared to the optimal actions in hindsight. New phenomena arise from these characteristics, including the well-known exploration/exploitation trade-off: algorithms should balance between exploiting known high-reward actions and exploring new actions which potentially could yield higher rewards. In the present work, we aim to shed light on the fundamental question of *learnability* in contextual bandits and unveil key differences from the classical full-feedback setting.

Universal consistency. We focus on the foundational notion of *consistency*. In the contextual bandit context, a learner is consistent if its long-term excess regret vanishes. Contexts are modeled by a stochastic process $\mathbb{X} = (X_t)_{t\geq 1}$. If \hat{a}_t is the selected action and r_t the reward function at time t, we ask that for any measurable policy π^* ,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t)) - r_t(\hat{a}_t) \le 0 \quad (a.s.).$$

As shown in the above equation, we follow a traditional regret analysis, where we compare the learner to a fixed policy (static regret) as opposed to switching regret where the comparison policy may also change. For robustness and generality, one commonly aims to design algorithms that ensure consistency for a large class of instances. In this paper, we consider the strongest notion of *universal consistency*, introduced in [2], which asks that a learning rule is consistent for any possible reward mechanism informally, any form of reward functions $(r_t)_{t\geq 1}$. The notion of universal consistency was mostly studied in the full-feedback supervised learning framework. In this context, a learner observes a stream of data $(X_t, Y_t)_{t\geq 1}$ and makes predictions \hat{Y}_t at each step. Thus, it is universally consistent if irrespective of the underlying mechanism relating values Y to contexts X, its average excess error compared to any measurable predictor function $f : \mathcal{X} \to \mathcal{Y}$ vanishes: $\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T} \ell(\hat{Y}_t, Y_t) - \ell(f(X_t), Y_t) \leq$ 0 (a.s.). Starting with the work of [3] which proved universal consistency for a large class of local average estimators in Euclidean spaces, a significant line of work focused on extending these results. Notably, one can achieve universal consistency for more general spaces and loss functions [4, 5]. More recently, [6–8] provided learning rules for universal learning under a provably-minimal assumption on the context space \mathcal{X} known as essential separability. While these works focused on independent identically distributed (i.i.d.) data, more restricted consistency results were also obtained for non-i.i.d. mixing, stationary ergodic data processes [5, 9, 10] or processes satisfying the law of large numbers [11–13].

Optimistic learning. Following these efforts to generalize results, a natural question arises: what are the fundamental limits of universal consistency? To answer this question, we adopt the framework of optimistic learning [2, 6, 14] which aims to study learning with *provably-minimal* assumptions. As originally introduced by [2], the notion of optimistically universal learning is motivated by the following reasoning. If we are interested in designing a learning algorithm that achieves a particular learning guarantee (in our case, universal consistency under the process X), to succeed we must necessarily assume that such a guarantee is at least *possible* (i.e., that there exists a learner achieving this guarantee). Since such an assumption typically cannot be verified empirically, making such an assumption is an act of op*timism.* As such, this is referred to as the *optimist's assumption* [2]. The main question in this literature is to determine whether there exists a learning algorithm which achieves the desired guarantee given only the assumption that it is possible to do so (in our case, this means making no additional assumptions about the process \mathbb{X}). Such a learning algorithm is said to be *optimistically universal*. Since the optimist's assumption is always *necessary* to achieve the desired guarantee, an optimistically universal learning algorithm succeeds under the *minimal* possible assumptions. Thus, in the present context, an algorithm is called optimistically universal if it is universally consistent under every process X for which there exists a universally consistent learner: that is, it *learns whenever learning is possible*. The key point is that the learner whose existence establishes that X admits universal consistency may depend on the distribution of X, whereas an optimistically universal learner must be consistent under *every* such X.

In the present work, we aim to understand whether optimistically universal learning is possible for contextual bandits under various categories of reward adversaries. It is useful first to understand and characterize the minimal assumptions for the existence of a universally consistent learning rule: that is, which processes X satisfy the optimist's assumption. Informally, we aim to characterize

 $C = \{X : \exists \text{ learning rule } f. \text{ s.t. } \forall \text{ rewards within a given model, } f. \text{ is consistent} \}.$

Second, we search for *optimistically universal* procedures: i.e., which are universally consistent under all processes where this is possible ($X \in C$). For any process X, if such an algorithm fails to be universally consistent, we are guaranteed that no other algorithm would be either.

Universal learning in contextual bandits. While the literature on universal learning in the case of full-feedback is very extensive, it is surprisingly sparse for partial-feedbacks. Previous literature mostly investigated stochastic contextual bandits under important structural assumptions on rewards, such as smoothness or margin conditions. Closest to universal learning—in which one relaxes assumptions on

the reward mechanism—[15] showed that for continuous rewards in the contexts, strong consistency can be achieved with traditional non-parametric methods, for Euclidean context spaces. [1] gave the first results for contextual bandits on universal consistency per se. They focus on stationary rewards—the underlying reward mechanism is invariant over time—and show in particular that for the main case of interest—finite action spaces \mathcal{A} —universal consistency is achievable under the same class of processes as for the noiseless full-feedback case. In contrast with previous literature, the proposed learning rules are consistent without any assumptions on the rewards, on general spaces and under large classes of noni.i.d. contexts. Further, they show that optimistically universal learning rules always exist for stationary bandits.

The present work challenges the stationarity assumption from [1]. In particular, this does not allow for changes in the underlying reward mechanism, a behavior ubiquitous in current applications. It is well-known that the distribution of contexts and rewards can shift over time, such as seasonal changes in consumer behavior and can be adversarial. Our analysis mainly focuses on two models for the strengh of the adversary: oblivious rewards for which the reward mechanism can depend on the past context history, but not the past actions of the learner; and the strongest online rewards for which the rewards can be adaptive on past contexts and selected actions. This study shows that having adversarial rewards—as opposed to stationary rewards—plays a crucial role in the fundamental limits of learnability for contextual bandits, and represents a significant advancement in the general analysis of more intricate decision-making processes, such as reinforcement learning.

1.1 Related works

Literature on optimistic supervised learning. Optimistic learning was first introduced by [2] for the *realizable* (noiseless) case when values are exactly given as $Y_t = f(X_t)$ for some unknown measurable function $f : \mathcal{X} \to \mathcal{Y}$, and provided necessary conditions and sufficient conditions for universal learning. The characterization was then completed in a subsequent line of work [14, 16, 17]. In particular, while nearest-neighbor is not consistent even for i.i.d. processes in general metric spaces [18], a simple variant with restricted memory is optimistically universal for general separable metric spaces. Notably, the corresponding class of learnable processes—which intuitively asks that the process visits sublinearly measurable partition of the ambient space—is significantly larger than previously considered relaxations of the i.i.d. assumptions. For more general noisy data generating processes [19, 20] gave complete characterizations and showed that universal learning can be achieved not only for noisy data but arbitrarily dependent values \mathbb{Y} on the contexts \mathbb{X} , possibly even adversarial to the learner's predictions. Specifically, [20] showed that under mild assumptions on the value space—including totally-bounded-metric spaces—optimistically universal learning with noisy values is possible on the exact same class of processes as for noiseless values. Hence, learning with arbitrary or adversarial responses comes at no generality expense for the full-feedback setting.

Literature on contextual bandits and non-stationarity. The concept of contextual bandits was first introduced in a limited context for single-armed bandits [21, 22]. Since then, considerable effort was made to generalize the framework and provide efficient methods under important structural assumptions on the rewards. Most of the literature considered parametric assumptions [23–28], but substantial progress has also been achieved in the non-parametric setting towards obtaining minimax guarantees under smoothness (e.g., Lipschitz) conditions or margin assumptions [29–32], with further refinements including [33, 34].

While the above-cited works mostly focus on i.i.d. data, the non-stationary case has also been studied in the literature. The fact that the reward distribution can change over time has been widely acknowledged in the established parametric setting for contextual bandits, and has been explored under various models including [35–41]. The non-parametric case, more relevant to our work has also been considered for Lipschitz rewards and margin conditions [31, 42]. We note however, that these works often consider non-static regret, where the baseline is also non-stationary, while we focus on the excess regret compared to *fixed* policies.

1.2 Summary of the present work

We mainly focus on bounded rewards. Our first main result shows that in the main case of interest of finite action spaces \mathcal{A} and separable metrizable spaces \mathcal{X} admitting a non-atomic probability measure, optimistic universal learning is impossible, even under the weakest adversarial model which we call *memoryless*: rewards conditionally on their selected action and context are independent but may follow different conditional distributions. This implies that adapting algorithms for specific context processes is necessary to ensure universal learning. This is the first example of such a phenomenon for online learning, for which previously considered settings always admitted optimistically universal learning rules, including realizable (noiseless) supervised learning [2, 14, 16], arbitrarily noisy (potentially adversarial rewards) supervised learning [19, 20], and stationary contextual bandits [1]. Intuitively, *personalization* and *generalization* are incompatible for contextual bandits with adversarial rewards.

Next, we study universally learnable processes for various adversarial reward models. On the negative side, we show that in the main case of interest, the set of learnable processes for stationary contextual bandits or supervised learning denoted C_2 is not anymore fully learnable even for memoryless rewards: learning with adversarial rewards is fundamentally more difficult. This comes as a surprising result since C_2 processes admitted universal learning in all previous learning settings. We further identify novel necessary and sufficient conditions, involving intricate behavior of duplicates in the context process. In particular, for memoryless, oblivious, and online rewards, the set of learnable processes is strictly between C_2 and a smaller class C_1 . For this same case of interest, we give an exact characterization of these learnable processes for online rewards: this characterization involves a sort of convergence rate of the instance process towards its limit distribution. Given the knowledge of this rate, universal learning is achievable with a learning rule that we provide; on the other hand, without a priori knowledge on this rate, universal learning is impossible since optimistic universal learning is not achievable. While we leave the exact characterization for memoryless and oblivious rewards as an open question for finite action spaces A and context spaces admitting a non-atomic probability measure, our characterizations in all other cases are complete.

Last, we give extensions of the above results, when the rewards are unbounded or satisfy some regularity constraints, namely uniform continuity.

1.3 Overview of contributions and techniques

Non-existence of optimistically universal learning rules. The proof involves several major steps. First, one needs to show that universal learning is achievable for a large class of processes. In particular, we show that deterministic C_2 processes are learnable, where C_2 is the characterization of learnable processes for supervised learning or stationary contextual bandits. This is achieved by assigning each distinct instance a multi-armed bandit learner designed to learn the best action for this instance, which corresponds to pure *personalization*. Next, we argue that C_1 processes—the characterization of learnable processes for countable action spaces A in stationary contextual bandits—can be learned with the same structural risk minimization approach introduced by [1] for stationary contextual bandits, which corresponds to generalization.

The main challenge is to show that one cannot universally learn both classes of processes (deterministic C_2 and C_1) with a unique algorithm. At the high level, we show that by contradiction, personalization and generalization are incompatible. We consider a C_1 -like algorithm, where instances are i.i.d. during a phase, then the same sequence is repeated many times. The reward is identical for each duplicate and has the following behavior: one *safe* action a_2 always has relatively high reward, and an *uncertain* action a_1 has random reward. We then show that because of the C_1 property, the algorithm needs to follow the safe action in order to be consistent: if it explores the uncertain action too often, the incurred loss is significant. More precisely, we show that the exploration rate of the unsafe action a_2 decays to 0. Once the algorithm reaches a certain threshold, we stop the stochastic process and consider a realization of the uncertain rewards and C_1 -like process. Once these are taken as deterministic, the optimal policy would be to use the action a_2 when it has high reward, which the algorithm did not perform. Repeating this process inductively with decaying threshold, we can show that on a deterministic C_2 process, the algorithm is not universally consistent.

New classes of stochastic processes for learning theory. We identify novel classes of processes that arise in the characterization of learnable processes. In the main case of interest, we give a new necessary condition C_4 . Informally, while C_2 processes only required that the process the process visits only a sublinear number of sets from any countable partition of the context space X, the necessary condition C_4 requires this sublinear behavior to be uniform spatially in X. Loosely speaking, when the convergence speed of the sublinear visit property is heterogeneous across space, one can take advantage of these discrepancies with adversarial rewards together with a somewhat similar personalization/generalization incompatibility phenomenon as the one described above. More precisely, if C_4 is not satisfied, locally in the context space X, one can find the following behavior: contexts are duplicated across phases of exponential time-length, for arbitrarily small exponent. One can then consider oblivious rewards—rewards that may depend on past contexts $X_{\leq t}$ but only the selected action \hat{a}_t at time t—that are identical on duplicates but with one safe and one uncertain option as above. Eventually, the algorithm's exploration rate of the uncertain action decays to 0. However, for a given fixed realization of the rewards, this is suboptimal. In this proof, the dependence of the rewards on past contexts was necessary to make sure that during each constructed phase, no information on the rewards of future local space zones is revealed.

On the positive side, we introduce a novel condition C_5 that is universally learnable, with $C_1 \subsetneq C_5$ in general. Intuitively, this asks that there is a specific rate at which we can add duplicates while still preserving the C_1 behavior. This should be related to the property observed in [1] that if we were to replace all duplicates with an arbitrary value $x_0 \in \mathcal{X}$, C_2 processes would belong to C_1 . The C_5 property provides an intermediary condition. We now briefly describe the algorithm we introduce to achieve universal consistency on C_5 processes. The learning rule heavily relies on the knowledge of the correct rate to add duplicates. For all points included within this addition rate, we can use the structural risk minimization approach since these points still have C_1 behavior. For the remaining duplicates, we use pure personalization by assigning a bandit learner to each distinct instance. In particular, all deterministic C_2 processes belong to C_5 . Further, we can show that for online rewards, condition C_5 is also necessary and as a result is an exact characterization of learnable processes in this setting. In particular, for online rewards, universal learning exactly requires the a priori knowledge of the correct rate to add duplicates.

Last, in an attempt to bridge the gap $C_5 \subsetneq C_4$ remaining for oblivious rewards, we propose a new condition C_6 on processes that is necessary for universal learning. In the general case of context spaces \mathcal{X} admitting non-atomic probability distributions, we have $C_5 \subset C_6 \subsetneq C_4$. This shows that further uniform continuity than the C_4 condition is necessary. The condition can be tightened using the same proof for a stronger type of adversary that we call prescient for which the rewards can also depend on the complete

sequence X instead of the past revealed contexts to the learner. For these rewards, we can show that a stronger C_7 —and simpler than C_6 —is necessary. We believe in general $C_7 \subsetneq C_6$ but more importantly, the question of whether $C_5 = C_7$, is open. Hence, possibly, our characterizations for prescient and stronger reward models are tight.

2 Preliminaries

Let $(\mathcal{X}, \mathcal{B})$ be a separable metrizable Borel context space and \mathcal{A} a separable metrizable Borel action space \mathcal{A} . When considering continuity assumptions, we suppose that \mathcal{A} is given with a metric d. For countable action spaces, we use the discrete topology. We are interested in the following sequential contextual bandit framework: at step $t \ge 1$, the learner observes a context $X_t \in \mathcal{X}$, then selects an action $\hat{a}_t \in \mathcal{A}$ and last, receives a reward $r_t \in \mathcal{R}$ which may be stochastic. Unless mentioned otherwise, we suppose that the rewards are bounded $\mathcal{R} = [0, \bar{r}]$ and that the upper bound \bar{r} is known. Hence, without loss of generality we may pose $\bar{r} = 1$. The learner is *online* and as such, can only use the current history to selects the action \hat{a}_t .

Definition 1 (Learning rule). A learning rule is a sequence $f_{\cdot} = (f_t)_{t \ge 1}$ of possibly randomized measurable functions $f_t : \mathcal{X}^{t-1} \times \mathcal{R}^{t-1} \times \mathcal{X} \to \mathcal{A}$. The action selected at t is $\hat{a}_t = f_t((X_s)_{s \le t-1}, (r_s)_{s \le t-1}, X_t)$.

We now precise the data generation process. We suppose that the contexts $\mathbb{X} = (X_t)_{t\geq 1}$ are generated from a general stochastic process. To define the rewards, $(r_t)_{t\geq 1}$, many models for the underlying reward mechanism are possible. [1] considered the case of *stationary* rewards when the rewards follow a conditional distribution $P_{r|a,x}$ conditionally on the selected action \hat{a}_t and the context X_t at the current time $t \geq 1$. We consider the considerably more general case of adversarial rewards. Of particular interest to the discussion of this paper will be 1. *oblivious* rewards which correspond to the case when the learner plays a game against an adversary oblivious to the player's actions and 2. *online* rewards when the adversary can choose rewards depending on the complete history of contexts, selected actions and received rewards. For a stochastic process \mathbb{X} , we will use the notation $\mathbb{X}_{\leq t} = (X_{t'})_{t' \leq t}$. Also, for a measurable set $A \in \mathcal{B}$, we will use the shorthand $\mathbb{X} \cap A = \{X_t : X_t \in A, t \geq 1\}$.

Definition 2 (Reward models). The reward mechanism is said to be

- stationary (stat.) if there is a conditional distribution $P_{r|a,x}$ such that the rewards $(r_t)_{t\geq 1}$ given their selected action a_t and context X_t are independent and follow $P_{r|a,x}$
- memoryless if there are conditional distributions $(P_{r|a,x,t})_{t\geq 1}$ such that $(r_t)_{t\geq 1}$ given their selected action a_t and context X_t are independent for $t \geq 1$ and respectively follow $P_{r|a,x,t}$
- oblivious if there are conditional distributions $(P_{r|a, \boldsymbol{x}_{\leq t}})_{t \geq 1}$ such that r_t given the selected action a_t and the past contexts $\mathbb{X}_{\leq t}$, follows $P_{r|a, \boldsymbol{x}_{\leq t}}$
- online if there are conditional distributions $(P_{r|a_{\leq t}, x_{\leq t}, r_{\leq t-1}})_{t\geq 1}$ such that r_t given the sequence of selected actions $a_{\leq t}$ and the sequence of contexts $\mathbb{X}_{\leq t}$ and received rewards $r_{\leq t-1}$, follows $P_{r|a_{\leq t}, x_{\leq t}, r_{\leq t-1}}$.

We refer to all the models except for the stationary one as *adversarial*. To emphasize the dependence of the reward in the selected action, and the conditional distributions, we may write $r_t(a \mid X_t)$, $r_t(a \mid X_{\leq t})$, $r_t(a \mid X_{\leq t})$, $r_t(a \mid X_{\leq t-1}, X_{\leq t}, r_{\leq t})$ for the corresponding reward models. When the conditioning is clear from context, we may simply write $r_t(a)$ for the reward if action a is selected.

The general goal in contextual bandits is to discover or approximate an optimal policy $\pi^* : \mathcal{X} \to \mathcal{A}$ if it exists. For adversarial rewards, there may not exist a single optimal policy π^* . Instead, we aim for consistent algorithms that have sublinear regret compared to any fixed measurable policy.

Definition 3 (Consistency and universal consistency). Let \mathbb{X} be a stochastic process on \mathcal{X} , $(r_t)_{t\geq 1}$ be a reward mechanism and f. be a learning rule. Denote by $(\hat{a}_t)_{t\geq 1}$ its selected actions. We say that f. is consistent under \mathbb{X} with rewards r if for any measurable policy $\pi^* : \mathcal{X} \to \mathcal{A}$,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t)) - r_t(\hat{a}_t) \le 0, \quad (a.s.).$$

We say that f is universally consistent for a given reward model if it is consistent under X with any reward within the considered reward model.

Even in the simplest case of full-feedback noiseless learning [2], universal consistency is not always achievable. For instance, if the process X visits a distinct instance at each step the learner, the information gathered on previous instances $X_{\leq t-1}$ does not provide information on the rewards for instance X_t . We are then interested in understanding the set of processes X on X for which universal learning is possible. More practically, we aim to provide optimistically universally consistent learning rules which, if they exist, would be universally consistent whenever this is possible.

Definition 4 (Optimistically universal learning rule). For a given reward model which we write $model \in \{stat, memoryless, oblivious, prescient, online\}$, we define

$$C_{model} = \{ \mathbb{X} : \exists \text{ learning rule universally consistent for model under } \mathbb{X} \}.$$

We say that a learning rule f. is optimistically universal for the reward model if it is universally consistent under any process $X \in C_{model}$ for that reward model.

In general $\mathcal{C}_{online} \subset \mathcal{C}_{oblivious} \subset \mathcal{C}_{memoryless} \subset \mathcal{C}_{stat}$.

2.1 Two main classes of stochastic processes

We give the definitions of two main conditions on stochastic processes arising in our characterizations of learnable processes. First, given a stochastic process X on \mathcal{X} , an extended process is given by $\tilde{X} = (X_t)_{t \in \mathcal{T}}$ where $\mathcal{T} \subset \mathbb{N}$ is a possibly random subset of times—which can depend on any random variable, the process X itself, rewards potentially observed by a learner, etc. We define the limit submeasure $\hat{\mu}_{\tilde{X}}$ as follows. For any $A \in \mathcal{B}$,

$$\hat{\mu}_{\tilde{\mathbb{X}}}(A) = \limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_A(X_t).$$

The first condition intuitively asks that the expected empirical limsup frequency of sets $A \in \mathcal{B}$ is a continuous sub-measure on \mathcal{B} .

Condition 1 (Blanchard et al. [1], Hanneke [2]). Let X be a stochastic process and $\tilde{X} = (X_t)_{t \in \mathcal{T}}$ an extended process. \tilde{X} satisfies the condition if for every monotone sequence $\{A_k\}_{k=1}^{\infty}$ of measurable subsets of X with $A_k \downarrow \emptyset$,

$$\lim_{k \to \infty} \mathbb{E}[\hat{\mu}_{\tilde{\mathbb{X}}}(A_k)] = 0.$$

We define C'_1 as the set of extended processes \tilde{X} satisfying this condition. For clarity, we also define C_1 as the set of (classical) processes X satisfying this condition (taking $T = \mathbb{N}$).

The next condition asks that X visits a sublinear number of sets of any measurable partition of \mathcal{X} .

Condition 2 (Hanneke [2]). For every sequence $\{A_k\}_{k=1}^{\infty}$ of disjoint measurable subsets of \mathcal{X} , $|\{k : \mathbb{X}_{\leq T} \cap A_k \neq \emptyset\}| = o(T)$ (a.s.). Denote by C_2 the set of all processes \mathbb{X} satisfying this condition.

Intuitively, this condition asks that the process does not keep exploring completely different regions of the space \mathcal{X} . This is known that even in the noiseless full-feedback setting, C_2 is a necessary condition for universal learning [2] since intuitively, the past history does not provide any information on newly visited regions for a learner. [2] showed that both classes above are very general classes of processes. Precisely, we have $C_1 \subset C_2$ and i.i.d. processes, stationary ergodic processes, stationary processes and processes satisfying the law of large numbers belong to C_1 .

2.2 Useful algorithms

Our learning rules will use as subroutine the following two algorithms. First, we will use the algorithm EXP3.IX for regret bounds with high-probability in adversarial bandits.

Theorem 5 ([43]). *There exists an algorithm* EXP3.IX *for adversarial multi-armed bandit with* $K \ge 2$ *arms such that for any* $\delta \in (0, 1)$ *and* $T \ge 1$ *,*

$$\max_{i \in [K]} \sum_{t=1}^{T} (r_t(a_i) - r_t(\hat{a}_t)) \le 4\sqrt{KT \ln K} + \left(2\sqrt{\frac{KT}{\ln K}} + 1\right) \ln \frac{2}{\delta},$$

with probability at least $1 - \delta$.

We will always use a very simplified version of this result: there exists a universal constant c > 0 such that

$$\max_{i \in [K]} \sum_{t=1}^{T} (r_t(a_i) - r_t(\hat{a}_t)) \le c\sqrt{KT \ln K} \ln \frac{1}{\delta},$$

with probability $1 - \delta$ for $\delta \leq \frac{1}{2}$. Second, we use the EXPINF algorithm from [1] which uses EXP3.IX as subroutine to achieve sublinear regret compared to an infinite countable sequence of experts.

Theorem 6 ([1]). There is an online learning rule EXPINF using bandit feedback such that for any countably infinite set of experts $\{E_1, E_2, \ldots\}$ (possibly randomized), for any $T \ge 1$ and $0 < \delta \le \frac{1}{2}$, with probability at least $1 - \delta$,

$$\max_{1 \le i \le T^{1/8}} \sum_{t=1}^{T} \left(r_t(E_{i,t}) - r_t(\hat{a}_t) \right) \le c T^{3/4} \sqrt{\ln T} \ln \frac{T}{\delta}.$$

where c > 0 is a universal constant. Further, with probability one on the learning and the experts, there exists \hat{T} such that for any $T \ge 1$,

$$\max_{1 \le i \le T^{1/8}} \sum_{t=1}^{T} \left(r_t(E_{i,t}) - r_t(\hat{a}_t) \right) \le \hat{T} + cT^{3/4} \sqrt{\ln T} \ln T.$$

3 Statement of results

Our first main result is that for contextual bandits with adversarial rewards, for generic metric spaces \mathcal{X} —that admit a non-atomic probability measure, e.g., any uncountable Polish space—there never exists an optimistically universal learning rule. On the other hand, if \mathcal{X} does not admit a non-atomic probability measure, optimistic learning is possible.

Theorem 7. Let \mathcal{X} be a separable metrizable Borel space.

- 1. Let \mathcal{A} be a finite action space with $|\mathcal{A}| \geq 2$.
 - If X admits a non-atomic probability measure, there does not exist an optimistically universal learning rule for any adversarial reward model considered in Definition 2 (i.e., all except stationary).
 - Otherwise, there exists an optimistically universal learning rule for all reward models from Definition 2 and C_{online} = C_{stat} = C₂.
- 2. Let A be a countably infinite action space, there exists an optimistically universal learning rule for all reward models from Definition 2 and $C_{online} = C_{stat} = C_1$.
- 3. Let \mathcal{A} be an uncountable separable metrizable Borel space, then universal learning is never achievable and $C_{online} = C_{stat} = \emptyset$.

The question of whether optimistic learning is possible for finite action spaces is answered in Section 4. The case of infinite action spaces is treated in Section 6.1. Thus, Theorem 7 is a concatenation of Theorems 15 and 16 and Section 6.1.

The fact that optimistic learning is impossible the main case of finite action space and spaces \mathcal{X} admitting a non-atomic probability measure comes in stark contrast with all learning frameworks that have been studied in the universal learning literature. Namely, for the noiseless full-feedback [2, 14], noisy/adversarial full-feedback [20] and stationary partial-feedback [1] learning frameworks, analysis showed that there always existed an optimistically universal learning rule. Precisely, the optimistically universal learning rule for stationary contextual bandits in finite action spaces provided by [1] combined two strategies:

- A strategy 0, which treats each distinct context completely separately by assigning a distinct bandit subroutine to each new instance. Informally, this corresponds to learning the optimal action for each new context without gathering population information.
- A strategy 1, in which the learning rule views context in an aggregate fashion: it tries to fit the policy which performed best on the complete historical data using learning-with-experts subroutines, from a set of pre-defined policies.

The procedure to combine these strategies estimates their performance, to implement the best strategy during pre-defined periods. We show that for adversarial rewards, balancing these two strategies is impossible. In particular, an adversarial reward mechanism can fool the estimation procedure by changing behavior between the estimation period and the implementation period.

The non-existence of an optimistically universal learning rule also provides another proof that model selection is impossible for contextual bandits. A formulation of this question was posed as a COLT 2020 open problem [44]. The impossibility of model selection was then recently proved first with a switching bandit problem [45]. Our results show this general impossibility in a completely different context. More

precisely, Proposition 8 below shows that universal consistency up to a fixed error tolerance $\epsilon > 0$ is always achievable under C_2 processes (which were necessary for universal learning even in the stationary case [1]). However, Theorem 7 implies that combining these learning rules for decaying ϵ to achieve vanishing excess error is not possible in general.

Proposition 8. Let \mathcal{X} be a separable metrizable Borel space and \mathcal{A} a finite action space. For any $\epsilon > 0$, there exists a learning rule f_{\cdot}^{ϵ} such that for any process $\mathbb{X} \in C_2$ and adversarial reward mechanism $(r_t)_{t\geq 1}$, for any measurable policy $\pi^* : \mathcal{X} \to \mathcal{A}$,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t)) - r_t(\hat{a}_t(\epsilon)) \le \epsilon, \quad (a.s.),$$

where $\hat{a}_t(\epsilon)$ denotes the action selected by the learning rule at time t.

The proof is given in Section 5.3. Theorem 7 provides the characterizations of universally learnable processes in all cases except the main case of interest when A is finite and \mathcal{X} admits a non-atomic probability measure. Giving exact characterizations for this case is rather complex and in the following, we only give necessary conditions and sufficient conditions. These require the introduction of novel classes of stochastic processes for online learning.

3.1 Additional classes of stochastic processes

We first give a significantly stronger assumption asking that the process only visits a finite number of distinct points. This very restrictive condition will only arise for unbounded rewards $\mathcal{R} = [0, \infty)$.

Condition 3 (Hanneke [2], Blanchard et al. [16]). $|\{x : X \cap \{x\} \neq \emptyset\}| < \infty$ (*a.s.*). Denote by C_3 the set of all processes X satisfying this condition.

We then introduce two novel conditions on stochastic processes. Before doing so, we need to introduce some exponential time scales. Intuitively, for $\alpha > 0$, the exponential time scale at rate α is the sequence of times given by $T^k(\alpha) \approx \lfloor (1+\alpha)^k \rfloor$ for $k \ge 0$. For convenience, we will instead consider for all integers $i \ge 0$ the sequence of times $T_i^k = \lfloor 2^u(1+v2^{-i}) \rfloor$ where $k = u2^i + v$ and $u \ge 0, 0 \le v < 2^i$ are integers. In particular, $u = \lfloor k2^{-i} \rfloor$ and $v = k \mod 2^i$. These times have an exponential behavior with rate oscillating between 2^{-i-1} and 2^{-i} but conveniently, they form periods $[T_i^k, T_i^{k+1})$ which become finer as *i* increases. For $t \ge 1$, we then define $k_i(t)$ as the index *k* such that $t \in [T_i^k, T_i^{k+1})$. This allows to consider the set of times *t* such that X_t is the first appearance of the instance on its period,

$$\mathcal{T}^{i} = \{ t \ge 1 : \forall T_{i}^{k_{i}(t)} \le t' < t, X_{t'} \neq X_{t} \}.$$

By construction, note that $\mathcal{T}^i \subset \mathcal{T}^{i+1}$ for all $i \geq 0$. We are now ready to define the next condition which intuitively asks that the process has a \mathcal{C}'_1 behavior uniformly at any exponential scale.

Condition 4. For any sequence of disjoint measurable sets $(A_i)_{i>1}$ of \mathcal{X} , we have

$$\lim_{i \to \infty} \mathbb{E} \left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^i} \mathbb{1}_{A_i}(X_t) \right] = 0.$$

Denote by C_4 the set of all processes X satisfying this condition.

Then, we define the next condition which asks that there exists a rate to include decreasing exponential scales while conserving the C'_1 property.

Condition 5. There exists an increasing sequence of integers $(T_i)_{i>0}$ such that letting

$$\mathcal{T} = \bigcup_{i \ge 0} \mathcal{T}^i \cap \{t \ge T_i\},\$$

we have $\tilde{\mathbb{X}} = (X_t)_{t \in \mathcal{T}} \in \mathcal{C}'_1$. Denote by \mathcal{C}_5 the set of all processes \mathbb{X} satisfying this condition.

We now introduce two new conditions on stochastic processes which we will show are necessary for some of the considered reward models. These build upon the definition of C_4 processes. Before introducing them, we need to analyze large deviations of the empirical measure in C'_1 processes. The next lemma intuitively shows that for a process $\tilde{\mathbb{X}} \in C'_1$, for large enough time steps, one can bound the deviations of the empirical measure of a set $A \in \mathcal{B}$ compared to the limit sub-measure $\hat{\mu}_{\mathbb{X}}(A)$ uniformly in the set A.

Lemma 9. Let \mathbb{X} be a stochastic process on \mathcal{X} and \mathcal{T} some random times such that $\tilde{\mathbb{X}} = (X_t)_{t \in \mathcal{T}} \in \mathcal{C}'_1$. Then, for any $\epsilon > 0$, there exists $T_{\epsilon} \ge 1$ and $\delta > 0$ such that for any measurable set $A \in \mathcal{B}$,

$$\mathbb{E}[\hat{\mu}_{\tilde{\mathbb{X}}}(A)] \leq \delta \Longrightarrow \mathbb{E}\left[\sup_{T \geq T_{\epsilon}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} \mathbb{1}_{A}(X_{t})\right] \leq \epsilon.$$

Now consider a process $\mathbb{X} \in \mathcal{C}_4$. For any integer $p \geq 0$, the definition of \mathcal{C}_4 implies $\mathbb{X}^p := (X_t)_{t \in \mathcal{T}^p} \in \mathcal{C}'_1$. Indeed, the sets \mathcal{T}^i are increasing in $i \geq 0$, hence for $i \geq p$ one has $\mathcal{T}^p \subset \mathcal{T}^i$. As a result, Condition 4 implies that for any disjoint measurable sets $(A_i)_{i\geq 1}$, one has $\mathbb{E}[\hat{\mu}_{\mathbb{X}^p}(A_i)] = \mathbb{E}[\limsup_{T\to\infty} \sum_{t\leq T,t\in\mathcal{T}^p} \mathbb{1}_{A_i}(X_t)] \to 0$ as $i \to \infty$. Now for any $\epsilon > 0$ and $T \geq 1$, we define

$$\begin{split} \delta^{p}(\epsilon;T) &:= \sup \left\{ 0 \leq \delta \leq 1 : \forall A \in \mathcal{B} \text{ s.t. } \sup_{l} \mathbb{E}[\hat{\mu}_{\mathbb{X}^{l}}(A)] \leq \delta, \\ \forall \tau \geq T \text{ online stopping time, } \mathbb{E}\left[\frac{1}{2\tau} \sum_{\tau \leq t < 2\tau, t \in \mathcal{T}^{p}} \mathbb{1}_{A}(X_{t})\right] \leq \epsilon \right\}, \end{split}$$

where the τ is a stopping time with respect to the filtration generated by the instance process X. In particular, τ can be seen as an online procedure which decides when to count the number of instances of \mathbb{X}^p falling in the considered set A. Note that $\delta^p(\epsilon; T)$ satisfies the property that for all measurable set A satisfying $\sup_l \mathbb{E}[\hat{\mu}_{\mathbb{X}^l}(A)] \leq \delta^p(\epsilon; T)$ and any stopping time $\tau \geq T$,

$$\mathbb{E}\left[\frac{1}{2\tau}\sum_{\tau\leq t<2\tau,t\in\mathcal{T}^p}\mathbb{1}_A(X_t)\right]\leq\epsilon,$$

which can be checked for all sets $A \in \mathcal{B}$ separately. Next, the quantity $\delta^p(\epsilon; T)$ is non-decreasing in T. Further, as a direct application of Lemma 9, because $\mathbb{X}^p \in \mathcal{C}'_1$, there exists $T^p(\epsilon) \ge 1$ and $\delta > 0$ such that for $T \ge T^p(\epsilon)$, we have $\delta^p(\epsilon; T) \ge \delta$. As a result, we have $\delta^p(\epsilon) := \lim_{T\to\infty} \delta^p(\epsilon; T) \ge \delta > 0$. Also, the quantity $\delta^p(\epsilon; T)$ is non-increasing in p since the sets \mathcal{T}^p are non-decreasing with p. Thus, $\delta^p(\epsilon)$ is also non-increasing in p. We are now ready to introduce the condition on stochastic processes based on the limit of the quantities $\delta^p(\epsilon)$.

	Stationary contextual bandits [1]		Contextual bandits with	
Learning setting			adversarial rewards [This paper]	
	\mathcal{C}_{stat}	OL?	Necessary and sufficient conditions on C	OL?
Finite $\mathcal{A}, \mathcal{A} \ge 2, \mathcal{X}$ with non-atomic proba. measure	\mathcal{C}_2	Yes	$\mathcal{C}_1 \subsetneq \mathcal{C}_5 \subset \mathcal{C} \subsetneq \mathcal{C}_2 \ \mathcal{C}_5 = \mathcal{C}_{online} \subset \mathcal{C}_{oblivious} \subset \mathcal{C}_6$	No
Finite $\mathcal{A}, \mathcal{A} \geq 2, \mathcal{X}$ without non-atomic proba. measure	\mathcal{C}_2	Yes	$\mathcal{C}=\mathcal{C}_2$	Yes
Countably infinite \mathcal{A}	\mathcal{C}_1	Yes	$\mathcal{C}=\mathcal{C}_1$	Yes
Uncountable \mathcal{A}	Ø	N/A	$\mathcal{C}=\emptyset$	N/A

Table 1: Characterization of learnable processes for universal learning in contextual bandits, depending on the action space A, context space X and reward model. When the model is not specified, C refers to any of the considered models. OL? = Is optimistic learning possible?

Condition 6. $X \in C_4$ and for any $\epsilon > 0$, we have $\lim_{p\to\infty} \delta^p(\epsilon) > 0$. Denote by C_6 the set of all processes X satisfying this condition.

Intuitively, this asks that the maximum deviations are also bounded in p, hence C_6 processes have more regularity than general C_4 processes. However, the maximum deviations are limited by the fact that they should be discernible through an online stopping time τ .

The following inclusions hold $C_3 \subset C_1 \subset C_5 \subset C_6 \subset C_4 \subset C_2$. Indeed, the inclusion $C_3 \subset C_1$ is known [2]. $C_1 \subset C_5$ and $C_6 \subset C_4$ are immediate from the definition of Condition 5 and Condition 6 respectively. The inclusion $C_4 \subset C_2$ is shown in Proposition 19. Last, the fact that for oblivious rewards, C_6 is necessary (Theorem 22) and C_5 is sufficient (Theorem 29) shows that $C_5 \subset C_6$.

3.2 Necessary and sufficient conditions for universal learning

Our second main contribution is giving necessary and sufficient conditions for universal learning with adversarial rewards. In addition to characterizations from Theorem 7, we have the following.

Theorem 10. Let \mathcal{X} be a separable metrizable Borel space admitting a non-atomic probability measure and \mathcal{A} a finite action space with $|\mathcal{A}| \geq 2$. Then $\mathcal{C}_1 \subsetneq \mathcal{C}_5 = \mathcal{C}_{online} \subset \mathcal{C}_{oblivious} \subset \mathcal{C}_{memoryless} \subsetneq \mathcal{C}_2$. Further, $\mathcal{C}_{oblivious} \subset \mathcal{C}_6 \subsetneq \mathcal{C}_2$.

These results are proved in Section 5. The fact that $C_{memoryless} \subseteq C_2$ is proved in Theorem 20. $C_{oblivious} \subset C_6$ is proved in Theorem 22 while $C_6 \subseteq C_2$ comes from Theorem 20 and the fact that $C_6 \subset C_4$ (Theorem 23 further gives an example of processes in $C_4 \setminus C_6$). $C_{online} \subset C_5$ is proved in Theorem 27 and $C_1 \subseteq C_5 \subset C_{online}$ is proved in Theorem 29 and Proposition 30. Here is the overview of relations we show between the classes of processes: for \mathcal{X} admitting non-atomic probability measures, $C_1 \subseteq C_5 \subset C_6 \subseteq C_4 \subseteq C_2$.

In particular, our characterization is complete for the strongest online rewards, unlike for memoryless and oblivious rewards. We believe that $C_5 \subsetneq C_6$ in general. In fact, the proof of Theorem 22 for the necessity of C_6 for oblivious rewards can be tightened given a stronger reward model in which the reward adversary can additionally take into account the *complete* sequence X—instead of the revealed contexts to the learner $X_{\leq t}$. We refer to this reward model as *prescient* rewards (see Definition 24 for a formal definition) and show that in this case, a stronger C_7 condition is necessary (Theorem 25). We leave open the question of whether $C_5 = C_7$. If this were true, then we also have an exact characterization for prescient rewards. Our findings are summarized in Table 1, which also compares learnable processes for stationary and adversarial contextual bandits. We leave open the exact characterization of learnable processes for memoryless and oblivious rewards in finite action spaces A and context spaces admitting a non-atomic probability measure.

Open question: Let \mathcal{X} be a separable metrizable Borel space admitting a non-atomic probability measure and \mathcal{A} a finite action space with $|\mathcal{A}| \geq 2$. What is an exact characterization of $C_{memoryless}$ or $C_{oblivious}$?

Finally, we also give results in a setting where we assume that rewards are unbounded. We answer the same questions: what are the learnable processes for which universal learning is possible, and can we obtain optimistically universal learning rules? We use a subscript $C^{unbounded}$ to specify that we consider the case of unbounded rewards. We show that in that case, results are identical to the case of stationary contextual bandits.

Proposition 11. Let \mathcal{X} be a separable metrizable Borel space. For all reward models,

- if A is uncountable, $C^{unbounded} = C_3$ for all reward models. Further, there is an optimistically universal learning rule,
- *if* A *is uncountable, universal learning for unbounded rewards is never achievable.*

Last, we extend our results to rewards with additional regularity assumptions. For a given metric d on A, we suppose that they are uniformly-continuous, generalizing a notion introduced in [1].

Let (\mathcal{A}, d) be a separable metric space. The reward mechanism $(r_t)_{t\geq 1}$ is uniformly-continuous if for any $\epsilon > 0$, there exists $\Delta(\epsilon) > 0$ such that

$$\begin{aligned} \forall t \geq 1, \forall (\boldsymbol{x}_{\leq t}, \boldsymbol{a}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}) \in \mathcal{X}^{t} \times \mathcal{A}^{t-1} \times \mathcal{R}^{t-1}, \forall a, a' \in \mathcal{A}, \\ d(a, a') \leq \Delta(\epsilon) \Rightarrow \left| \mathbb{E}[r_t(a) - r_t(a') \mid \mathbb{X}_{\leq t} = \boldsymbol{x}_{\leq t}, \boldsymbol{a}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}] \right| \leq \epsilon, \end{aligned}$$

For uniformly-continuous rewards we use a reduction to the case of rewards without regularity assumptions, which we refer to as *unrestricted* rewards. Then, we recover the same results for uniformlycontinuous rewards, in totally-bounded (resp. non-totally-bounded) action spaces as for unrestricted rewards in finite (resp. countably infinite) action spaces. We adopt the subscript C^{uc} to emphasize that we consider uniformly-continuous rewards.

Theorem 12. Let \mathcal{X} be a metrizable Borel space and $model \in \{memoryless, oblivious, online\}$.

- If \mathcal{A} is a totally-bounded metric space, all properties for \mathcal{C}_{model} for finite action spaces described in Theorem 10 hold for \mathcal{C}_{model}^{uc} . Further, there is an optimistically universal learning rule for uniformly-continuous rewards if and only if there is one for finite action spaces for unrestricted rewards as in Theorem 7.
- If \mathcal{A} is a non-totally-bounded metric space, all properties for \mathcal{C}_{model} for countable action spaces described in Theorem 10 hold for \mathcal{C}_{model}^{uc} . Further, there is always an optimistically universal learning rule for uniformly-continuous rewards.

This result is proved in Section 6.3 and is a concatenation of Proposition 33 for necessary conditions and Theorem 35 and Theorem 36 for sufficient conditions for universal learning.

4 Existence or non-existence of an optimistically universal learning rule

In this section, we ask the question of whether there exists an optimistically universal learning rule for finite action spaces. In fact, in all the frameworks considered for universal learning—noiseless [14] or noisy/adversarial responses [20] in the full-feedback setting and stationary partial-feedback responses [1]—analysis showed that optimistically universal learning always existed. However, the learning rule provided by [1] for stationary rewards under C_2 processes heavily relies on the assumption that the rewards are stationary in order to make good estimates of the performance of different learning strategies. In particular, one can easily check that this learning rule would not be universally consistent under adversarial rewards even in the weakest memoryless setting. Instead, we will show that for contextual bandits with adversarial rewards, in general there does not exist optimistically universal learning rules.

To do so, we first need to argue that the set of learnable processes even in the online setting C_{online} contains a reasonably large class of processes. We first show that using the EXP3.IX algorithm for adversarial bandits [43] as subroutine yields a universally consistent learning rule for processes X which visit a sublinear number of distinct instances.

Proposition 13. Let \mathcal{X} be a metrizable separable Borel space and \mathcal{A} a finite action space. There exists a learning rule which is universally consistent for online rewards under any process \mathbb{X} satisfying $|\{x \in \mathcal{X} : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\}| = o(T)$ (a.s.).

Proof. Consider the learning rule f. which simply performs independent copies of the EXP3.IX algorithm in parallel such that to each distinct instance visited is assigned a EXP3.IX. More precisely, for any $t \ge 1$, instances $x_{\le t}$ and observed rewards $r_{\le t-1}$, we define

$$f_t(\boldsymbol{x}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}, x_t) = \text{EXP3.IX}(\hat{\boldsymbol{a}}_{S_t}, \boldsymbol{r}_{S_t}),$$

where $S_t = \{t' < t : x_{t'} = x_t\}$ is the set of times that x_t was visited previously and $\hat{a}_{t'}$ denotes the action selected at time t' for t' < t. We now show that this learning rule is universally consistent on any process \mathbb{X} which visits a sublinear number of distinct instances almost surely. For simplicity we denote \hat{a}_t the action selected by f. at time t. Let \mathbb{X} such that almost surely, $\frac{1}{T} | \{x \in \mathcal{X} : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\} | \to 0$. Denote by \mathcal{E} this event, and for any $T \ge 1$ we define $\epsilon(T) = \frac{1}{T} | \{x \in \mathcal{X} : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\} |$ and $S_T = \{x \in \mathcal{X} : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\}$, hence $|S_T| = T\epsilon(T)$. Further, for any $x \in S_T$ we pose $\mathcal{T}_T(x) = \{t \le T : X_t = x\}$. Let $\mathcal{H}_0(T) = \{x \in S_T : |\mathcal{T}_T(x)| < \frac{1}{\sqrt{\epsilon(T)}}\}, \mathcal{H}_1(T) = \{x \in S_T : \frac{1}{\sqrt{\epsilon(T)}} \le |\mathcal{T}_T(x)| < \ln^2 T\}$ and $\mathcal{H}_2(T) = \{x \in S_T : |\mathcal{T}_T(x)| \ge \ln^2 T\}$, so that $S_T = \mathcal{H}_0(T) \cup \mathcal{H}_1(T) \cup \mathcal{H}_2(T)$. Note that

$$\sum_{x \in \mathcal{H}_0(T)} \sum_{t \in \mathcal{T}_T(x)} r_t(\pi(X_t)) - r_t(\hat{a}_t) \le \frac{|\mathcal{H}_0(T)|}{\sqrt{\epsilon(T)}} \le \sqrt{\epsilon(T)}T.$$

Now fix a measurable policy $\pi : \mathcal{X} \to \mathcal{A}$. Then,

$$\sum_{x \in \mathcal{H}_2(T)} \sum_{t \in \mathcal{T}_T(x)} r_t(\pi(X_t)) - r_t(\hat{a}_t) \le \sum_{x \in \mathcal{H}_2(T)} \max_{a \in \mathcal{A}} \sum_{t \in \mathcal{T}_T(x)} (r_t(a) - r_t(\hat{a}_t)).$$

Now recall that for any $x \in S_T$, on $\mathcal{T}_T(x)$ the algorithm EXP3.IX was performed. As a result, by Theorem 5, conditionally on the realization \mathbb{X} , for any $x \in \mathcal{H}_2(T)$, with probability $1 - \frac{1}{T^3}$, conditionally on \mathbb{X} ,

$$\max_{a \in \mathcal{A}} \sum_{t \in \mathcal{T}_T(x)} (r_t(a) - r_t(\hat{a}_t)) \le 3c\sqrt{|\mathcal{A}||\mathcal{T}_T(x)|\ln|\mathcal{A}|} \ln T \le |\mathcal{T}_T(x)| \cdot 3c\frac{\sqrt{|\mathcal{A}|\ln|\mathcal{A}|}}{\ln T}.$$

Noting that $|\mathcal{H}_2(T)| \leq T$, we obtain by the union bound that (conditionally on X) with with probability $1 - \frac{1}{T^2}$,

$$\sum_{x \in \mathcal{H}_2(T)} \max_{a \in \mathcal{A}} \sum_{t \in \mathcal{T}_T(x)} \left(r_t(a) - r_t(\hat{a}_t) \right) \le 3c \frac{\sqrt{|\mathcal{A}| \ln |\mathcal{A}|}}{\ln T} \sum_{x \in \mathcal{H}_2(T)} |\mathcal{T}_T(x)| \le 3c \sqrt{|\mathcal{A}| \ln |\mathcal{A}|} \frac{T}{\ln T}$$

We denote by \mathcal{F}_T the event when the above equation holds. We have $\mathbb{P}[\mathcal{F}_T] \geq 1 - \frac{1}{T^2}$ where the probability is also taken over X. We now turn to points in $\mathcal{H}_1(T)$ for which we need to go back to the proof of Theorem 5 from [43]. Taking the same notations as in the original proof, for $u \geq 1$, let $\eta_u = 2\gamma_u = \sqrt{\frac{\ln |\mathcal{A}|}{|\mathcal{A}|u}}$, and for any $t \geq 1$, $a \in \mathcal{A}$ denote by $p_{t,a}$ the probability that the learning rule selects action a at time t, and let $\ell_{t,a} = 1 - r_t(a)$. Next, let $u(t) = |\{s \leq t : X_s = X_t\}|$ and pose $\tilde{\ell}_{t,a} = \frac{1 - r_t(a)}{p_{t,a} + \gamma_u} \mathbb{1}[\hat{a}_t = a]$. Using the derivations of the proof of Theorem 5, for any $x \in S_T$, writing $\mathcal{T}_T(x) = \{t_1(x), \ldots, t_{|\mathcal{T}_T(x)|}\}$, for any $a' \in \mathcal{A}$,

$$\sum_{u=1}^{|\mathcal{T}_T(x)|} \left(\ell_{t_u,\hat{a}} - \tilde{\ell}_{t_u,a'} \right) \le \frac{\ln |\mathcal{A}|}{\eta_{|\mathcal{T}_T(x)|}} + \sum_{u=1}^{|\mathcal{T}_T(x)|} \eta_u \sum_{a \in \mathcal{A}} \tilde{\ell}_{t_u,a}$$

Summing these equations with $a' = \pi(x)$, we obtain

$$\sum_{x \in \mathcal{H}_1(T)} \sum_{t \in \mathcal{T}_T(x)} (1 - \tilde{\ell}_{t,\pi(X_t)}) - r_t(\hat{a}_t) \le \sum_{x \in \mathcal{H}_1(T)} \sqrt{|\mathcal{A}| \ln |\mathcal{A}| |\mathcal{T}_T(x)|} + \sum_{x \in \mathcal{H}_1(T)} \sum_{t \in \mathcal{T}_T(x)} \eta_{u(t)} \sum_{a \in \mathcal{A}} \tilde{\ell}_{t,a}.$$

Now let for any $a \in \mathcal{A}$, conditionally on \mathbb{X} , the sequence $(\sum_{x \in \mathcal{H}_1(T')} \sum_{t \in \mathcal{T}_{T'}(x)} \eta_{u(t)}(\tilde{\ell}_{t,a} - \ell_{t,a}))_{T' \leq T}$ is a super-martingale (the immediate expected value of $\tilde{\ell}_{t,a}$ is $\frac{p_{u(t)}}{p_{u(t)} + \gamma_{u(t)}} \ell_{t,a}$) and each increment is upperbounded by 2 in absolute value: $0 \leq \eta_{u(t)} \tilde{\ell}_{t,a} \leq \eta_{u(t)} \frac{\ell_{t,a}}{p_{u(t),a} + \gamma_{u(t)}} \leq \frac{\eta_{u(t)}}{\gamma_{u(t)}} \leq 2$. Therefore, Azuma's inequality implies

$$\mathbb{P}\left[\sum_{x\in\mathcal{H}_1(T)}\sum_{t\in\mathcal{T}_T(x)}\eta_{u(t)}\sum_{a\in\mathcal{A}}(\tilde{\ell}_{t,a}-\ell_{t,a})\leq 4T^{3/4}\mid\mathbb{X}\right]\geq 1-e^{-2\sqrt{T}}.$$

Similarly, because $0 \leq \tilde{\ell}_{t,a} \leq \frac{1}{\gamma_{u(t)}} = 2\sqrt{\frac{|\mathcal{A}|u(t)}{\ln |\mathcal{A}|}}$, we have

$$\mathbb{P}\left[\sum_{x\in\mathcal{H}_1(T)}\sum_{t\in\mathcal{T}_T(x)}\sum_{a\in\mathcal{A}}(\tilde{\ell}_{t,\pi(X_t)}-\ell_{t,\pi(X_t)})\leq 4\sqrt{\frac{|\mathcal{A}|}{\ln|\mathcal{A}|}}T^{3/4}\ln T\mid\mathbb{X}\right]\geq 1-e^{-2\sqrt{T}}.$$

As a result, on an event \mathcal{G}_T of probability at least $1 - (1 + |\mathcal{A}|)e^{-2\sqrt{T}}$, we have

$$\begin{split} \sum_{x \in \mathcal{H}_{1}(T)} \sum_{t \in \mathcal{T}_{T}(x)} r_{t}(\pi(X_{t})) - r_{t}(\hat{a}_{t}) &\leq \sum_{x \in \mathcal{H}_{1}(T)} \sqrt{|\mathcal{A}| \ln |\mathcal{A}| |\mathcal{T}_{T}(x)|} + \sum_{x \in \mathcal{H}_{1}(T)} \sum_{t \in \mathcal{T}_{T}(x)} \eta_{u(t)} \sum_{a \in \mathcal{A}} \ell_{t,a} \\ &+ 4\sqrt{\frac{|\mathcal{A}|}{\ln |\mathcal{A}|}} T^{3/4} \ln T + 4T^{3/4} \\ &\leq \sum_{x \in \mathcal{H}_{1}(T)} \sqrt{|\mathcal{A}| \ln |\mathcal{A}| |\mathcal{T}_{T}(x)|} + \sum_{x \in \mathcal{H}_{1}(T)} |\mathcal{A}| \sum_{t \in \mathcal{T}_{T}(x)} \eta_{u(t)} \\ &+ 4\sqrt{\frac{|\mathcal{A}|}{\ln |\mathcal{A}|}} T^{3/4} \ln T + 4T^{3/4} \\ &\leq \sum_{x \in \mathcal{H}_{1}(T)} 3\sqrt{|\mathcal{A}| \ln |\mathcal{A}| |\mathcal{T}_{T}(x)|} + 8\sqrt{|\mathcal{A}|} T^{3/4} \ln T \\ &\leq 3\sqrt{|\mathcal{A}| \ln |\mathcal{A}|} \epsilon(T)^{1/4} T + 8\sqrt{|\mathcal{A}|} T^{3/4} \ln T. \end{split}$$

Combining all our estimates, we showed that on $\mathcal{F}_T \cap \mathcal{G}_T$,

$$\sum_{t \le T} r_t(\pi(X_t)) - r_t(\hat{a}_t) \le 8|\mathcal{A}| T^{3/4} \ln T + 3c\sqrt{|\mathcal{A}| \ln |\mathcal{A}|} \frac{T}{\ln T} + (\sqrt{\epsilon(T)} + 3\sqrt{|\mathcal{A}| \ln |\mathcal{A}|} \epsilon(T)^{1/4}) T$$

Now note that $\sum_{T\geq 1} \mathbb{P}[\mathcal{F}_T^c] + \mathbb{P}[\mathcal{G}_T^c] < \infty$. Hence, the Borel-Cantelli lemma implies that on an event \mathcal{A} of probability one, there exists $\hat{T} \geq 1$ such that for any $T \geq \hat{T}$, the event $\mathcal{F}_T \cap \mathcal{G}_T$ is satisfied. As a result, on the event $\mathcal{E} \cap \mathcal{A}$, since $\epsilon(T) \to 0$, we obtain

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi(X_t)) - r_t(\hat{a}_t) \le 0.$$

By union bound, $\mathcal{E} \cap \mathcal{A}$ has probability one, hence we proved that the learning rule f is universally consistent on \mathbb{X} . This ends the proof of the proposition.

As a simple consequence of Proposition 13, deterministic C_2 processes are always universally learnable even in the online rewards setting.

Proposition 14. Let \mathcal{X} be a metrizable separable Borel space and \mathcal{A} a finite action space. There exists a learning rule which is universally consistent for any deterministic process $\mathbb{X} \in C_2$ under online rewards.

Proof. We first show that any deterministic process $X \in C_2$ visits a sublinear number of distinct instances almost surely. Denote $S_T = \{X_t : t \leq T\}$ the set of visited instances until time T and let $S = \bigcup_{T \to \infty} S_T$. Then, $\{x\}_{x \in S}$ forms a countable sequence of disjoint sets. Hence, by the C_2 property and because X is deterministic, we have that

$$|\{x : \{x\} \cap \mathbb{X}_{\le T} \neq \emptyset\}| = |S_t| = |\{x \in S : \{x\} \cap \mathbb{X}_{\le T} \neq \emptyset\}| = o(T), \quad (a.s.).$$

Hence, by Proposition 13, the learning rule which performs EXP3.IX independently for each distinct visited instance is universally consistent under X. This ends the proof of the proposition. ■

Next, we argue that C_1 processes are also universally learnable in the online rewards setting. In the case of countable action sets A, [1] gave a universally consistent learning rule EXPINF under C_1 processes using Theorem 6. Precisely, the learning rule uses a result from [2] showing that there exists a countable set of policies $\Pi = \{\pi^i : X \to A, i \ge 1\}$ that is empirically dense within measurable policies under any C_1 process. As a result, to yield a universally consistent learning rule under C_1 processes, it suffices to have a learning rule with sublinear regret compared to any policy $\pi \in \Pi$. The algorithm EXPINF achieves this property using restarted EXP3.IX subroutines with slowly increasing finite set of experts from the sequence Π . Because the subroutines EXP3.IX have guarantees in the adversarial bandit framework, EXPINF directly inherits this guarantee and is a result universally consistent under C_1 processes for online rewards. Thus, $C_1 \subset C_{online}$.

We are now ready to show that for spaces \mathcal{X} on which there exists a non-atomic probability measure on the space \mathcal{X} , there does not exist any optimistically universally consistent learning rule. Precisely, we show that there is no learning rule that is universally consistent both on \mathcal{C}_1 and deterministic \mathcal{C}_2 processes. Note that most context spaces \mathcal{X} of interest would admit a non-atomic probability measure, in particular any uncountable Polish space.

Theorem 15. Let \mathcal{X} a metrizable separable Borel space such that there exists a non-atomic probability measure μ on \mathcal{X} , i.e., such that $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$. If \mathcal{A} is a finite action space with $|\mathcal{A}| \ge 2$, then there does not exist an optimistically universal learning rule for memoryless rewards (a fortiori for oblivious, prescient or online rewards).

Proof. We fix $a_1, a_2 \in A$ two distinct actions. Suppose that there exists an optimistically universal learning rule f_{\cdot} . For simplicity, we will denote by \hat{a}_t the action chosen by this learning rule at step t. We will construct a deterministic process $\mathbb{X} \in C_2$ and rewards r_t for which f does not achieve universal consistency.

We construct the process \mathbb{X} and rewards $(r_t)_{t\geq 1}$ recursively. Let $\epsilon_k = 2^{-k}$ for $k \geq 1$. The process and rewards are constructed together with times T_k such that a significant regret is incurred to the learner between times T_k and T_{k+1} for all $k \geq 1$. We pose $T_0 = 0$. We are now ready to start the induction. Suppose that we have already defined T_l for l < k and the deterministic process $\mathbb{X}_{\leq T_{k-1}}$ as well as the deterministic rewards r_t for $t \geq T_{k-1}$. Let $\mathbb{Z} = (Z_i)_{i\geq 1}$ be an i.i.d. sequence on \mathcal{X} with distribution μ . Pose $T^i = \frac{(1+i)!}{\epsilon_k} T_{k-1}$ for $i \geq 0$ and $k_i = \epsilon_k T^i (= (1+i)!T_{k-1}), n_i = \sum_{j < i} k_j$ for $i \geq 0$. Letting $\bar{x} \in \mathcal{X}$ an arbitrary instance, we now consider the following process $\tilde{\mathbb{X}}$:

$$\tilde{X}_{t} = \begin{cases} X_{t}, & t \leq T_{k-1}, \\ \bar{x}, & T_{k-1} < t < T^{0}, \\ Z_{n_{i}+l}, & t = T^{i} + p \cdot k_{i} + l, \quad 0 \leq p < \frac{1}{\epsilon_{k}}, \ 0 \leq l < k_{i}, i \geq 0, \\ \bar{x}, & 2T^{i} \leq t < T^{i+1}, \quad i \geq 0. \end{cases}$$

The process is deterministic until time T^0 . From this point, the process is constructed by periods, where period $i \ge 0$ corresponds to times $T^i \le t < T^{i+1} = (1+i)T^i$. Each period *i* has a first phase $T^i \le t < 2T^i$ composed of $\frac{1}{\epsilon_k}$ sub-phases of length $k_i = \epsilon_k T^i$ on which the process repeats exactly. We can therefore focus on the first sub-phase $T^i \le t < T^i(1 + \epsilon_k)$, which is constructed as an i.i.d. process following distribution μ independent from the past samples. In the second phase of period *i* for $2T^i \le T^{i+1}$ the process is idle equal to \bar{x} . This ends the construction of the process \tilde{X} .

We now argue that $\tilde{\mathbb{X}} \in C_1$. Indeed, note that forgetting about the part for $t \leq T^0$, and idle phases where the process visits \bar{x} only, this process takes values from an i.i.d. process \mathbb{Z} and each value is duplicated $\frac{1}{\epsilon_k}$ times throughout the whole process. Formally, let $(A_p)_{p\geq 1}$ be a decreasing sequence of measurable sets with $A_p \downarrow \emptyset$. Then for any $T^i < T \leq T^{i+1}$ with $i \geq 1$ we have, for p sufficiently large so that $\bar{x} \notin A_p$,

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{A_p}(\tilde{X}_t) \leq \frac{2T^{i-1}}{T^i} + \frac{1}{\epsilon_k T^i} \sum_{l=n_i}^{n_i+k_i-1} \mathbb{1}_{A_p}(Z_l)$$
$$\leq \frac{2}{1+i} + \frac{n_i+k_i}{k_i} \frac{1}{n_i+k_i} \sum_{l=0}^{n_i+k_i-1} \mathbb{1}_{A_p}(Z_l)$$

Last, we note that $\frac{n_i+k_i}{k_i} \to 1$ as $i \to \infty$. As a result, we obtain $\hat{\mu}_{\mathbb{X}}(A_p) \leq \hat{\mu}_{\mathbb{Z}}(A_p)$. Because $\mathbb{Z} \in \mathcal{C}_1$, we have $\mathbb{E}[\hat{\mu}_{\mathbb{Z}}(A_p)] \to 0$ as $p \to \infty$, which proves $\mathbb{E}[\hat{\mu}_{\mathbb{X}}(A_p)] \to 0$ as well. This ends the proof that $\mathbb{X} \in \mathcal{C}_1$.

We now construct rewards. Before doing so, for any $i \ge 0$, let δ_i such that

$$\mathbb{P}\left[\min_{1 \le u < v < n_{i+1}} \rho(Z_i, Z_j) \le \delta_i\right] \le 2^{-i-2}.$$

This is possible because μ is non-atomic, as a result with probability one, all Z_k for $k \ge 1$ are distinct. Then, by the union bound, with probability at least $1 - \frac{1}{2} = \frac{1}{2}$, for all $i \ge 0$ we have

$$\min_{1 \le u < v < n_{i+1}} \rho(Z_u, Z_v) > \delta_i.$$

We denote by \mathcal{E} the event where the above inequality holds for all $i \geq 1$ and for all $u \geq 1$, $Z_u \neq \bar{x}$. Because μ is non-atomic, we still have $\mathbb{P}[\mathcal{E}] \geq \frac{1}{2}$. We now construct a partition of \mathcal{X} as follows. Let $(x^k)_k$ be a dense sequence of \mathcal{X} . We denote by $B(x,r) = \{x' \in \mathcal{X}, \rho(x,x') < r\}$ the ball centered at x of radius r > 0. For any $k \geq 1$ and $\delta > 0$ let $P_k(\delta) = B(x^k, \delta) \setminus \bigcup_{l < k} B(x^l, \delta)$. Then, $(P_k(\delta))_k$ forms a partition of \mathcal{X} . For any $\delta > 0$ and sequence $\mathbf{b} = (b_k)_{k \geq 1}$ in $\{0, 1\}$ we consider the following deterministic rewards

$$r_{\delta,\mathbf{b}}(a \mid x) = \begin{cases} b_k & a = a_1, \ x \in P_k(\delta) \\ \frac{3}{4} & a = a_2, \\ 0 & a \notin \{a_1, a_2\}. \end{cases}$$

Now for any sequence of binary sequences $\mathbf{b} = (\mathbf{b}^i)_{i\geq 0}$ where $\mathbf{b}^i = (b_k^i)_{k\geq 1}$, we will consider the memoryless rewards $r^{\mathbf{b}}$ defined as follows. The deterministic rewards r_t being constructed for $t \leq T_{k-1}$, we pose $r_t^{\mathbf{b}} = r_t$ for $t \leq T_{k-1}$. For all idle phases, i.e., $T_{k-1} < t < T^0$ or $2T^i \leq T^{i+1}$ for $i \geq 0$, we pose $r_t^{\mathbf{b}} = 0$. Last, for any $i \geq 0$ and $T^i \leq t < 2T^i$ we pose $r_t^{\mathbf{b}} = r_{\delta_i, \mathbf{b}^i}$. Now let \mathbf{b} be a random sequence such that all \mathbf{b}^i are independent i.i.d. Bernouilli $\mathcal{B}(\frac{1}{2})$ sequences in $\{0, 1\}$. On the event \mathcal{E} , all new instances fall in distinct sets of the partitions defining the rewards. Hence, with this perspective, the reward of the action a_2 is always $\frac{3}{4}$ while on the event \mathcal{E} , for each new instance value, the reward of a_1 is a random Bernouilli $\mathcal{B}(\frac{1}{2})$. Intuitively, for a specific instance x, if the learner has not yet explored the arm a_1 , selecting a_1 incurs an average regret $\frac{1}{4}$ compared to selecting the fixed arm a_2 . We will then argue that there is a time T_k and a realization of $\tilde{\mathbb{X}}_{\leq T_k}$ and rewards, such that on this realization, the regret compared to the best actions for each instance in hindsight is significantly large. We now formalize these ideas.

Because $\tilde{\mathbb{X}}$ is a C_1 process, there exists a universally consistent learning rule under $\tilde{\mathbb{X}}$. Then, because f is optimistically universal, it is universally consistent under $\tilde{\mathbb{X}}$. Now fix a specific realization of the sequences in b, considering the policy which always plays action a_2 , i.e. $\pi_0 : x \in \mathcal{X} \mapsto a_2 \in \mathcal{A}$, we have

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t^{\mathbf{b}}(a_2 \mid X_t) - r_t^{\mathbf{b}}(\hat{a}_t \mid X_t) \le 0, \quad (a.s.)$$

In particular, since $\mathbb{P}[\mathcal{E}] \geq \frac{1}{2}$, we have

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t^{\mathbf{b}}(a_2 \mid X_t) - r_t^{\mathbf{b}}(\hat{a}_t \mid X_t) \mid \mathcal{E}, \mathbf{b}\right] \le 0.$$

As a result, taking the expectation over b then applying Fatou's lemma gives

$$\limsup_{T \to \infty} \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} r_t^{\mathbf{b}}(a_2 \mid X_t) - r_t^{\mathbf{b}}(\hat{a}_t \mid X_t) \mid \mathcal{E}\right] \le 0.$$

Now let $\alpha_k := \frac{1}{16 \cdot 4^{1/\epsilon_k}}$. In particular, there exists $i \ge \frac{4}{\alpha_k}$ such that for all $T \ge T^i$,

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}r_t^{\mathbf{b}}(a_2 \mid X_t) - r_t^{\mathbf{b}}(\hat{a}_t \mid X_t) \mid \mathcal{E}\right] \le \frac{\alpha_k}{4}.$$
(1)

For simplicity, we may write $r_t^{\mathbf{b}}(a)$ instead of $r_t^{\mathbf{b}}(a \mid x)$, when it is clear from context that $x = X_t$. We now focus on period $[T^i, 2T^i)$ and denote by $S_p^i := \{T^i + (p-1) \cdot \epsilon_k T^i \leq t < T^i + p \cdot \epsilon_k T^i\}$ the sub-phase p for $1 \leq p \leq \frac{1}{\epsilon_k}$ of this period. Also note by A_p^i the number of new exploration steps for arm a_1 during S_p^i , i.e., times when the learner selected a_1 for an instance that had not previously been explored

$$\mathcal{A}_{p}^{i} = \{ t \in \mathcal{S}_{p}^{i} : \hat{a}_{t} = a_{1}, \forall 1 \le q$$

We show by induction that $\mathbb{E}[A_p^i | \mathcal{E}] \leq 4^{p+1} \alpha_k T^i$ for all $1 \leq p \leq \frac{1}{\epsilon_k}$. Let $1 \leq p \leq \frac{1}{\epsilon_k}$. Suppose that the result was shown for $1 \leq q < p$ (if p = 1 this is directly satisfied). We have

$$\begin{split} & \mathbb{E}\left[\sum_{t=1}^{T^{i}(1+p\epsilon_{k})-1} r_{t}^{\mathbf{b}}(a_{2}) - r_{t}^{\mathbf{b}}(\hat{a}_{t}) \mid \mathcal{E}\right] \\ & \geq -2T^{i-1} + \mathbb{E}\left[\sum_{t=T^{i}(1+(p-1)\epsilon_{k})}^{T^{i}(1+p\epsilon_{k})-1} (r_{t}^{\mathbf{b}}(a_{2}) - r_{t}^{\mathbf{b}}(\hat{a}_{t}))\mathbb{1}_{\mathcal{A}_{p}^{i}}(t) - \sum_{q < p} \frac{(p+1-q)A_{q}^{i}}{4} \mid \mathcal{E}\right] \\ & = -2T^{i-1} - \sum_{q < p} \frac{p+1-q}{4}\mathbb{E}[A_{q}^{i} \mid \mathcal{E}] + \mathbb{E}\left[\sum_{t=T^{i}(1+(p-1)\epsilon_{k})}^{T^{i}(1+p\epsilon_{k})-1} \mathbb{1}_{\mathcal{A}_{p}^{i}}(t)\mathbb{E}[r_{t}^{\mathbf{b}}(a_{2}) - r_{t}^{\mathbf{b}}(\hat{a}_{t})|t \in \mathcal{A}_{p}^{i}, \mathcal{E}] \middle| \mathcal{E}\right] \end{split}$$

where in the first inequality we discard times from phase S_p^i for which an exploration of the corresponding instance during phases $S_1^i, \ldots S_{p-1}^i$: these yield a regret least (3/4 - 1) = -1/4 compared to the fixed arm a_2 . For each instance newly explored during phase S_q^i , i.e. $t \in S_q^i$, it affects potentially the (p+1-q) next times with the same instance in phases S_q^i, \ldots, S_p^i . Now, note that all elements in **b** are together independent, and independent from the process \mathbb{X} , in particular independent from \mathcal{E} . As a result, the rewards at a time \mathcal{A}_p^i are independent from the past because X_t visits a set of the partition $(P_k(\delta_i))_k$ which has never been visited. Thus, we have

$$\mathbb{E}[r_t^{\mathbf{b}}(a_2) - r_t^{\mathbf{b}}(\hat{a}_t) | t \in \mathcal{A}_p, \mathcal{E}] = \frac{3}{4} - \frac{0+1}{2} = \frac{1}{4}.$$

Combining the above estimates with Eq(1) then gives

$$-2T^{i-1} - \frac{1}{4}\sum_{q < p}(p+1-q)\mathbb{E}[A_q^i \mid \mathcal{E}] + \frac{1}{4}\mathbb{E}[A_p^i \mid \mathcal{E}] \le \mathbb{E}\left[\sum_{t=1}^{T^i(1+p\epsilon_k)-1} r_t^{\mathbf{b}}(a_2) - r_t^{\mathbf{b}}(\hat{a}_t) \mid \mathcal{E}\right]$$
$$\le \frac{\alpha_k}{4}T^i(1+p\epsilon_k) \le \frac{\alpha_k}{2}T^i$$

Thus,

$$\mathbb{E}[A_{p}^{i} \mid \mathcal{E}] \leq \left(\frac{8}{1+i} + 2\alpha_{k}\right) T^{i} + \sum_{q < p} (p+1-q) \mathbb{E}[A_{q}^{i} \mid \mathcal{E}]$$

$$\leq 4\alpha_{k} T^{i} \left(1 + \sum_{q=1}^{p-1} (p+1-q)4^{q}\right)$$

$$\leq 4\alpha_{k} T^{i} \left(1 + \sum_{q=1}^{p-1} 2^{p-q}4^{q}\right) = 4\alpha_{k} T^{i} \left(1 + 2^{p}(2^{p}-1)\right) \leq 4^{p+1}\alpha_{k} T^{i}.$$

This completes the induction.

For any time t, denote $a_t^* = \arg \max_{a \in \mathcal{A}} r_t^{\mathbf{b}}(a)$ the optimal arm in hindsight. Note that $a_t^* \in \{a_1, a_2\}$. We lower bound the regret of the learner compared to the best action in hindsight until time T^{i+1} . To do so, define $\mathcal{B} = \bigcup_{p=1}^{1/\epsilon_k} \{t \in \mathcal{S}_p^i : \forall 1 \leq q \leq p, t + (q-p)\epsilon_k T^i \notin \mathcal{A}_q^i\}$ the set of times t such that the learner never explored a_1 on the present and past appearances of the instance X_t . We also define $\mathcal{C} = \{T^i \leq t < 2T^i : a_t^* = a_1\}$ the set of times when a_1 was the optimal action. One can observe that for any time in \mathcal{B} , because no exploration on a_1 was performed up for the corresponding instance X_t in the past history, $\mathbb{P}[t \in \mathcal{C} | t \in \mathcal{B}, \mathcal{E}] = \frac{1}{2}$. Hence, if $t \in \mathcal{B} \cap \mathcal{C} \cap \mathcal{E}$, the learner incurs a regret at least $\frac{1}{4}$ compared to the best arm $a_t^* = a_1$. Therefore,

$$\mathbb{E}\left[\sum_{t=1}^{2T^{i}-1} r_{t}^{\mathbf{b}}(a_{t}^{*}) - r_{t}^{\mathbf{b}}(\hat{a}_{t}) \mid \mathcal{E}\right] \geq \frac{1}{4} \mathbb{E}\left[\sum_{t \in \mathcal{B}} \mathbb{1}_{\mathcal{C}}(t) \mid \mathcal{E}\right] = \frac{1}{8} \mathbb{E}[|\mathcal{B}| \mid \mathcal{E}].$$

where by construction, we have $|\mathcal{B}| + \sum_{p=1}^{1/\epsilon_k} \left(\frac{1}{\epsilon_k} - p + 1\right) A_p^i = 2T^i - T^i = T^i$. As a result,

$$\mathbb{E}\left[\sum_{t=1}^{2T^{i}-1} r_{t}^{\mathbf{b}}(a_{t}^{*}) - r_{t}^{\mathbf{b}}(\hat{a}_{t}) \mid \mathcal{E}\right] \geq \frac{T^{i}}{8} - \frac{\alpha_{k}}{2} T^{i} \sum_{p=1}^{1/\epsilon_{k}} \left(\frac{1}{\epsilon_{k}} - p + 1\right) 4^{p}$$
$$\geq \frac{T^{i}}{8} - \alpha_{k} T^{i} 4^{1/\epsilon_{k}}$$
$$\geq \frac{T^{i}}{16} \geq \frac{2T^{i} - 1}{32}.$$

Hence, there exist a realization of instances $X_{<2T^i} \leq \tilde{X}_{<2T^i}$ falling in \mathcal{E} and of rewards $(r_t)_{<2T^i}$ such that the regret compared to the best action in hindsight for on this specific instance sequence and for these rewards is at least $\frac{T^i}{16}$. We then pose $T_k := 2T^i - 1$, and use the realization $X_{\leq T_k}$, $(r_t)_{\leq T_k}$ for the deterministic process $\mathbb{X}_{\leq T_k}$ and $(r_t)_{t\leq T_k}$. We recall that by construction, the realizations are consistent with the previously constructed process $\mathbb{X}_{\leq T_{k-1}}$ and rewards $(r_t)_{\leq T_{k-1}}$. Further, to each new instance

between times T^i and $2T^i - 1$ corresponded a best action in hindsight: this gives a collection of pairs (x, a) where $x \in \mathcal{X}$ is an instance visited by the deterministic process \mathbb{X} between times T^i and $2T^i - 1$ and $a \in \{a_1, a_2\}$ is the corresponding best action. Let \mathcal{D}_k denote this collection. This ends the recursive construction of the deterministic process \mathbb{X} and rewards.

Because we enforced that the samples of μ be always distinct and different from \bar{x} across the construction of X, the countable collection $\bigcup_{k\geq 1} \mathcal{D}_k$ of pairs instance/optimal-action never contains pairs with the same instance x. Hence, we can consider the following measurable policy $\pi^* : \mathcal{X} \to \mathcal{A}$ defined by

$$\pi^*(x) = \begin{cases} a & \text{if } (x,a) \in \bigcup_{k \ge 1} \mathcal{D}_k, \\ a_2 & \text{otherwise.} \end{cases}$$

This policy always performs the optimal action in hindsight. Hence by construction, for any $k \ge 1$,

$$\mathbb{E}\left[\frac{1}{T_k}\sum_{t=1}^{T_k} r_t(\pi^*(X_t) \mid X_t) - r_t(\hat{a}_t \mid X_t)\right] \ge \frac{1}{32}$$

where \hat{a}_t refers to the learner's decisions on the constructed process X and rewards $(r_t)_{t\geq 1}$. Note that the expectation is taken only with respect to the learner's randomness given that X and $(r_t)_{t\geq 1}$ are deterministic. Because the above equation holds for all $k \geq 1$ and $(T_k)_{k\geq 1}$ is an increasing sequence of times, we have

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t)) - r_t(\hat{a}_t)\right] \ge \limsup_{T \to \infty} \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t)) - r_t(\hat{a}_t)\right] \ge \frac{1}{32},$$

where we used Fatou's lemma. This proves that f is not universally consistent on X.

We now show that $\mathbb{X} \in C_2$. It suffices to check that it visits a sublinear number of distinct points—this is also necessary since \mathbb{X} is deterministic. For $t \ge 1$, denote by N_t the number of distint instances visited by the process $\mathbb{X}_{\le t}$. Fix $k \ge 1$. The process $\mathbb{X}_{\le T_k}$ being constructed from the process $\mathbb{X}_{\le T_k}$ above, we re-use the same notations. Let $i \ge 1$ such that $T_k = 2T^i - 1$. For $1 \le j \le i$ and $T^j \le t < \min(T^{j+1}, T_k)$ we have $N_t \le T_{k-1} + 1 + n_j + k_j \le 1 + \epsilon_k T^0 + 2k_j \le 1 + 3\epsilon_k T^j \le 1 + 3\epsilon_k t$. (The additional 1 accounts for \bar{x} .) For $T_{k-1} < t < T^0$, we have $N_t \le 1 + N_{t_{k-1}} \le 2 + 3\epsilon_{k-1} t$. As a result for all $T_{k-1} < t \le T_k$ we have

$$N_t \le 2 + 3\epsilon_{k-1}t.$$

Because $\epsilon_k \to 0$ as $k \to \infty$, we obtain that $\frac{N_t}{t} \to 0$ as $t \to \infty$. This shows that $\mathbb{X} \in C_2$. Because \mathbb{X} is deterministic and in C_2 , Proposition 14 shows that there exists an universally consistent learning rule on \mathbb{X} . However f is not universally consistent under \mathbb{X} which contradicts the hypothesis. This ends the proof that there does not exist an optimistically universal learning rule.

We now turn to the case of spaces \mathcal{X} which do not have a non-atomic measure and show that in this case, the learning rule for processes visiting a sublinear number of distinct instances in Proposition 13 is optimistically universal learning rule for all settings including online rewards.

Theorem 16. Let \mathcal{X} a metrizable separable Borel space such that there does not exist a non-atomic probability measure on \mathcal{X} , and \mathcal{A} a finite action space. Then, learnable processes are exactly $C_{stat} = C_{online} = C_2$ and there exists an optimistically universal learning rule for all settings.

Proof. We show that any process $\mathbb{X} \in C_2$ visits a sublinear number of distinct instances almost surely. Fix $\mathbb{X} \in C_2$. Using [1, Lemma 5.1], because \mathcal{X} does not admit a non-atomic probability measure, there exists a countable set $Supp(\mathbb{X})$ such that on an event \mathcal{E} of probability one, for all $t \ge 1$, $X_t \in Supp(\mathbb{X})$. Then consider the sequence $(\{x\})_{x \in Supp(\mathbb{X})}$ of disjoint measurable sets of \mathcal{X} . Applying the C_2 property of \mathbb{X} to this sequence yields $|\{x \in Supp(\mathbb{X}) : \{x\} \cap \mathbb{X}_{\le T}\}| = o(T), (a.s.)$. We denote by \mathcal{F} the corresponding event of probability one. By union bound $\mathbb{P}[\mathcal{E} \cap \mathcal{F}] = 1$. Now on the event \mathcal{E} , for any $T \ge 1$ we have

$$|\{x \in \mathcal{X} : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\}| = |\{x \in Supp(\mathbb{X}) : \{x\} \cap \mathbb{X}_{\leq T}\}|.$$

As a result, on the event $\mathcal{E} \cap \mathcal{F}$ we have $|\{x \in \mathcal{X} : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\}| = o(T)$, which proves the claim that \mathcal{C}_2 visit a sublinear number of distinct instances almost surely. As a result, the learning rule f from Proposition 13 which simply performs independent copies of the EXP3.IX algorithm for each distinct visited instance is universally consistent under all processes $\mathbb{X} \in \mathcal{C}_2$. Now recall that in the stationary case, the condition \mathcal{C}_2 is already necessary for universal learning. In fact, this condition is already necessary for universal learning in the noiseless full-feedback setting [2]. As a result, $\mathcal{C}_{online} \subset \mathcal{C}_{stat} = \mathcal{C}_2$. Therefore, universally learnable processes are exactly \mathcal{C}_2 even in the online rewards setting and f is optimistically universal, which completes the proof.

5 Universally learnable processes for context spaces with non-atomic probability measures

5.1 Necessary conditions on learnable processes

In the previous section, we showed that for spaces \mathcal{X} that do not have non-atomic probability measures, the set of learnable processes is exactly C_2 , independently of the learning setting. Here, we focus on the remaining case of universal learning for spaces \mathcal{X} that admit a non-atomic probability measure for adversarial rewards and aim to understand which processes admit universal learning. We focus here on necessary conditions; sufficient conditions are given in the next section.

5.1.1 Condition 4 is necessary for universal learning with oblivious rewards

We quickly recall the definition of condition C_4 . For an integer $i \ge 0$ and any $k \ge 1$, we define $T_i^k = \lfloor 2^u(1+v2^{-i}) \rfloor$ where $k = u2^i + v$ and $u \ge 0, 0 \le v < 2^i$ are integers. In particular, $u = \lfloor k2^{-i} \rfloor$ and $v = k \mod 2^i$. These times form periods $[T_i^k, T_i^{k+1})$ which become finer as *i* increases. Then consider the set of times *t* such that X_t is the first appearance of the instance on its period,

$$\mathcal{T}^{i} = \{ t \ge 1 : T_{i}^{k} \le t < T_{i}^{k+1}, \ \forall T_{i}^{k} \le t' < t, X_{t'} \neq X_{t} \}.$$

We note that the sets \mathcal{T}^p are increasing with p. Condition \mathcal{C}_4 is defined as follows.

Condition 4. For any sequence of disjoint measurable sets $(A_i)_{i>1}$ of \mathcal{X} , we have

$$\lim_{i \to \infty} \mathbb{E} \left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^i} \mathbb{1}_{A_i}(X_t) \right] = 0.$$

Denote by C_4 the set of all processes X satisfying this condition.

We first give an alternative definition of C_4 which will be useful in the next results.

Proposition 17. Let \mathcal{X} be a metrizable separable Borel space and \mathbb{X} a stochastic process on \mathcal{X} . The following are equivalent.

- $\mathbb{X} \in \mathcal{C}_4$,
- For any sequence of decreasing measurable sets $(A_i)_{i\geq 1}$ with $A_i \downarrow \emptyset$,

$$\sup_{p\geq 0} \mathbb{E} \left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{A_i}(X_t) \right] \xrightarrow[i \to \infty]{} 0.$$

• For any sequence of decreasing measurable sets $(A_i)_{i\geq 1}$ with $A_i\downarrow \emptyset$,

$$\mathbb{E}\left[\sup_{p\geq 0}\limsup_{T\to\infty}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}^p}\mathbb{1}_{A_i}(X_t)\right]\xrightarrow[i\to\infty]{} 0.$$

Proof. Suppose that the second proposition is not satisfied. We aim to show that $\mathbb{X} \notin C_4$. By hypothesis, there exists measurable sets $A_i \downarrow \emptyset$, $\epsilon > 0$, and an increasing sequence of indices $(i_p)_{p \ge 1}$ such that

$$\sup_{l\geq 0} \mathbb{E} \left[\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T, t\in \mathcal{T}^l} \mathbb{1}_{A_{i_p}}(X_t) \right] \geq \epsilon.$$

Now let $i \ge 1$ and $p \ge 1$ such that $i_p \ge i$. We observe that because $A_{i_p} \subset A_i$,

$$\sup_{l\geq 0} \mathbb{E}\left[\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}^l} \mathbb{1}_{A_i}(X_t)\right] \geq \sup_{l\geq 0} \mathbb{E}\left[\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}^l} \mathbb{1}_{A_{i_p}}(X_t)\right] \geq \epsilon.$$

Hence, for any $i \ge 1$, there exists p(i) > 0 such that

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{p(i)}} \mathbb{1}_{A_i}(X_t)\right] \ge \frac{\epsilon}{2}$$

Case 1. We consider a first case where there exists $\eta_i > 0$ such that for any $j \ge i$,

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{p(i)}} \mathbb{1}_{A_j}(X_t)\right] \ge \eta_i.$$

For simplicity, we will write $T^k = T^k_{p(i)}$. We will also drop the indices i of p(i) and η_i for conciseness. We now construct by induction a sequence of indices $(k(l))_{l\geq 0}$ together with indices $(j(l))_{l\geq 0}$ with k(0) = 1, j(0) = i and such that for any $l \geq 1$,

$$\mathbb{E}\left[\sup_{T^{k(l-1)} < T \le T^{k(l)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_{j(l-1)} \setminus A_{j(l)}}(X_t)\right] \ge \frac{\eta}{2}.$$

Suppose that we have already constructed $j(0), \ldots, j(l-1)$ and $k(0), \ldots, k(l-1)$. Note that

$$\mathbb{E}\left[\sup_{T>T^{k(l-1)}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}^p}\mathbb{1}_{A_{j(l-1)}}(X_t)\right] \geq \mathbb{E}\left[\limsup_{T\to\infty}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}^p}\mathbb{1}_{A_{j(l-1)}}(X_t)\right] \geq \eta.$$

Therefore, by the dominated convergence theorem, there exists k(l) > k(l-1) such that

$$\mathbb{E}\left[\sup_{T^{k(l-1)} < t \le T^{k(l)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_{j(l-1)}}(X_t)\right] \ge \frac{3\eta}{4}$$

Now because $A_i \downarrow \emptyset$, there exists j(l) > j(l-1) such that $\mathbb{P}[A_{j(l)} \cap \mathbb{X}_{\leq T^{k(l)}} = \emptyset] \geq 1 - \frac{\eta}{4}$. Let us denote by \mathcal{E} this event. Then,

$$\mathbb{E}\left[\sup_{T^{k(l-1)} < t \leq T^{k(l)}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{A_{j(l-1)} \setminus A_{j(l)}}(X_t)\right]$$

$$\geq \mathbb{E}\left[\mathbb{1}[\mathcal{E}] \sup_{T^{k(l-1)} < t \leq T^{k(l)}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{A_{j(l-1)} \setminus A_{j(l)}}(X_t)\right]$$

$$= \mathbb{E}\left[\mathbb{1}[\mathcal{E}] \sup_{T^{k(l-1)} < t \leq T^{k(l)}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{A_{j(l-1)}}(X_t)\right]$$

$$\geq \mathbb{E}\left[\sup_{T^{k(l-1)} < t \leq T^{k(l)}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{A_{j(l-1)}}(X_t)\right] - \frac{\eta}{4} \geq \frac{\eta}{2}.$$

This ends the construction of the indices k(l) and j(l) for $l \ge 1$. Now for any $u \ge 1$, let $S_u = \{l \ge 1 : l \equiv 2^{u-1} \mod 2^u\}$. The main remark is that S_u is infinite for all $u \ge 1$ and they are all disjoint. We then pose $B_u = \bigcup_{l \in S_u} A_{j(l-1)} \setminus A_{j(l)}$. Because all S_u are disjoint, this implies that the sets $(B_u)_u$ are also disjoint. Then, using Fatou's lemma together with the fact that all S_u are infinite, we obtain

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{B_u}(X_t)\right] \geq \limsup_{k \in S_u} \mathbb{E}\left[\sup_{T^{k(l-1)} < T \leq T^{k(l)}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{B_u}(X_t)\right]$$
$$\geq \limsup_{k \in S_u} \mathbb{E}\left[\sup_{T^{k(l-1)} < T \leq T^{k(l)}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{A_{j(l-1)} \setminus A_{j(l)}}(X_t)\right]$$
$$\geq \frac{\eta}{2}.$$

We obtain therefore for any $u \ge p$

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^u} \mathbb{1}_{B_u}(X_t)\right] \ge \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{B_u}(X_t)\right] \ge \frac{\eta}{2}.$$

This ends the proof that $\mathbb{X} \notin C_4$.

Case 2. Recalling that the sets $(A_i)_i$ are decreasing, we can now suppose that for all $i \ge 1$, one has $\mathbb{E}\left[\limsup_{T\to\infty} \frac{1}{T} \sum_{t\le T, t\in\mathcal{T}^{p(i)}} \mathbb{1}_{A_j}(X_t)\right] \to 0$ as $j \to \infty$. We now construct a sequence of indices $(i(u))_{u\ge 1}$ as follows such that i(1) = 1 and for any $u \ge 1$,

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{p(i(u))}} \mathbb{1}_{A_{i(u)} \setminus A_{i(u+1)}}(X_t)\right] \ge \frac{\epsilon}{4}$$

Suppose we have constructed i(u). Then, by the hypothesis of this case, there exists i(u + 1) > i(u) such that

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{p(i(u))}} \mathbb{1}_{A_{i(u+1)}}(X_t)\right] \le \frac{\epsilon}{4}$$

Now note that

$$\begin{aligned} \frac{\epsilon}{2} &\leq \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^{p(i(u))}} \mathbb{1}_{A_{i(u)}}(X_t)\right] \leq \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^{p(i(u))}} \mathbb{1}_{A_{i(u)} \setminus A_{i(u+1)}}(X_t)\right] \\ &+ \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^{p(i(u))}} \mathbb{1}_{A_{i(u+1)}}(X_t)\right]\end{aligned}$$

As a result, the induction at step p is complete. We then define a sequence of measurable sets $(B_j)_{j\geq 1}$ such that for any $u \geq 1$, $B_{p(i(u))} = A_{i(u)} - A_{i(u+1)}$, and for all other indices $j \notin \{p(i(u)), u \geq 1\}$ we set $B_j = \emptyset$. All these sets are disjoint, and we have for any $u \geq 1$,

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{p(i(u))}} \mathbb{1}_{B_{p(i(u))}}(X_t)\right] \ge \frac{\epsilon}{4}.$$

Therefore, $\mathbb{X} \notin C_4$.

We now show that if X satisfies the second property, then $X \in C_4$. Let $(A_i)_i$ be a sequence of disjoint measurable sets, and define $B_i = \bigcup_{j>i} A_j$. Then,

$$0 \leq \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^i} \mathbb{1}_{A_i}(X_t)\right] \leq \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^i} \mathbb{1}_{B_i}(X_t)\right]$$
$$\leq \sup_{p \geq 0} \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{B_i}(X_t)\right].$$

Hence, because $B_i \downarrow \emptyset$, the second property implies that $\mathbb{E}\left[\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}^i} \mathbb{1}_{A_i}(X_t)\right] \to 0$ as $i\to\infty$.

Now for any Borel set A, by the dominated convergence theorem and the fact that the sets \mathcal{T}^p are increasing for $p \ge 0$, we obtain

$$\lim_{p \to \infty} \mathbb{E} \left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_A(X_t) \right] = \mathbb{E} \left[\lim_{p \to \infty} \limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_A(X_t) \right],$$

where both terms are bounded by 1. In other terms,

$$\sup_{p\geq 0} \mathbb{E} \left[\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}^p} \mathbb{1}_A(X_t) \right] = \mathbb{E} \left[\sup_{p\geq 0} \limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}^p} \mathbb{1}_A(X_t) \right].$$

As a result, the second and third condition of the proposition are equivalent.

The main result of this section is that the C_4 condition is necessary for universal learning with oblivious rewards.

Theorem 18. Let \mathcal{X} a metrizable separable Borel space, and a finite action space \mathcal{A} with $|\mathcal{A}| \geq 2$. Then, $\mathcal{C}_{oblivious} \subset \mathcal{C}_4$.

Proof. Fix, $a_1, a_2 \in A$ two distinct actions. By contradiction, let $\mathbb{X} \notin C_4$ and f. a universally consistent learning rule under \mathbb{X} for oblivious rewards. For simplicity, we will denote by \hat{a}_t the action selected by the learning rule at time t. By hypothesis, let $(A_i)_{i\geq 1}$ be a sequence of disjoint measurable sets and $0 < \epsilon \leq 1$ such that

$$\limsup_{i \to \infty} \mathbb{E} \left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^i} \mathbb{1}_{A_i}(X_t) \right] \ge \epsilon.$$

Then, there exists an increasing sequence $(j(i))_{i\geq 1}$ such that for any $p\geq 1$,

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{j(i)}} \mathbb{1}_{A_{j(i)}}(X_t)\right] \ge \frac{\epsilon}{2}.$$

We write $\mathcal{I} = \{j(i), i \geq 1\}$. Without loss of generality, we can suppose $A_j = \emptyset$ if $j \in \mathcal{I}$. We now construct recursively rewards $(r_t)_{t\geq 1}$ on which this algorithm is not consistent, as well as a policy $\pi^* : \mathcal{X} \to \mathcal{A}$ compared to which the algorithm has high regret. The reward functions and policy are constructed recursively together with an increasing sequence of times $(T^p)_{p\in\mathcal{I}}$ such that after the p-th iteration of the construction process, the rewards r_t for $t \leq T^p$ have been defined such that $r_t(\cdot \mid x_{\leq t}) = 0$ if $x \notin \bigcup_{i < p} A_i$, the policy $\pi^*(\cdot)$ is defined on $\bigcup_{i < p} A_i$ and always the best action in hindsight until T^{p-1} . For p = j(p'), suppose that we have performed p' - 1 iterations of this construction and have constructed the times $T^{j(1)}, \ldots, T^{j(p'-1)}$. For convenience, let $\alpha_p = 2^{-p-1}$ and define $K_p = \left\lceil \frac{2}{\alpha_p} \log \frac{2^6}{\epsilon} \right\rceil$, $\beta_p = \frac{\epsilon}{2^{10}(1+2\alpha_p)^{(K_p-1)K_p}4^{K_p}}$, $\tilde{K}_p = \left\lceil \frac{2}{\alpha_p} \log \frac{8}{\beta_p} \right\rceil$ and $M_p = \max(\frac{8}{\epsilon\alpha_p}, (1+2\alpha_p)^{K_p+\tilde{K}_p})$. We first construct by induction an increasing sequence of indices $(k(l))_{l\geq 0}$ with $k(0) = \min\{k \geq 2^p : T_p^k > M_p T^{j(p'-1)}\}$ and such that for any $l \geq 1$, $T_p^{k(l)} > M_p T_p^{k(l-1)}$ and

$$\mathbb{E}\left[\max_{M_pT_p^{k(l-1)} < T \le T_p^{k(l)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_p}(X_t)\right] \ge \frac{\epsilon}{4}.$$

To do so, suppose that we have constructed k(l') for $0 \le l' < l$. Note that

$$\mathbb{E}\left[\sup_{T>M_p T_p^{k(l-1)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_p}(X_t)\right] \ge \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_p}(X_t)\right] \ge \frac{\epsilon}{2}.$$

Then, by dominated convergence theorem, there exists k(l) > k(l-1) such that $T_p^{k(l)} > M_p T_p^{k(l-1)}$ and

$$\mathbb{E}\left[\max_{M_p T_p^{k(l-1)} < T \le T_p^{k(l)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_p}(X_t)\right] \ge \mathbb{E}\left[\sup_{T > M_p T_p^{k(l-1)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_p}(X_t)\right] - \frac{\epsilon}{4} \ge \frac{\epsilon}{4}.$$

This ends the construction of the sequence $(k(l))_{l\geq 0}$. We then denote by $\hat{k}(l)$ the index of a phase $(T_p^{k-1}, T_p^k]$ where the max is attained, i.e.

$$\hat{k}(l) = \operatorname*{argmax}_{k \le k(l)} \left(\max_{M_p T_p^{k(l-1)}, T_p^{k-1} < T \le T_p^k} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A_p}(X_t) \right).$$

Ties can be broken with alphabetical order. Because $T_p^k \leq 2T_p^{k-1}$, we have in particular,

$$\mathbb{E}\left[\frac{1}{T_p^{\hat{k}(l)}}\sum_{t\leq T_p^{\hat{k}(l)},t\in\mathcal{T}^p}\mathbb{1}_{A_p}(X_t)\right]\geq\frac{\epsilon}{8}.$$

Now for any $l \ge 1$, let δ_l such that

$$\mathbb{P}\left[\min_{1 \le t, t' \le T_p^{k(l)}, X_t \ne X_{t'}} \rho(X_t, X_{t'}) \le \delta_l\right] \le \frac{\epsilon}{2^{l+10}}$$

Then, let \mathcal{E} be the event when for all $l \geq 1$, we have $\min_{1 \leq t, t' \leq T_p^{k(l)}, X_t \neq X_{t'}} \rho(X_t, X_{t'}) > \delta_l$. By the union bound, $\mathbb{P}[\mathcal{E}] \geq 1 - \frac{\epsilon}{2^{10}}$. As a result, we have

$$\mathbb{E}\left[\frac{1}{T_p^{\hat{k}(l)}}\sum_{t\leq T_p^{\hat{k}(l)}, t\in\mathcal{T}^p}\mathbb{1}_{A_p}(X_t) \mid \mathcal{E}\right] \geq \frac{\epsilon}{16}.$$
(2)

Now for $\delta > 0$ and $u \ge 1$, define the sets $P_u(\delta) = (A_p \cap B(x^u, \delta)) \setminus \bigcup_{v < u} B(x^v, \delta)$ which form a partition of A_p . For any $\delta > 0$ and sequence $\mathbf{b} = (b_u)_{u \ge 1}$ in $\{0, 1\}$ we consider the following deterministic rewards

$$r_{\delta,\boldsymbol{b}}(a \mid x) = \begin{cases} b_u & a = a_1, \ x \in P_u(\delta), \\ \frac{3}{4} & a = a_2, \\ 0 & a \notin \{a_1, a_2\}, \end{cases} \text{ if } x \in A_p, \qquad r_{\delta,\boldsymbol{b}}(\cdot \mid x) = 0 \text{ if } x \notin A_p.$$

For any sequence of binary sequences $\mathbf{b} = (\mathbf{b}^k)_{k\geq 0}$ where $\mathbf{b}^k = (b_u^k)_{u\geq 1}$, and binary sequence $\mathbf{c} = (c_k)_{k\geq 0}$ we construct the rewards $\mathbf{r}^{\mathbf{b},\mathbf{c}}$ as follows. For $t \leq T^{j(p'-1)}$ we pose $r_t^{\mathbf{b},\mathbf{c}} = r_t$ so that the rewards $\mathbf{r}^{\mathbf{b},\mathbf{c}}$ coincide with those constructed by induction so far. For $T^{j(p'-1)} < t \leq T_p^{k(0)}$ we pose $r_t^{\mathbf{b},\mathbf{c}} = 0$. For $t > T_p^{k(0)}$ let $l \geq 1$ such that $T_p^{k(l-1)} < t \leq T_p^{k(l)}$ and k > k(0) such that $T_p^{k-1} < t \leq T_p^k$. Then, we pose

$$r_t^{\mathbf{b}, \mathbf{c}}(a \mid x_{\leq t}) = \begin{cases} 0 & \exists t' \leq T_p^{k(l-1)} : x_{t'} = x_t \\ 0 & \text{o.w. } c_k = 0, \\ r_{\delta_l, \mathbf{b}^l}(a \mid x_t) & \text{o.w. } c_k = 1, \forall T_p^{k-1} < t' < t : x_{t'} \neq x_t, \\ 0 & \text{o.w. } c_k = 1, \exists T_p^{k-1} < t' < t : x_{t'} = x_t, \end{cases}$$

for $a \in \mathcal{A}, x_{\leq t} \in \mathcal{X}^t$. Note that these rewards coincide on the rewards that have been constructed by induction so far. Now let b be generated such that all b^k are independent i.i.d. Bernouilli $\mathcal{B}(\frac{1}{2})$ random sequences in $\{0, 1\}$, and c is also an independent i.i.d. $\mathcal{B}(\frac{1}{2})$ process. The sequence is used to delete some periods $(T_p^{k-1}, T_p^k]$. Precisely, for any $l \geq 1$, we consider the following event where we deleted the periods between $\hat{k}(l) - K_p - \tilde{K}_p$ and $\hat{k}(l) - K_p$ but did not delete periods after this phase until period $\hat{k}(l)$,

$$\mathcal{F}_{l}^{p} = \bigcap_{\hat{k}(l) - K_{p} - \tilde{K}_{p} < k \le \hat{k}(l) - K_{p}} \{c_{k} = 0\} \cap \bigcap_{\hat{k}(l) - K_{p} < k \le \hat{k}(l)} \{c_{k} = 1\}.$$

One can note that the events \mathcal{F}_l^p for $l \ge 1$ are together independent. Indeed, $\hat{k}(l) \le k(l)$ and $T_p^{\hat{k}(l)} > M_p T^{k(l-1)} \ge (1+2\alpha_p)^{K_p+\tilde{K}_p}T^{k(l-1)}$, which yields $\hat{k}(l) > k(l-1) + K_p + \tilde{K}_p$. As a result, the indices of c considered in the events \mathcal{F}^p all lie in distinct intervals (k(l-1), k(l)], hence their independence. Further, we have $\mathbb{P}[\mathcal{F}_l^p] = 2^{-K_p-\tilde{K}_p}$. Then, the Borel-Cantelli implies that on an event \mathcal{F}^p of probability one, there is an infinite number of $l \ge 1$ such that \mathcal{F}_l^p is satisfied.

Next, define $\pi_0 : x \in \mathcal{X} \mapsto a_2 \in \mathcal{A}$, the policy which always selects arm a_2 . Fix any realization of **b** and **c**. Because f is universally consistent for oblivious rewards, it has in particular sublinear regret compared to π_0 under rewards $\mathbf{r}^{\mathbf{b},\mathbf{c}}$, i.e., almost surely $\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^T r_t^{\mathbf{b},\mathbf{c}}(a_2 \mid X_t) - r_t^{\mathbf{b},\mathbf{c}}(\hat{a}_t \mid X_t)) \leq 0$. Now observe that the event \mathcal{F}^p only depends on \mathbf{c} and \mathbb{X} and is in particular independent from **b**. Therefore, $\mathbb{P}[\mathcal{E} \cap \mathcal{F}^p \mid \mathbf{b}] = \mathbb{P}[\mathcal{E} \cap \mathcal{F}^p] \geq 1 - \frac{\epsilon}{2^{10}}$, where we used $\mathbb{P}[\mathcal{F}^p] = 1$. Therefore,

$$\mathbb{E}\left[\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}r_t^{\mathbf{b},\mathbf{c}}(a_2\mid \mathbb{X}_{\leq t}) - r_t^{\mathbf{b},\mathbf{c}}(\hat{a}_t\mid \mathbb{X}_{\leq t})\mid \mathcal{E}, \mathcal{F}^p, \mathbf{b}\right] \leq 0.$$

For conciseness, we will omit the terms $X_{\leq t}$ in the rest of the proof. We then take the expectation over b and c. Thus, by the dominated convergence theorem, there exists $l_0 \geq 1$ such that

$$\mathbb{E}\left[\sup_{T>T_p^{k(l_0)}}\frac{1}{T}\sum_{t=1}^T r_t^{\mathbf{b},\boldsymbol{c}}(a_2) - r_t^{\mathbf{b},\boldsymbol{c}}(\hat{a}_t) \mid \mathcal{E}, \mathcal{F}^p\right] \leq \frac{\beta_p}{8}.$$

On the event \mathcal{F}^p , there exists $\hat{l} > l_0$ such that the event $\mathcal{F}^p_{\hat{l}}$ is met. For convenience, we take \hat{l} the minimum index satisfying these conditions. Then, we have

$$\mathbb{E}\left[\sup_{T_p^{\hat{k}(\hat{l})-K_p} < T \le T_p^{\hat{k}(\hat{l})}} \frac{1}{T} \sum_{t=1}^T r_t^{\mathbf{b}, \mathbf{c}}(a_2) - r_t^{\mathbf{b}, \mathbf{c}}(\hat{a}_t) \mid \mathcal{E}, \mathcal{F}^p\right]$$
$$\leq \mathbb{E}\left[\sup_{T_p^{k(\hat{l}-1)} < T \le T_p^{k(\hat{l})}} \frac{1}{T} \sum_{t=1}^T r_t^{\mathbf{b}, \mathbf{c}}(a_2) - r_t^{\mathbf{b}, \mathbf{c}}(\hat{a}_t) \mid \mathcal{E}, \mathcal{F}^p\right] \le \frac{\beta_p}{8}.$$

Now let l^p such that $\mathbb{P}[\hat{l} \leq l^p \mid \mathcal{F}^p] \geq \frac{1}{2}$. Then,

$$\mathbb{E}\left[\sup_{T_p^{\hat{k}(\hat{l})-K_p} < T \le T_p^{\hat{k}(\hat{l})}} \frac{1}{T} \sum_{t=1}^T r_t^{\mathbf{b}, \mathbf{c}}(a_2) - r_t^{\mathbf{b}, \mathbf{c}}(\hat{a}_t) \mid \mathcal{E}, \mathcal{F}^p, \hat{l} \le l_p\right] \le \frac{\beta_p}{4}.$$
(3)

For conciseness, we will write \hat{k} for $\hat{k}(\hat{l})$, let $\mathcal{G}^p = \mathcal{E} \cap \mathcal{F}^p \cap \{\hat{l} \leq l_p\}$. We now use similar same arguments as in the proof of Theorem 15, to show that the learning rule incurs a large regret compared to

the best action in hindsight, before time $T_p^{\hat{k}}$. We focus on the period $(T_p^{\hat{k}-K_p}, T_p^{\hat{k}}]$, which we decompose using the sets

$$S_q = \{T_p^{\hat{k} - K_p - 1 + q} < t \le T_p^{\hat{k} - K_p + q} : X_t \in A_p\} \cap \mathcal{T}^p, \quad 1 \le q \le K_p.$$

We also define E_q the number of new exploration steps for arm a_1 during S_q ,

$$Exp_q = \left\{ t \in \mathcal{S}_q : \hat{a}_t = a_1 \text{ and } \forall t' \in \bigcup_{q' < q} \mathcal{S}_{q'} : X_{t'} = X_t, \ \hat{a}_{t'} \neq a_1 \right\} \setminus \{t : \exists t' \leq T_p^{\hat{k} - \tilde{K}_p}, X_{t'} = X_t\},$$

and $E_q = |Exp_q|$. We now show by induction on i that $\mathbb{E}\left[\frac{E_q}{T_p^k} \mid \mathcal{G}^p\right] \leq (1 + 2\alpha_p)^{(q-1)K_p} 4^{q+1}\beta_p$ for all $1 \leq q \leq K_p$. Suppose that this is shown for all 1 < q' < q. Recalling that on the event \mathcal{G}^p , for any $T_p^{\hat{k}-K_p-\tilde{K}_p} < t \leq T_p^{\hat{k}-K_p}$ we have $r_t^{\mathbf{b},\mathbf{c}} = 0$, we can use the same arguments as in Theorem 15 to obtain

$$\begin{split} & \mathbb{E}\left[\frac{1}{T_{p}^{\hat{k}-K_{p}+q}}\sum_{t=1}^{T_{p}^{\hat{k}-K_{p}+q}}r_{t}^{\mathbf{b},\mathbf{c}}(a_{2})-r_{t}^{\mathbf{b},\mathbf{c}}(\hat{a}_{t})\mid\mathcal{G}^{p}\right] \\ &\geq -\mathbb{E}\left[\frac{T_{p}^{\hat{k}-K_{p}-\tilde{K}_{p}}}{T_{p}^{\hat{k}-K_{p}+q}}\mid\mathcal{G}^{p}\right]+\sum_{q'=1}^{q}\mathbb{E}\left[\frac{1}{T_{p}^{\hat{k}-K_{p}+q}}\sum_{t=T_{p}^{\hat{k}-K_{p}-1+q'}+1}^{T_{p}^{\hat{k}-K_{p}+q'}}r_{t}^{\mathbf{b},\mathbf{c}}(a_{2})-r_{t}^{\mathbf{b},\mathbf{c}}(\hat{a}_{t})\mid\mathcal{G}^{p}\right] \\ &= -\mathbb{E}\left[\frac{T_{p}^{\hat{k}-K_{p}-\tilde{K}_{p}}}{T_{p}^{\hat{k}-K_{p}+q}}\mid\mathcal{G}^{p}\right]+\sum_{q'=1}^{q}\mathbb{E}\left[\frac{1}{T_{p}^{\hat{k}-K_{p}+q}}\sum_{t\in\mathcal{S}^{q'}}r_{t}^{\mathbf{b},\mathbf{c}}(a_{2})-r_{t}^{\mathbf{b},\mathbf{c}}(\hat{a}_{t})\mid\mathcal{G}^{p}\right] \\ &\geq -(1+q)\mathbb{E}\left[\frac{T_{p}^{\hat{k}-K_{p}-\tilde{K}_{p}}}{T_{p}^{\hat{k}-K_{p}+q}}\mid\mathcal{G}^{p}\right]-\sum_{q'$$

where the additional terms $-T_p^{\hat{k}-\tilde{K}_p}$ compared to the computations in Theorem 15 are due to the fact that in Exp_q we also discard times of instances that were visited before $T^{\hat{k}-\tilde{K}_p}$, and that in a single period S_q , there are no duplicates. Now for any $T^{\hat{k}-K_p-1+q} < t \leq T^{\hat{k}-K_p+q}$ such that a pure exploration was performed $t \in Exp_q$, we have

$$\mathbb{E}[r_t^{\mathbf{b},\mathbf{c}}(a_2) - r_t^{\mathbf{b},\mathbf{c}}(\hat{a}_t) \mid t \in Exp_q, \mathcal{G}^p, \hat{k}] = \frac{3}{4} - \frac{0+1}{2} = \frac{1}{4},$$

because X_t visits a set of the partition $(P_u(\delta_{\hat{k}-K_p+q}))_u$ which has never been visited in the past, hence the reward of a_1 on this set is equally likely to be 0 or 1 (depending on b), and \mathcal{G}^p is independent from b. Also, using the inequality $\log(1+z) \geq \frac{z}{2}$ for $0 \leq z \leq 1$ we obtain $T_p^{\hat{k}-\tilde{K}_p} \leq (1+\alpha_p)^{-\tilde{K}_p}(1+T_p^{\hat{k}-K_p}) \leq \frac{\beta_p}{8}(1+T_p^{\hat{k}-K_p}) \leq \frac{\beta_p}{4}T_p^{\hat{k}-K_p}$. Lastly, $T_p^{\hat{k}-K_p} \geq T_p^{\hat{k}}/(1+2\alpha_p)^{K_p}$. Combining these results with Eq (3) yields

$$\frac{\beta_p}{4} \ge -(1+q)\frac{\beta_p}{4} - \frac{1}{4}\sum_{q' < q}(q+1-q')(1+2\alpha_p)^{K_p}\mathbb{E}\left[\frac{E_{q'}}{T_p^{\hat{k}}} \mid \mathcal{G}^p\right] + \frac{1}{4}\mathbb{E}\left[\frac{E_q}{T_p^{\hat{k}}} \mid \mathcal{G}^p\right].$$

Thus,

$$\begin{split} \mathbb{E}\left[\frac{E_q}{T_p^{\hat{k}}} \mid \mathcal{G}^p\right] &\leq (2+q)\beta_p + (1+2\alpha_p)^{K_p} \sum_{q' < q} (q+1-q') \mathbb{E}\left[\frac{E_{q'}}{T_p^{\hat{k}}} \mid \mathcal{G}^p\right] \\ &\leq (1+2\alpha_p)^{(q-1)K_p}\beta_p \left(2+q+4\sum_{q'=1}^{q-1} (q+1-q')4^{q'}\right) \\ &\leq (1+2\alpha_p)^{(q-1)K_p}4^{q+1}\beta_p. \end{split}$$

This completes the induction. Now for any $t \ge 1$, denote by $a_t^* = \operatorname{argmax}_{a \in \mathcal{A}} r_t^{\mathbf{b}, \mathbf{c}}(a \mid \mathbb{X}_{\le t})$ the optimal action in hindsight. In particular, $a_t^* \in \{a_1, a_2\}$. Now define

$$\mathcal{B} = \bigcup_{q=1}^{K_0} \left\{ t \in \mathcal{S}_q : \forall t' \in \bigcup_{q' < q} \mathcal{S}_{q'} : X_{t'} = X_t, t \notin Exp_{q'} \right\}.$$

These are times such that we never explored the action a_2 . In particular, on \mathcal{G}^p , the learner incurs an average regret of at least $\frac{1}{8}$ on these times since action a_2 would be optimal with probability $\frac{1}{2}$ with a reward excess $\frac{1}{4}$ over action a_1 . Therefore,

$$\mathbb{E}\left[\frac{1}{T_p^{\hat{k}}}\sum_{t=1}^{T_p^{\hat{k}}} r_t^{\mathbf{b},\mathbf{c}}(a_t^*) - r_t^{\mathbf{b},\mathbf{c}}(\hat{a}_t) \mid \mathcal{G}^p\right] \ge \mathbb{E}\left[\frac{1}{T_p^{\hat{k}}}\sum_{T_p^{\hat{k}-K_p} < t \le T_p^{\hat{k}}} r_t^{\mathbf{b},\mathbf{c}}(a_t^*) - r_t^{\mathbf{b},\mathbf{c}}(\hat{a}_t) \mid \mathcal{G}^p\right]$$
$$\ge \frac{1}{8}\mathbb{E}\left[\frac{|\mathcal{B}|}{T_p^{\hat{k}}} \mid \mathcal{G}^p\right].$$

Now denote by $T_p^* = |\{t \leq T_p^{\hat{k}} : X_t \in A_p\} \cap \mathcal{T}^p|$. Recall that because \mathcal{F}^p and \hat{l} are independent from \mathcal{E} , by Eq (2), we have $\mathbb{E}\left[\frac{T_p^*}{T_p^k} \mid \mathcal{G}^p\right] = \left[\frac{T_p^*}{T_p^k} \mid \mathcal{E}\right] \geq \frac{\epsilon}{16}$. By construction, we have $|\mathcal{B}| + \sum_{q=1}^{K_p} (K_p - q + 1)E_q + K_pT_p^{\hat{k}-K_p-\tilde{K}_p} \geq T_p^* - T_p^{\hat{k}-K_p}$. Thus,

$$\mathbb{E}\left[\frac{1}{T_{p}^{\hat{k}}}\sum_{t=1}^{T_{p}^{\hat{k}}}r_{t}^{\mathbf{b},\mathbf{c}}(a_{t}^{*})-r_{t}^{\mathbf{b},\mathbf{c}}(\hat{a}_{t})\mid\mathcal{G}^{p}\right] \geq \frac{\epsilon}{2^{7}}-\frac{K_{p}}{4}\mathbb{E}\left[\frac{T_{p}^{\hat{k}-K_{p}-\tilde{K}_{p}}}{T_{p}^{\hat{k}}}\right] \\ -\frac{\beta_{p}}{2}\sum_{q=1}^{K_{p}}(K_{p}-q+1)(1+2\alpha_{p})^{(q-1)K_{p}}4^{q} \\ \geq \frac{\epsilon}{2^{7}}-\frac{\beta_{p}K_{p}}{16}-\frac{\beta_{p}}{2}(1+2\alpha_{p})^{(K_{p}-1)K_{p}}4^{K_{p}+1} \\ \geq \frac{\epsilon}{2^{8}}.$$

Recall that by construction $\mathbb{P}[\hat{l} \leq l^p | \mathcal{F}^p] \geq \frac{1}{2}$. Also, $\mathbb{P}[\mathcal{F}^p] = 1$ and both these events are independent from \mathcal{E} , hence, letting $T^p = T_p^{k(l^p)}$ we have

$$\mathbb{E}\left[\sup_{T^{p-1} < T \le T^p} \frac{1}{T} \sum_{t=1}^T r_t^{\mathbf{b}, \mathbf{c}}(a_t^*) - r_t^{\mathbf{b}, \mathbf{c}}(\hat{a}_t) \mid \mathcal{E}\right] \ge \frac{1}{2} \mathbb{E}\left[\frac{1}{T_p^{\hat{k}}} \sum_{t=1}^{T_p^{\hat{k}}} r_t^{\mathbf{b}, \mathbf{c}}(a_t^*) - r_t^{\mathbf{b}, \mathbf{c}}(\hat{a}_t) \mid \mathcal{G}^p\right] \ge \frac{\epsilon}{2^9}$$

This ends the construction of the sequence T^p . Then, for any binary sequences \boldsymbol{b} and \boldsymbol{c} we introduce slightly different rewards $(\tilde{r}_t^{\boldsymbol{b},\boldsymbol{c}})_{t\leq T^p}$ as follows: for $t\leq T^{j(p'-1)}_p$, $\tilde{r}_t^{\boldsymbol{b},\boldsymbol{c}}=r_t$, for $T^{j(p'-1)}< t\leq T^{k(0)}_p$ let $\tilde{r}_t^{\boldsymbol{b},\boldsymbol{c}}=0$. For $t>T^{k(0)}_p$ let $l\geq 1$ such that $T^{k(l-1)}_p < t\leq T^{k(l)}_p$ and k>k(0) such that $T^{k-1}_p < t\leq T^k_p$. Then, we pose

$$\tilde{r}_t^{\boldsymbol{b},\boldsymbol{c}}(a \mid x_{\leq t}) = \begin{cases} 0 & \exists t' \leq T_p^{k(l-1)} : x_{t'} = x_t \\ 0 & \text{o.w. } c_k = 0, \\ r_{\delta_{l^p},\boldsymbol{b}}(a \mid x_t) & \text{o.w. } c_k = 1, \forall T_p^{k-1} < t' < t : x_{t'} \neq x_t, \\ 0 & \text{o.w. } c_k = 1, \exists T_p^{k-1} < t' < t : x_{t'} = x_t, \end{cases}$$

for $a \in \mathcal{A}, x_{\leq t} \in \mathcal{X}^t$. The only difference with the previous oblivious rewards is that we use the same reward function $r_{\delta_{l^p}, \mathbf{b}}$ across phases $(T_p^{k(l-1)}, T_p^{k(l)}]$ for $l \leq l^p$. Then, consider the following policy,

$$\pi^{\mathbf{b}}(x) = \begin{cases} a_1 & \text{if } b_u = 1, x \in P_u(\delta_{l^p}) \cap A_p, \\ a_2 & \text{if } b_u = 0, x \in P_u(\delta_{l^p}) \cap A_p, \\ \pi^*(x) & \text{if } x \in \bigcup_{i < p} A_i \\ a_1 & \text{if } x \notin \bigcup_{i \le p} A_i. \end{cases}$$

Note that by induction hypothesis on the rewards r_t for $t \leq T^{j(p'-1)}$, using the rewards $\tilde{r}^{b,r}$, π^b always selects the best action in hindsight for times $t \leq T^{j(p'-1)}$. Also, by construction, π^b also selects the best action in hindsight for times $T^{j(p'-1)} < t \leq T^p$.

Similarly to before, suppose that b, c are generated as independent i.i.d. $\mathcal{B}(\frac{1}{2})$ processes. We now argue that on the event \mathcal{E} , the learning process with rewards $r^{\mathbf{b},c}$ until T^p is stochastically equivalent to the learning process with rewards $\tilde{r}^{b,c}$ until T^p . Indeed, these rewards only differ in that for different periods $(T_p^{k(l-1)}, T_p^{k(l)}]$, we may have reward r_{δ_l, b^l} instead of $r_{\delta_{l^p}, b}$. However, on the event \mathcal{E} , new instances always fall in portions where the reward of a_1 is still $\mathcal{B}(\frac{1}{2})$ conditionally on the current available history. This holds for both reward sequences. Further, duplicates can only affect rewards during the same period $(T_p^{k(l-1)}, T_p^{k(l)}]$ by construction—if x_t is a duplicate from a previous period, the reward function is 0. Hence, even though for $r^{\mathbf{b},c}$, we have distinct sequences b^l , these are all consistent with a

single sequence **b** based on a finer partition at scale δ_{l^p} . Precisely, we have

$$\begin{split} \mathbb{E}_{\boldsymbol{b},\boldsymbol{c}} \left[\mathbb{E}_{\mathbb{X},\hat{a}} \left[\sup_{T^{j(p'-1)} < T \leq T^{p}} \frac{1}{T} \sum_{t=1}^{T} \tilde{r}_{t}^{\boldsymbol{b},\boldsymbol{c}}(\pi^{\boldsymbol{b}}(X_{t})) - \tilde{r}^{\boldsymbol{b},\boldsymbol{c}}(\hat{a}_{t}) \mid \mathcal{E} \right] \right] \\ &= \mathbb{E}_{\mathbb{X}} \left[\mathbb{E}_{\boldsymbol{b},\boldsymbol{c}} \mathbb{E}_{\hat{a}} \left[\sup_{T^{j(p'-1)} < T \leq T^{p}} \frac{1}{T} \sum_{t=1}^{T} \tilde{r}_{t}^{\boldsymbol{b},\boldsymbol{c}}(a_{t}^{*})) - \tilde{r}^{\boldsymbol{b},\boldsymbol{c}}(\hat{a}_{t}) \mid \mathbb{X}, \mathcal{E} \right] \mid \mathcal{E} \right] \\ &= \mathbb{E}_{\mathbb{X}} \left[\mathbb{E}_{\boldsymbol{b},\boldsymbol{c}} \mathbb{E}_{\hat{a}} \left[\sup_{T^{j(p'-1)} < T \leq T^{p}} \frac{1}{T} \sum_{t=1}^{T} r_{t}^{\boldsymbol{b},\boldsymbol{c}}(a_{t}^{*}) - r^{\boldsymbol{b},\boldsymbol{c}}(\hat{a}_{t}) \mid \mathbb{X}, \mathcal{E} \right] \mid \mathcal{E} \right] \\ &= \mathbb{E} \left[\sup_{T^{j(p'-1)} < T \leq T^{p}} \frac{1}{T} \sum_{t=1}^{T} r_{t}^{\boldsymbol{b},\boldsymbol{c}}(a_{t}^{*}) - r^{\boldsymbol{b},\boldsymbol{c}}(\hat{a}_{t}) \mid \mathcal{E} \right] \geq \frac{\epsilon}{2^{9}}. \end{split}$$

As a result, there exists a specific realization of b and c such that

$$\mathbb{E}_{\mathbb{X},\hat{a}}\left[\sup_{T^{j(p'-1)} < T \le T^p} \frac{1}{T} \sum_{t=1}^T \tilde{r}_t^{\boldsymbol{b},\boldsymbol{c}}(\pi^{\boldsymbol{b}}(X_t)) - \tilde{r}^{\boldsymbol{b},\boldsymbol{c}}(\hat{a}_t) \mid \mathcal{E}\right] \ge \frac{\epsilon}{2^9}.$$

Hence, because $\mathbb{P}[\mathcal{E}^c] \leq \frac{\epsilon}{2^{10}}$, we obtain

$$\mathbb{E}_{\mathbb{X},\hat{a}}\left[\sup_{T^{j(p'-1)} < T \le T^p} \frac{1}{T} \sum_{t=1}^T \tilde{r}_t^{\boldsymbol{b},\boldsymbol{c}}(\pi^{\boldsymbol{b}}(X_t)) - \tilde{r}^{\boldsymbol{b},\boldsymbol{c}}(\hat{a}_t)\right] \ge \frac{\epsilon}{2^9} \left(1 - \frac{\epsilon}{2^{10}}\right) - \frac{\epsilon}{2^{10}} \ge \frac{\epsilon}{2^{11}}$$

Now for all $t \leq T^p$ we pose $r_t = \tilde{r}_t^{b,c}$, and complete the definition of π^* by setting $\pi^*(x) = \pi^b(x)$ on $\bigcup_{i \leq p} A_i$. Note that these definitions are consistent with the previously constructed rewards and the actions selected by the policy on $\bigcup_{i < p} A_i$. This ends the recursive construction of the rewards $r = (r_t)_{t \geq 1}$ and the policy π^* on $\bigcup_{i \geq 1} A_i$. We close the definition of π^* by setting $\pi^*(x) = a_1$ for $x \notin \bigcup_{i \geq 1} A_i$ arbitrarily. The constructed policy π^* is measurable because it is measurable on each A_i for $i \geq 1$.

We now analyze the regret of the algorithm compared to π^* for the rewards $(r_t)_t$. First, note that the rewards are deterministic and that π^* is the optimal policy, i.e., which always selects the best arm in hindsight. Also, if **b**, **c** denote the realizations used in the iteration p = j(p') of the above recursion, for any $t \leq T^p$ we have $r_t = \tilde{r}_t^{b,c}$. As a result,

$$\mathbb{E}\left[\sup_{T^{j(p'-1)} < T \le T^{j(p')}} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^b(X_t)) - r_t(\hat{a}_t)\right] \ge \frac{\epsilon}{2^{11}}.$$

Now by Fatou's lemma, we have

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^b(X_t)) - r_t(\hat{a}_t)\right]$$

= $\mathbb{E}\left[\limsup_{p' \to \infty} \sup_{T^{j(p'-1)}} < T \le T^{j(p')} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^b(X_t)) - r_t(\hat{a}_t)\right]$
 $\ge \limsup_{p \to \infty} \mathbb{E}\left[\sup_{T^{j(p'-1)}} < T \le T^{j(p')} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^b(X_t)) - r_t(\hat{a}_t)\right]$
 $\ge \frac{\epsilon}{2^{11}}.$

As a result, f is not consistent on the oblivious rewards $(r_t)_t$ under \mathbb{X} , which contradicts the hypothesis that f is universally consistent under \mathbb{X} . This ends the proof of the theorem.

Recall that the condition C_2 is necessary for universal learning because this is already the case for noiseless online learning [2] and is also a sufficient for universal learning in noiseless online learning [14], online learning with adversarial responses [16] and stationary contextual bandits [1]. In the next proposition, we show that our new necessary condition C_4 is a stronger condition than C_2 .

Proposition 19. Let \mathcal{X} be a metrizable separable Borel space. Then, $C_4 \subset C_2$.

Proof. Suppose that $\mathbb{X} \notin C_2$, then there exists a sequence of disjoint sets $(A_i)_{i\geq 1}$ and $\epsilon > 0$ such that $\mathbb{E}[\limsup_{T\to\infty} \frac{1}{T} | \{i \geq 1, A_i \cap \mathbb{X}_{\leq T} \neq \emptyset\} |] \geq \epsilon$. We now let $B_i = \bigcup_{j\geq i} A_j$. We define $\overline{\mathcal{T}} = \{t \geq 1 : \forall t' < t, X_{t'} \neq X_t\}$ the set of new instances times. Then, for any $i \geq 1$,

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^i} \mathbb{1}_{B_i}(X_t)\right] \geq \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \bar{\mathcal{T}}} \mathbb{1}_{B_i}(X_t)\right]$$
$$\geq \mathbb{E}\left[\limsup_{T \to \infty} \frac{|\{j \geq i : A_j \cap \mathbb{X}_{\leq T} \neq \emptyset\}|}{T}\right]$$
$$= \mathbb{E}\left[\limsup_{T \to \infty} \frac{|\{j \geq 1 : A_j \cap \mathbb{X}_{\leq T} \neq \emptyset\}|}{T}\right]$$
$$\geq \epsilon.$$

This holds for all $i \ge 1$ and $B_i \downarrow \emptyset$. Hence, the second property of Proposition 17 implies $\mathbb{X} \notin C_4$.

In fact, C_4 is a strictly stronger condition than C_2 provided that \mathcal{X} admits a non-atomic probability measure. More precisely, in the next result, we explicitly construct a process $\mathbb{X} \in C_2 \setminus C_4$ which does not admit universal learning even in the memoryless setting. As a result, for memoryless, oblivious, prescient and online rewards, one cannot universally learn all C_2 processes, while this was achievable for stationary rewards. Thus having adversarial partial-feedback on the losses of each action strictly reduces the set of learnable processes $C_{online} \subset C_{oblivious} \subset C_{memoryless} \subseteq C_2$.

Theorem 20. Let \mathcal{X} be a metrizable separable Borel space such that there exists a non-atomic probability measure on \mathcal{X} , and a finite action space \mathcal{A} with $|\mathcal{A}| \ge 2$. Then, $\mathcal{C}_4 \subsetneq \mathcal{C}_2$ and the set of learnable processes also satisfies $\mathcal{C}_{memoryless} \subsetneq \mathcal{C}_2$.

Before proving this result, we present a lemma which allows to have a countable sequence of nonatomic measures with disjoint support.

Lemma 21. Let \mathcal{X} be a metrizable separable Borel space such that there exists a non-atomic probability measure on \mathcal{X} . Then, there exists a sequence of disjoint non-empty measurable sets $(A_i)_{i\geq 0}$ and probability measures $(\nu_i)_{i\geq 0}$ on \mathcal{X} such that $\nu_i(A_i) = 1$.

Proof. Let ρ denote the metric on \mathcal{X} . First, let $(x^i)_{i\geq 1}$ be a dense sequence on \mathcal{X} . For any $x \in \mathcal{X}$ and r > 0 we denote by $B(x, r) = \{x' \in \mathcal{X} : \rho(x, x') < \delta\}$ the open ball centered at x of radius r. Then, for any $\delta > 0$, we define the partition $\mathcal{P}(\delta) = (P_i(\delta))_{i\geq 1}$ by $P_i(\delta) = B(x^i, \delta) \setminus \bigcup_{i < i} B(x^j, \delta)$.

Let μ_{-1} a non-atomic probability measure on \mathcal{X} . We construct the disjoint measures and sets recursively. We pose $B_0 = \mathcal{X}$. Suppose for $p \ge 1$ that we have constructed disjoint sets $(A_i)_{i \le p-1}$, disjoint with B_{p-1} , as well as non-atomic probability measures $(\nu_i)_{i \le p-1}$ and μ_{p-1} satisfying $\nu_i(A_i) = 1$ for

 $i \leq p-1$ and $\mu_{p-1}(B_{p-1}) = 1$. Now let $Z_1, Z_2 \sim \mu_{p-1}$ two independent random variables with distribution μ_{p-1} . Because μ_{p-1} is non-atomic, $Z_1 \neq Z_2$ almost surely. Thus, there exists $\delta_p > 0$ such that $\mathbb{P}[\rho(Z_1, Z_2) \leq \delta_p] \leq \frac{1}{2}$. As a result, with probability at least $\frac{1}{2}, Z_1$ and Z_2 fall in distinct sets of the partition $\mathcal{P}(\delta_p)$. Hence, there exists at least two indices i < j such that $\mathbb{P}[Z_1 \in P_i(\delta_p)], \mathbb{P}[Z_2 \in P_j(\delta_p)] > 0$. We then pose $A_p = B_{p-1} \cap P_i(\delta_p)$ and $B_p = B_{p-1} \cap P_j(\delta_p)$. Because $\mu_{p-1}(B_{p-1}) = 1$, we have $\mu_{p-1}(A_p) = \mu_{p-1}(P_i(\delta_p)) > 0$. Similarly, $\mu_{p-1}(B_p) > 0$. Hence, we can consider the probability measure ν_p of μ_{p-1} conditionally on A_p (i.e. $\nu_p(A) = \frac{\mu_{p-1}(A \cap A_p)}{\mu_{p-1}(A_p)}$ for all measurable A). Similarly, let μ_p the probability measure of μ_{p-1} conditionally on B_p . Both are non-atomic because the original measure μ_{p-1} is non-atomic. This ends the recursion and the proof of the lemma.

We are now ready to prove the theorem.

Proof of Theorem 20. Fix $a_1, a_2 \in \mathcal{A}$ two distinct actions. Let $(x^i)_{i\geq 1}$ be a dense sequence of \mathcal{X} and denote by B(x, r) denotes the open ball centered at $x \in \mathcal{X}$ with radius r > 0. Using, Lemma 21, let $(A_i)_{i\geq 0}$ disjoint measurable sets together with non-atomic probability measures $(\nu_i)_{i\geq 0}$ such that $\nu_i(A_i) = 1$. We then fix $x_0 \in A_0$ (we will not use the set A_0 any further and from now will only reason on the sets $(A_i)_{i\geq 1}$) and for $i \geq 1$, we define $S_i = \{k \geq 1 : k \equiv 2^{i-1} \mod 2^i\}$. Then let \mathbb{Z}^i for $i \geq 1$ be independent processes where \mathbb{Z}^i is an i.i.d. process following the distribution ν_i . We now construct a process \mathbb{X} on \mathcal{X} . For any $k \geq 1$, let $T_k = 2^k k!$, $n_i = 2^{\lfloor \log_2 i \rfloor}$ for $i \geq 1$, and $l_k = \sum_{l \in S_i, l < k} \frac{T_k}{n_i}$, where $k \equiv 2^{i-1} \mod 2^i$. For any $t \geq 1$, we pose

$$X_t = \begin{cases} Z_{l_k+r}^i & \text{if } T_k \le t < 2T_k, k \equiv 2^{i-1} \mod 2^i, t-T_k \equiv r \mod \frac{T_k}{n_i}, 1 \le r \le \frac{T_k}{n_i}, \\ x_0 & \text{otherwise.} \end{cases}$$

This ends the construction of X. We now argue that $X \in C_2$. Let $(B_l)_{l\geq 1}$ be a sequence of disjoint measurable sets of \mathcal{X} . Because \mathbb{Z}^i is an i.i.d. process for any $i \geq 1$, the event \mathcal{E}_i where $|\{l : \mathbb{Z}_{\leq T}^i \cap B_l \neq \emptyset\}| = o(T)$ has probability one. Now define $\mathcal{E} = \bigcap_{i\geq 1} \mathcal{E}_i$, which has probability one by the union bound. Fix $\epsilon > 0$ and $i^* = \lceil \frac{2}{\epsilon} \rceil$ so that $\epsilon \leq \frac{1}{n_i^*}$. On the event \mathcal{E} for any $i \leq i^*$ there exists T_i such that for any $T \geq T_i$ we have $|\{l : \mathbb{Z}_{\leq T}^i \cap B_l \neq \emptyset\}| \leq \frac{\epsilon}{2^i}T$. Now let $T^0 = \max_{i\leq i^*} T_i n_i$. Then, on \mathcal{E} , for any $T \geq T^0$,

$$\begin{split} |\{l : \mathbb{X}_{\leq T} \cap B_l \neq \emptyset\}| &\leq 1 + \sum_{i=1}^{i^*} |\{l : \mathbb{Z}_{\leq \lfloor T/n_i \rfloor}^i \cap B_l \neq \emptyset\}| \\ &+ |\{l : \exists t \leq T : X_t \in B_l, T_k \leq t < 2T_k, k \equiv 0 \mod 2^{i^*}\}| \\ &\leq 1 + \epsilon T + |\{X_t, \quad t \leq T, T_k \leq t < 2T_k, k \equiv 0 \mod 2^{i^*}\}| \\ &\leq 1 + \epsilon T + \frac{T}{n_{i^*}} + \frac{T}{n_{i^*}} \\ &\leq 3\epsilon T + 1. \end{split}$$

In the first inequality, the additional 1 is due to the visit of x_0 , and in the third inequality, we used the fact that in a phase $i > i^*$, each point is duplicated $n_i \ge n_{i^*}$ times. This yields a term $\frac{T}{n_{i^*}}$. The second term $\frac{T}{n_{i^*}}$ in the third inequality is due to boundary effects for times close to T, the worst-case scenarios being attained for T of the form $T_k(1 + \frac{1}{n_i})$. As a result, on \mathcal{E} , we have $\limsup_{T\to\infty} \frac{1}{T} |\{l : \mathbb{X}_{\le T} \cap B_l \neq \emptyset\}| \le 3\epsilon$, which holds for any $\epsilon > 0$. Thus, $\frac{1}{T} |\{l : \mathbb{X}_{\le T} \cap B_l \neq \emptyset\}| \to 0$ on \mathcal{E} , which ends the proof that $\mathbb{X} \in C_2$.

We now show that there does not exist an universally consistent algorithm under X for memoryless rewards. One can easily check that $X \notin C_4$, since for any $i \ge 1$, we have

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{\lfloor \log_2 i \rfloor}} \mathbb{1}_{A_i}(X_t)\right] \ge \mathbb{E}\left[\limsup_{k \to \infty} \frac{\mathbb{1}_{S_i}(k)}{2T_k} \sum_{t \le 2T_k, t \in \mathcal{T}^{\lfloor \log_2 i \rfloor}} \mathbb{1}_{A_i}(X_t)\right]$$
$$\ge \mathbb{E}\left[\limsup_{k \to \infty} \frac{\mathbb{1}_{S_i}(k)}{2}\right] \ge \frac{1}{2}.$$

This already shows that $C_{online} \subset C_{oblivious} \subset C_4 \subsetneq C_2$. However, we will show a stronger statement that $\mathbb{X} \notin C_{memoryless}$. The proof uses the same techniques as Theorem 18, but leverages the fact that the phases S^i are deterministic and instances from previous phases $[T_k, 2T_k)$ do not appear in future phases. By contradiction, suppose that f is a universally consistent learning rule. We will refer to its decision at time t as \hat{a}_t for simplicity. We will construct recursively rewards $(r_t)_{t\geq 1}$ on which this algorithm is not consistent, as well as a policy $\pi^* : \mathcal{X} \to \mathcal{A}$ compared to which the algorithm has high regret. The rewards and policy are constructed recursively together with an increasing sequence of times $(T^p)_{p\geq 1}$ and indices $(i_p)_{p\geq 1}$ with $i_1 = 1$ such that after the p-th iteration of the construction process, the rewards will be deterministic and stationary, hence we may omit the subscript t. Suppose that we have performed p-1 iterations of this construction for $p\geq 1$. We will drop the subscripts p for simplicity and simply assume that we have defined the reward $r(a \mid \cdot)$ and the value of the policy $\pi^*(\cdot)$ on $\bigcup_{j < i} A_j$ for some $i \geq 1$ ($i = i_p$). We now construct the rewards on A_i . To do so, we will first introduce other memoryless rewards. For any $k \in S_i$, because ν_i is non-atomic, there exists δ_k such that

$$\mathbb{P}\left[\min_{1 \le u < v \le l_k + \frac{T_k}{n_i}} \rho(Z_u^i, Z_v^i) \le \delta_k\right] \le 2^{-k-5}$$

Then, let \mathcal{E}^i be the event when for all $k \in S_i$, we have $\min_{1 \le u < v \le l_k + \frac{T_k}{n_i}} \rho(Z_u^i, Z_v^i) > \delta_k$, and \mathbb{Z}^i takes values in A_i only—this is almost sure since $\nu_i(A_i) = 1$. By the union bound, $\mathbb{P}[\mathcal{E}^i] \ge 1 - \frac{1}{32}$. Now for $\delta > 0$ and $u \ge 1$, define the sets $P_u(\delta) = (A_i \cap B(x^u, \delta)) \setminus \bigcup_{v < u} B(x^v, \delta)$ which form a partition of A_i . For any $\delta > 0$ and sequence $\mathbf{b} = (b_u)_{u \ge 1}$ in $\{0, 1\}$ we consider the following deterministic rewards

$$r_{\delta,\mathbf{b}}(a \mid x) = \begin{cases} b_u & a = a_1, \ x \in P_u(\delta), \\ \frac{3}{4} & a = a_2, \\ 0 & a \notin \{a_1, a_2\}, \end{cases} \text{ if } x \in A_i, \qquad r_{\delta,\mathbf{b}}(a \mid x) = r(a \mid x) \text{ if } x \in \bigcup_{j < i} A_j,$$

and $r_{\delta,\mathbf{b}}(\cdot \mid x) = 0$ if $x \notin \bigcup_{j \leq i}$. Now for any sequence of binary sequences $\mathbf{b} = (\mathbf{b}^k)_{k \in S_i}$ where $\mathbf{b}^k = (b^k_u)_{u \geq 1}$, we will consider the memoryless rewards $r^{\mathbf{b}}$ defined as follows. For any $t \geq 2$, let $k \geq 1$ such that $T^k \leq t < T^{k+1}$, and $k' = \min\{l \in S_i : l \geq k\}$. We pose $r^{\mathbf{b}}_t = r_{\delta_{k'}, \mathbf{b}^{k'}}$, and $r^{\mathbf{b}}_1 = r^{\mathbf{b}}_2$. Now let \mathbf{b} be generated such that all \mathbf{b}^i are independent i.i.d. Bernouilli $\mathcal{B}(\frac{1}{2})$ random sequences in $\{0, 1\}$. Next, define $\pi_0 : x \in \mathcal{X} \mapsto a_2 \in \mathcal{A}$, the policy which always selects arm a_2 . Now fix any realization of $r^{\mathbf{b}}$. Because f is universally consistent for memoryless rewards, it has in particular sublinear regret compared to π_0 under rewards $r^{\mathbf{b}}$, i.e., almost surely $\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^T r^{\mathbf{b}}_t(a_2 \mid X_t) - r^{\mathbf{b}}_t(\hat{a}_t \mid X_t)) \leq 0$. The same arguments as in Theorem 15 with Fatou's lemma give

$$\limsup_{T \to \infty} \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} r_t^{\mathbf{b}}(a_2 \mid X_t) - r_t^{\mathbf{b}}(\hat{a}_t \mid X_t) \mid \mathcal{E}^i\right] \le 0$$
where the expectation is now also taken over **b**. Therefore, with $\alpha_i := \frac{1}{16 \cdot 4^{n_i}}$, there exists t_0 such that for all $T \ge t_0$, we have $\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^T r_t^{\mathbf{b}}(a_2 \mid X_t) - r_t^{\mathbf{b}}(\hat{a}_t \mid X_t) \mid \mathcal{E}^i\right] \le \frac{\alpha_i}{4}$. In particular, there exists $k \in S_i$ such that $k \ge \frac{4}{\alpha_i}$ and $T_k \ge t_0$ and the above inequality holds for all $T_k \le T < 2T_k$. Then, using the same arguments as in the proof of Theorem 15, if a_t^* denotes the best action in hindsight at time t, we have

$$\mathbb{E}\left[\sum_{t=T_k}^{2T_k-1} r_t^{\mathbf{b}}(a_t^* \mid X_t) - r_t^{\mathbf{b}}(\hat{a}_t \mid X_t) \mid \mathcal{E}^i\right] \ge \frac{T_k}{16}.$$

For any binary sequence **b**, we will write for conciseness $r^{\mathbf{b}} = r_{\delta_k, \mathbf{b}}$. We also define the following policy, restricted to instances in A_i :

$$\pi^{\boldsymbol{b}}: x \in A_i \mapsto \begin{cases} a_1 & \text{if } b_u = 1, x \in P_u(\delta_k), \\ a_2 & \text{if } b_u = 0, x \in P_u(\delta_k). \end{cases}$$

Now consider the case where **b** is an i.i.d. sequence of Bernouillis $\mathcal{B}(\frac{1}{2})$. We argue that on the event \mathcal{E}^i , the learning process before time $2T_k - 1$ and under rewards $\mathbf{r}^{\mathbf{b}}$ is stochastically equivalent to the learning under stationary rewards $\mathbf{r}^{\mathbf{b}} := (\mathbf{r}^{\mathbf{b}})_{t \geq 1}$ before $2T_k - 1$. Precisely, we have

$$\begin{split} \mathbb{E}_{\boldsymbol{b}\sim\mathcal{B}(\frac{1}{2})} \left[\mathbb{E}_{\mathbb{X},\hat{a}} \left[\sum_{t=T_{k}}^{2T_{k}-1} r^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_{t}) \mid X_{t}) - r^{\boldsymbol{b}}(\hat{a}_{t} \mid X_{t}) \mid \mathcal{E}^{i} \right] \right] \\ &= \mathbb{E}_{\mathbb{X}} \left[\mathbb{E}_{\boldsymbol{b}\sim\mathcal{B}(\frac{1}{2})} \mathbb{E}_{\hat{a}} \left[\sum_{t=T_{k}}^{2T_{k}-1} r^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_{t}) \mid X_{t}) - r^{\boldsymbol{b}}(\hat{a}_{t} \mid X_{t}) \mid \mathbb{X}, \mathcal{E}^{i} \right] \mid \mathcal{E}^{i} \right] \\ &= \mathbb{E}_{\mathbb{X}} \left[\mathbb{E}_{\mathbf{b}} \mathbb{E}_{\hat{a}} \left[\sum_{t=T_{k}}^{2T_{k}-1} r^{\boldsymbol{b}}_{t}(a^{*}_{t} \mid X_{t}) - r^{\boldsymbol{b}}_{t}(\hat{a}_{t} \mid X_{t}) \mid \mathbb{X}, \mathcal{E}^{i} \right] \mid \mathcal{E}^{i} \right] \\ &= \mathbb{E} \left[\sum_{t=T_{k}}^{2T_{k}-1} r^{\boldsymbol{b}}_{t}(a^{*}_{t} \mid X_{t}) - r^{\boldsymbol{b}}_{t}(\hat{a}_{t} \mid X_{t}) \mid \mathcal{E}^{i} \right] \\ &\geq \frac{T_{k}}{16}, \end{split}$$

where in the second inequality we used the fact that on the event \mathcal{E}^i , until time $2T_k - 1$ all distinct instances in A_i fall in distinct sets of the partition $(P_u(\delta_k))_u$: for both rewards \mathbf{r}^b and \mathbf{r}^b , the reward on a new instance A_i is independent from the past and has the distribution $\mathcal{B}(\frac{1}{2})$ for action a_1 and deterministic $\frac{3}{4}$ for action a_2 . As a result, there exists a specific realization of \mathbf{b} such that

$$\mathbb{E}_{\mathbb{X},\hat{a}}\left[\sum_{t=T_k}^{2T_k-1} r^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_t) \mid X_t) - r^{\boldsymbol{b}}(\hat{a}_t \mid X_t) \mid \mathcal{E}^i\right] \geq \frac{T_k}{16}.$$

Hence, because $\mathbb{P}[(\mathcal{E}^i)^c] \leq \frac{1}{32}$, we obtain

$$\mathbb{E}_{\mathbb{X},\hat{a}}\left[\sum_{t=T_{k}}^{2T_{k}-1} r^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_{t}) \mid X_{t}) - r^{\boldsymbol{b}}(\hat{a}_{t} \mid X_{t})\right] \ge \frac{T_{k}}{16} \left(1 - \frac{1}{32}\right) - \frac{T_{k}}{32} \ge \frac{2T_{k} - 1}{2^{7}}$$

Now denote $T^p = 2T_k - 1$, and let $i_{p+1} = 1 + \max\{j \ge i : \exists l \in S_i, T_l \le T^i\} = 1 + \max\{j \ge i : T_{2^{j-1}} \le T^i\}$. The index i_{p+1} is chosen so that until time T^p , the process \mathbb{X} has not visited $\bigcup_{j\ge i_p} A_j$ yet. Note that this index is well defined since $T_k \to \infty$ as $k \to \infty$. We then pose $r(\cdot \mid x) = r^b(\cdot \mid x)$ for all $x \in \bigcup_{i\le j < i_{p+1}} A_j$. In particular, we have $r(a \mid x) = 0$ for all $x \in \bigcup_{i_p < j < i_{p+1}} A_j$. Then pose

$$\pi^*(x) = \begin{cases} \pi^{\boldsymbol{b}}(x) & x \in A_i \\ a_2 & x \in \bigcup_{i < j < i^{p+1}} A_j. \end{cases}$$

This ends the recursive construction of the reward r and the policy π^* , i.e., we have constructed $r(\cdot | x)$ and $\pi^*(x)$ for all $x \in \bigcup_{i \ge 1} A_i$. We end the definition of the rewards by posing $r_t(\cdot | x) = 0$ and $\pi^*(x) = a_2$ if $x \notin \bigcup_{i \ge 1} A_i$. Note that $(r_t)_{t \ge 1}$ forms a valid sequence of rewards since by construction on each A_i they are deterministic. Similarly, π^* is measurable because it is measurable on each A_i .

We now analyze the regret of the algorithm compared to π^* for the rewards $(r_t)_t$. First, note that the rewards are deterministic, time independent, and that π^* is the optimal policy, i.e., which always selects the best arm in hindsight. Then, for any $p \ge 1$, we have

$$r(\cdot \mid x) = r^{\mathbf{b}}(\cdot \mid x), \quad \forall x \in \mathcal{X} \setminus \bigcup_{i \ge i_{p+1}} A_i.$$

where r^{b} denotes the rewards defined at the *p*-th iteration of the construction process. Now recall that by construction, the sets A_i visited by the process $\mathbb{X}_{\leq T^p}$ all satisfy $i < i_{p+1}$, which is the first index for which the rewards would differ. As a result, we have

$$\mathbb{E}\left[\frac{1}{T^{p}}\sum_{t=1}^{T^{p}}r(\pi^{*}(X_{t})\mid X_{t}) - r(\hat{a}_{t}\mid X_{t})\right] \geq \mathbb{E}\left[\frac{1}{T^{p}}\sum_{t=(T_{p}+1)/2}^{T^{p}}r(\pi^{*}(X_{t})\mid X_{t}) - r(\hat{a}_{t}\mid X_{t})\right]$$
$$= \mathbb{E}\left[\frac{1}{T^{p}}\sum_{t=(T_{p}+1)/2}^{T^{p}}r^{b}(\pi^{b}(X_{t})\mid X_{t}) - r^{b}(\hat{a}_{t}\mid X_{t})\right]$$
$$\geq \frac{1}{2^{7}},$$

where in the first inequality we used the fact that π^* always selects the best action in hindsight. Because this holds for any $p \ge 1$, we can use Fatou's lemma to obtain

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t) \mid X_t) - r_t(\hat{a}_t \mid X_t)\right]$$
$$\geq \limsup_{T \to \infty} \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t) \mid X_t) - r_t(\hat{a}_t \mid X_t)\right] \geq \frac{1}{2^7}.$$

As a result, f is not consistent on the stationary rewards $(r)_t$ under X, which ends the proof of the theorem.

5.1.2 A tighter necessary condition 6 for oblivious rewards

This section proves that C_6 is necessary for stochastic processes, which is tighter than the family C_4 . We first prove the lemma on large deviations of the empirical measure in C'_1 processes.

Proof of Lemma 9. Let $\epsilon > 0$ and suppose by contradiction that for all $T \ge 1$ and $\delta > 0$ there exists a measurable set $A(\delta; T)$ such that $\mathbb{E}[\hat{\mu}_{\tilde{\mathbb{X}}}(A(\delta; T))] \le \delta$ and

$$\mathbb{E}\left[\sup_{T'>T}\frac{1}{T'}\sum_{t\leq T',t\in\mathcal{T}}\mathbb{1}_{A(\delta;T)}(X_t)\right]>\epsilon.$$

We now construct by induction a sequence of sets $(A_i)_{i\geq 1}$ together with times $(T_i)_{i\geq 0}$ such that $T_0 = 0$. Now suppose that we have constructed T_{i-1} for $i \geq 1$. We take $A_i = A(\epsilon 2^{-i-2}; T_{i-1})$. Then, because $\mathbb{E}[\hat{\mu}_{\tilde{\mathbb{X}}}(A_i)] \leq \epsilon 2^{-i-2}$, by the dominated convergence theorem, there exists $T_i > T_{i-1}$ such that

$$\mathbb{E}\left[\sup_{T>T_i} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_{A_i}(X_t)\right] \le \frac{\epsilon}{2^{i+1}}$$

This ends the construction of the sequences. For any $i \ge 1$, let $B_i = A_i \setminus \bigcup_{j < i} A_j$ and note that

$$\mathbb{E}\left[\sup_{T>T_{i-1}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}}\mathbb{1}_{A_{i}}(X_{t})\right]$$

$$\leq \mathbb{E}\left[\sup_{T>T_{i-1}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}}\mathbb{1}_{B_{i}}(X_{t})\right] + \sum_{jT_{i-1}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}}\mathbb{1}_{A_{j}}(X_{t})\right]$$

$$\leq \mathbb{E}\left[\sup_{T>T_{i-1}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}}\mathbb{1}_{B_{i}}(X_{t})\right] + \sum_{jT_{j}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}}\mathbb{1}_{A_{j}}(X_{t})\right]$$

$$\leq \mathbb{E}\left[\sup_{T>T_{i-1}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}}\mathbb{1}_{B_{i}}(X_{t})\right] + \frac{\epsilon}{2}.$$

By construction $\mathbb{E}\left[\sup_{T>T_{i-1}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} \mathbb{1}_{A_i}(X_t)\right] > \epsilon$. Hence, letting $C_i = \bigcup_{j \geq i} B_j$, we obtain that for any $j \geq i$,

$$\mathbb{E}\left[\sup_{T>T_j} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} \mathbb{1}_{C_i}(X_t)\right] \geq \mathbb{E}\left[\sup_{T>T_j} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} \mathbb{1}_{B_{j+1}}(X_t)\right] \geq \frac{\epsilon}{2}.$$

As a result, by the dominated convergence theorem we have $\mathbb{E}[\hat{\mu}_{\tilde{\mathbb{X}}}(C_i)] \geq \frac{\epsilon}{2}$. Further, all sets B_i are disjoint. But $C_i \downarrow \emptyset$, which contradicts the hypothesis that $\tilde{\mathbb{X}} \in \mathcal{C}'_1$. This ends the proof of the lemma.

We recall the necessary definitions to introduce condition C_6 . For a process $X \in C_4$, any $\epsilon > 0$ and $T \ge 1$,

$$\begin{split} \delta^p(\epsilon;T) &:= \sup \left\{ 0 \leq \delta \leq 1 : \forall A \in \mathcal{B} \text{ s.t. } \sup_l \mathbb{E}[\hat{\mu}_{\mathbb{X}^l}(A)] \leq \delta, \\ \forall \tau \geq T \text{ online stopping time, } \mathbb{E}\left[\frac{1}{2\tau} \sum_{\tau \leq t < 2\tau, t \in \mathcal{T}^p} \mathbbm{1}_A(X_t) \right] \leq \epsilon \right\}, \end{split}$$

and $\delta^p(\epsilon) := \lim_{T \to \infty} \delta^p(\epsilon; T) > 0$. We recall condition \mathcal{C}_6 .

Condition 6. $X \in C_4$ and for any $\epsilon > 0$, we have $\lim_{p\to\infty} \delta^p(\epsilon) > 0$. Denote by C_6 the set of all processes X satisfying this condition.

The main result of this section is that this condition is necessary for oblivious rewards.

Theorem 22. Let \mathcal{X} be a metrizable separable Borel space, and a finite action space \mathcal{A} with $|\mathcal{A}| \geq 2$. Then, $C_{oblivious} \subset C_6$.

Proof. Fix $\mathbb{X} \in C_4 \setminus C_6$. By hypothesis, there exists $\epsilon > 0$ such that $\delta^p(\epsilon) \to 0$ as $p \to \infty$. Let $(p(i))_{i \ge 1}$ be the set of increasing indices such that $\delta^{p(i)}(\epsilon) \le \epsilon 2^{-i-3}$. Similarly to the proof of Theorem 18, we suppose by contradiction that there is a universally consistent learning rule f under \mathbb{X} and we will construct by induction some rewards on which the learning rule is not consistent. We will denote by \hat{a}_t the action selected by the learning rule at time t. Precisely, suppose that we have performed i - 1 iterations of the construction process for some $i \ge 1$, and have constructed times T^1, \ldots, T^{i-1} as well as rewards $(r_t)_{t \le T^{i-1}}$, disjoint sets A^1, \ldots, A^{i-1} satisfying

$$\sup_{l} \mathbb{E}[\hat{\mu}_{\mathbb{X}^{l}}(A^{j})] \le \epsilon 2^{-j-2}$$

for all j < i, and a policy π^* on $\bigcup_{j < i} A^i$. We will now focus on the times $\mathcal{T}^{p(i)}$. For convenience, in the rest of the proof, when clear from context, we will write p instead of p(i).

First, by hypothesis, for any $1 \le j < i$, we have $\mathbb{E}[\hat{\mu}_{\mathbb{X}^p}(A^j)] \le \epsilon 2^{-j-2}$. Thus, by the dominated convergence theorem, there exists t(j) such that

$$\mathbb{E}\left[\sup_{T \ge t(j)} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{A^j}(X_t)\right] \le \frac{\epsilon}{2^{j+1}}.$$

Therefore, summing these equations yields

$$\mathbb{E}\left[\sup_{T \ge \max_{j < i} t(j)} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_{\bigcup_{j < i} A^j}(X_t)\right] \le \frac{\epsilon}{2}.$$

We define $\tilde{T}^{i-1} = \max(T^{i-1}, t(1), \dots, t(i-1))$. Now by construction, $\delta^{p(i)}(\epsilon) \leq \epsilon 2^{-i-3}$. Therefore, there exists $T_0 \geq \tilde{T}^{i-1}$ such that for any $T \geq T_0$, we have $\delta^p(\epsilon; T) \leq \epsilon 2^{-i-2}$. Now for $T \geq T_0$, let $A^i(T) \in \mathcal{B}$ and $\tau^i(T) \geq T$ be a stopping time such that

$$\sup_{l} \mathbb{E}[\hat{\mu}_{\mathbb{X}^{l}}(A^{i}(T))] \leq \epsilon 2^{-i-2} \quad \text{and} \quad \mathbb{E}\left[\frac{1}{2\tau^{i}(T)} \sum_{\tau^{i}(T) \leq t < 2\tau^{i}(T), t \in \mathcal{T}^{p}} \mathbb{1}_{A^{i}(T)}(X_{t})\right] > \epsilon$$

Last, let U(T) be such that

$$\mathbb{P}[2\tau^{i}(T) > U(T)] \ge \frac{\epsilon}{2^{T+10}}.$$

Then, by the union bound, with probability at least $1 - \epsilon 2^{-10}$, for all $T \ge T_0$, we have $2\tau^i(T) \le U(T)$. Denote by \mathcal{H} this event. Next, let $k_i = 2^p + 1$, $\alpha_i = 2^{-p-1}$, $\beta_i = \frac{\epsilon}{2^{10}(1+2\alpha_i)^{(k_i-1)k_i}4^{k_i}}$, $\tilde{K}_i = \left\lceil \frac{2}{\alpha_i} \log \frac{8}{\beta_i} \right\rceil$ and $M_i = \max((1+2\alpha_i)^{\tilde{K}_i}, \frac{2^{10}}{\epsilon})$. We first construct by induction of increasing times $(T(l))_{l\ge 0}$ with $T(0) = M_i T_0$ and $T(l) \ge M_i U(T(l-1))$. For convenience, we use the notation $\tau_l^i = \tau^i(T(l))$, $A_l^i = A^i(T(l)) \setminus \bigcup_{1 \le j < i} A^j$ for $l \ge 0$. Then, by construction, $\tau^i(T(l)) \ge M_i U(T(l-1))$ and

$$\mathbb{E}\left[\frac{1}{2\tau_l^i}\sum_{\substack{\tau_l^i \leq t < 2\tau_l^i, t \in \mathcal{T}^p}} \mathbb{1}_{A_l^i}(X_t)\right] \\
\geq \mathbb{E}\left[\frac{1}{2\tau_l^i}\sum_{\substack{\tau_l^i \leq t < 2\tau_l^i, t \in \mathcal{T}^p}} \mathbb{1}_{A^i(T(l))}(X_t)\right] - \mathbb{E}\left[\frac{1}{2\tau_l^i}\sum_{\substack{\tau_l^i \leq t < 2\tau_l^i, t \in \mathcal{T}^p}} \mathbb{1}_{\bigcup_{j < i} A^j}(X_t)\right] \\
> \epsilon - \mathbb{E}\left[\sup_{T \geq \tilde{T}^{i-1}}\frac{1}{T}\sum_{t \leq T, t \in \mathcal{T}^p} \mathbb{1}_{\bigcup_{j < i} A^j}(X_t)\right] \\
> \frac{\epsilon}{2}.$$

For any $l \ge 1$, let $\delta_l > 0$ such that

$$\mathbb{P}\left[\min_{1 \le t, t' \le U(T(l)), X_t \ne X_{t'}} \rho(X_t, X_{t'}) \le \delta_l\right] \le \frac{\epsilon}{2^{l+10}}$$

Let \mathcal{E} be the event when for all $l \geq 1$, we have $\min_{1 \leq t,t' \leq U(T(l)), X_t \neq X_{t'}} \rho(X_t, X_{t'}) > \delta_l$ and \mathcal{H} is satisfied. By the union bound, $\mathbb{P}[\mathcal{E}] \geq 1 - \frac{\epsilon}{2^9}$. We now construct similar rewards to those in the proof of Theorem 18. Then, for any $\delta > 0$ and $u \geq 1$, define the sets $P_u(\delta) = B(x^u, \delta) \setminus \bigcup_{v < u} B(x^v, \delta)$ where $(x^u)_{u \geq 1}$ is a dense sequence of \mathcal{X} , which form a partition of \mathcal{X} . For any binary sequence $\boldsymbol{b} = (b_u)_{u \geq 1}$ in $\{0, 1\}$ define the deterministic rewards

$$r_{\delta, \mathbf{b}; l}(a \mid x) = \begin{cases} b_u \mathbb{1}_{x \in A_l^i} & a = a_1, x \in P_u(\delta), \\ \frac{3}{4} \mathbb{1}_{x \in A_l^i} & a = a_2, \\ 0 & a \notin \{a_1, a_2\}. \end{cases}$$

Next, for any sequence of binary sequences $\mathbf{b} := (\mathbf{b}^l)_{l \ge 1}$, we construct the deterministic rewards $\mathbf{r}^{\mathbf{b}}$ as follows. First, for $t \le T^{i-1}$, $r^{\mathbf{b}}_t = r_t$ the rewards already constructed. Also, for $T^{i-1} < t \le U(T(0))$, we pose $r^{\mathbf{b}}_t = 0$. Next, observe that τ^i_l is an online stopping time. Therefore, for any $l \ge 0$, $U(T(l-1)) < t < \tau^i_l$ or $2\tau^i_l \le t \le U(T(l))$, we pose $r^{\mathbf{b}}_t = 0$. Finally, for $\tau^i_l \le t < 2\tau^i_l$, U(T(l)) and k such that $T^{k-1}_p < t \le T^k_p$, we pose

$$r_t^{\mathbf{b}}(a \mid x_{\leq t}) = \begin{cases} 0 & \exists t' \leq U(T(l-1)) : x_{t'} = x_t, \\ 0 & \text{o.w., } \exists T_p^{k-1} < t' \leq t : x_{t'} = x_t, \\ r_{\delta_l, \mathbf{b}^l; l}(a \mid x_t) & \text{o.w., } \forall T_p^{k-1} < t' \leq t : x_{t'} \neq x_t, \end{cases}$$

for any $a \in \mathcal{A}$ and $x_{\leq t} \in \mathcal{X}^t$. Now generate **b** as independent i.i.d. Bernouilli $\mathcal{B}(\frac{1}{2})$ processes. We now compare the predictions of the learning rule compared to the constant policy which selects action a_2 . Because the learning rule is consistent under any rewards $r^{\mathbf{b}}$ for any realization **b**, and because $\mathbb{P}[\mathcal{E}] > 0$, taking the expectation over **b**, we obtain

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t^{\mathbf{b}}(a_2) - r_t^{\mathbf{b}}(\hat{a}_t) \mid \mathcal{E}\right] \le 0.$$

Next, we use the dominated convergence theorem to find $l^i \ge 1$ such that

$$\mathbb{E}\left[\sup_{T\geq T(l_i)/2}\frac{1}{T}\sum_{t=1}^T r_t^{\mathbf{b}}(a_2) - r_t^{\mathbf{b}}(\hat{a}_t) \mid \mathcal{E}\right] \leq \frac{\beta_p}{4}.$$

We now define $A^i = A^i_{l^i}$, $T^i = U(T(l^i))$ and focus on the period $[\tau^i_l, 2\tau^i_l)$. Let $\hat{k} = \max\{k : \tau^i_l \ge T^k_p\}$. Then, $[\tau^i_l, 2\tau^i_l) \subset [T^{\hat{k}}_p, T^{\hat{k}+2^p+1}_p)$ and we construct the following sets

$$\mathcal{S}_q = \{T_p^{\hat{k}+q-1} < t \le T_p^{\hat{k}+q} : X_t \in A^i\} \cap \mathcal{T}^p, \quad 1 \le q \le 2^p + 1 = k_i.$$
(4)

We also define Exp_q the exploration steps of arm a_1 during S_q .

$$Exp_q = \left\{ t \in \mathcal{S}_q : \hat{a}_t = a_1 \text{ and } \forall t' \in \bigcup_{q' < q} \mathcal{S}_{q'} : X_{t'} = X_t, \ \hat{a}_{t'} \neq a_1 \right\}$$
$$\setminus \{t : \exists t' \leq U(T(l^i - 1)), X_{t'} = X_t\},$$

and $E_q = |Exp_q|$. The same arguments as in Theorem 18 show that for all $1 \le q \le k_1$, we have $\mathbb{E}\left[\frac{E_q}{T_p^{k+k_i}} \mid \mathcal{E}\right] \le 4^{q+1}(1+2\alpha_i)^{(k_i-1)k_i}\beta_p$. For any $t \ge 1$, let a_t^* be the optimal action in hindsight and define

$$\mathcal{B}_q = \bigcup_{q \le \hat{q}} \left\{ t \in \mathcal{S}_q : \forall t' \in \bigcup_{q' < q} \mathcal{S}_{q'} : X_{t'} = X_t, t \notin Exp_{q'} \right\},\$$

the times such that we never explored action a_2 , before time $T_p^{\hat{k}+q}$. As in the proof of Theorem 18, for times in \mathcal{B} , the learner incurs an average regret at least $\frac{1}{8}$. Therefore,

$$\mathbb{E}\left[\frac{1}{T_p^{\hat{k}+k_i}}\sum_{t=1}^{T_p^{\hat{k}+k_i}}r_t^{\mathbf{b}}(a_t^*)-r_t^{\mathbf{b}}(\hat{a}_t)\mid \mathcal{E}\right] \ge \frac{1}{8}\mathbb{E}\left[\frac{|\mathcal{B}_q|}{T_p^{\hat{k}+k_i}}\mid \mathcal{E}\right].$$

Finally, let $T_p^* = |\{t \le T_p^{\hat{k}+k_i} : X_t \in A^i\} \cap \mathcal{T}^p|$. Noting that we have $\mathbb{E}\left[\frac{T_p^*}{T_p^{\hat{k}+k_i}} \mid \mathcal{E}\right] \ge \frac{1}{2}\mathbb{E}\left[\frac{T_p^*}{2\tau_l^i} \mid \mathcal{E}\right] \ge \frac{\epsilon}{4} \ge \frac{\epsilon}{16}$, the same arguments as in the original proof give directly

$$\mathbb{E}\left[\frac{1}{T_p^{\hat{k}+k_i}}\sum_{t=1}^{T_p^{\hat{k}+k_i}}r_t^{\mathbf{b}}(a_t^*)-r_t^{\mathbf{b}}(\hat{a}_t)\mid \mathcal{E}\right] \geq \frac{\epsilon}{2^8}.$$

As a result, there exists a realization of b such that the above equation holds for this specific realization. We then pose $r_t = r_t^{\mathbf{b}}$ for all $t \leq T^i$ and define a policy π^i on A^i as follows,

$$\pi^{i}(x) = \begin{cases} a_{1} & \text{if } b_{u}^{l} = 1, x \in P_{u}(\delta_{l^{i}}) \cap A^{i}, \\ a_{2} & \text{if } b_{u}^{l} = 0, x \in P_{u}(\delta_{l^{i}}) \cap A^{i}. \end{cases}$$

for any $x \in A^i$, which is possible because A^i is disjoint from $\bigcup_{j < i} A^j$. Now observe that the policy selects the best action in hindsight during the interval $[T(l^i), U(T(l^i)))$, irrespective on how it is defined outside of A^i . As a result, we have

$$\begin{split} & \mathbb{E}\left[\sup_{T^{i-1} < T \le T^{i}} \frac{1}{T} \sum_{t=1}^{T} r_{t}(\pi^{*}(X_{t})) - r_{t}(\hat{a}_{t}) \mid \mathcal{E}\right] \\ & \geq \mathbb{E}\left[\frac{1}{T^{\hat{k}+k_{i}}} \sum_{t=1}^{T^{\hat{k}+k_{i}}} r_{t}^{\mathbf{b}}(\pi^{*}(X_{t})) - r_{t}^{\mathbf{b}}(\hat{a}_{t}) \mid \mathcal{E}\right] \\ & \geq \mathbb{E}\left[-\frac{2U(T(l^{i}-1))}{T^{\hat{k}+k_{i}}} + \frac{1}{T^{\hat{k}+k_{i}}} \sum_{t=1}^{T^{\hat{k}+k_{i}}} r_{t}^{\mathbf{b}}(a_{t}^{*}) - r_{t}^{\mathbf{b}}(\hat{a}_{t}) \mid \mathcal{E}\right] \\ & \geq -\frac{2}{M_{i}} + \frac{\epsilon}{2^{8}} \\ & \geq \frac{\epsilon}{2^{9}}. \end{split}$$

This ends the recursive construction of the rewards. We close the definition of π^* by setting $\pi^*(x) = a_1$ for $x \notin \bigcup_{i>1} A^i$ arbitrarily. The constructed policy is measurable and we showed that for all $i \ge 1$,

$$\mathbb{E}\left[\sup_{T^{i-1} < T \le T^i} \frac{1}{T} \sum_{t=1}^T r_t(\pi^*(X_t)) - r_t(\hat{a}_t)\right] \ge \frac{\epsilon}{2^9}$$

Using Fatou's lemma, this shows that $\mathbb{E}\left[\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\tilde{r}_t(\pi^*(X_t)) - \tilde{r}_t(\hat{a}_t)\right] \geq \frac{\epsilon}{2^9}$. This ends the proof that f is not universally consistent under \mathbb{X} and ends the proof of the theorem.

We now give an example of process $\mathbb{X} \in C_4 \setminus C_6$.

Theorem 23. For $\mathcal{X} = [0, 1]$ with usual topology, $\mathcal{C}_6 \subsetneq \mathcal{C}_4$.

Proof. We construct a process \mathbb{X} on [0,1] by phases $[2^l, 2^{l+1})$ for $l \ge 0$. We set $X_1 = 0$ arbitrarily and divide phases by categories $S_p = \{l \ge 1 : l \equiv 2^{p-1} \mod 2^p\}$ for any $p \ge 1$. Next, for any $l \in S_p$, let

$$A_p(l) = \bigcup_{0 \le i < 2^l} \left[\frac{i2^p}{2^{p+l}}, \frac{i2^p + 1}{2^{p+l}} \right].$$

Importantly, $A_p(l)$ has Lebesgue measure 2^{-p} . Next, noting that $l \ge 2^{p-1} \ge p$, for $2^l \le t < 2^{l+1}$ we define

$$X_t = \begin{cases} \mathcal{U}_t(A_p(l)) & 2^l \le t < 2^l + 2^{l-p}, \\ X_{t'} & t \ge 2^l + 2^{l-p}, 2^l \le t' < 2^l + 2^{l-p}, t' \equiv t \mod 2^{l-p} \end{cases}$$

where $\mathcal{U}_t(A_p(l))$ denotes a uniform random variable on $A_p(l)$ independent from all past random variables. The process on S_p is constructed so that it has 2^p duplicates. This ends the construction of X.

We now show that $\mathbb{X} \in C_4$. For convenience, for any $l \ge 1$, let p(l) be the index such that $l \in S_{p(l)}$. Next, let $\mathbb{X}^p := (X_t)_{t \in \mathcal{T}^p}$ for $p \ge 0$. we will show the stronger statement that for any measurable set $A \in \mathcal{B}$, we have $\hat{\mu}_{\mathbb{X}^p}(A) \le \mu(A)$ (a.s.), where μ is the Lebesgue measure. To do so, fix $A \in \mathcal{B}$ and $\epsilon > 0$. Since A is Lebesgue measurable, there exists a sequence of disjoint intervals $(I_k)_{k\geq 0}$ within $\mathcal{X} = [0, 1]$ such that $A \subset \bigcup_{k\geq 0} I_k$ and

$$\sum_{k\geq 0} \ell(I_k) \leq \mu(A) + \epsilon,$$

where $\ell(I)$ is the length of an interval I. Then, let k_0 such that $\sum_{k \ge k_0} I_k \le \frac{\epsilon^2}{2^{p+1}}$ and pose $\ell_0 = \min_{k < k_0} \ell(I_k)$. Then, for any $l \ge \max(2, \log_2 \frac{k_0}{\epsilon}) := l_0$, with $l \in S_q$,

$$\frac{\mu(A \cap A_q(l))}{\mu(A_q(l))} \le \sum_{k < k_0} \frac{\mu(I_k \cap A_q(l))}{\mu(A_q(l))} + 2^q \mu\left(\bigcup_{k \ge k_0} I_k\right)$$
$$\le \sum_{k < k_0} (\ell(I_k) + 2^{-l}) + \epsilon^2 2^{q-p-1}$$
$$\le \mu(A) + 2\epsilon + \epsilon^2 2^{q-p-1}.$$

Let $q_0 = p + \log_2 \frac{1}{\epsilon}$. For any $l \ge l_0$ with $l \in \bigcup_{q < q_0} S_q$, we have $\frac{\mu(A \cap A_q(l))}{\mu(A_q(l))} \le \mu(A) + 3\epsilon$. Now for any $l \ge l_0$, if $l \in \bigcup_{q < q_0} S_q$, Hoeffding's inequality implies that for any $l \le r \le 2^{l-q}$,

$$\mathbb{P}\left[\sum_{2^{l} \le t < 2^{l} + r} \mathbb{1}_{A}(X_{t}) \le r(\mu(A) + 4\epsilon)\right] \ge 1 - e^{-2\epsilon^{2}r^{2}} \ge 1 - e^{-2\epsilon^{2}lr}.$$

Note that we always have $2^{l-q} \ge l$ since $l \ge 2^{q-1}$ and $l \ge 2$. In particular, because we have $\sum_{r\ge 1}\sum_{l\ge 1}e^{-2\epsilon^2 lr} < \infty$, on an event $\mathcal{E}(\epsilon)$ of probability one, there exists $\hat{l} \ge l_0$ such that the above equation holds for all $l \ge \hat{l}$ with $l \in \bigcup_{q < q_0} S_q$ and $l \le r \le 2^{l-q}$. Then, for $T \ge 2^{\hat{l}}$, letting $l(T) \ge 1$ such that $2^{l(T)} \le T < 2^{l(T)+1}$, we have

$$\sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_A(X_t) = \sum_{l < l(T)} \min(2^{p(l)}, 2^p) \sum_{2^l \le t < 2^l + 2^{l-p(l)}} \mathbb{1}_A(X_t) + \sum_{2^{l(T)} \le t \le T, t \in \mathcal{T}^p} \mathbb{1}_A(X_t)$$

$$\leq \sum_{l < l(T)} \epsilon 2^l \mathbb{1}[p(l) \ge q_0] + 2^{\hat{l}} + \sum_{\hat{l} \le l < l(T)} 2^l (\mu(A) + 4\epsilon) \mathbb{1}[p(l) < q_0]$$

$$\epsilon 2^{l(T)} \mathbb{1}[p(l(T)) \ge q_0] + [(T - 2^{l(T)} + 1)(\mu(A) + 4\epsilon) + l(T)] \mathbb{1}[p(l(T)) < q_0]$$

$$\leq 2^{\hat{l}} + l(T) + 2\epsilon 2^{l(T)} + (\mu(A) + 4\epsilon)T$$

$$\leq 2^{\hat{l}} + \log_2 T + (\mu(A) + 6\epsilon)T.$$

where in the first inequality, we used the fact that for $q \ge q_0$, $2^p \le \epsilon 2^q$. Further, the additional term l(T) comes from the fact that the estimates on $\mathcal{E}(\epsilon)$ held for $r \ge l$: writing $T = 2^{l(T)} + u2^{l(T)-p(l(T))} + v$, we first use $\mathcal{E}(\epsilon)$ with $r = 2^{l(T)-p(l(T))}$, then with $r = \max(v, l(T))$. As a result, on $\mathcal{E}(\epsilon)$, we have $\hat{\mu}_{\mathbb{X}^p}(A) \le \mu(A) + 6\epsilon$. Thus, on $\bigcap_{j\ge 0} \mathcal{E}(2^{-j})$ of probability one, we have $\hat{\mu}_{\mathbb{X}^p}(A) \le \mu(A)$, and this holds for all $p \ge 1$ and $A \in \mathcal{B}$. Using this property, verifying the \mathcal{C}_4 condition is straightforward. For disjoint measurable sets A_i , we have $\mathbb{E}[\hat{\mu}_{\mathbb{X}^i}(A_i)] \le \mu(A_i) \to 0$ because $\sum_i \mu(A_i) \le 1$.

We now show that $\mathbb{X} \notin C_6$. First, on an event \mathcal{F} of probability one, all samples $\mathcal{U}_t(A_p(l))$ are distinct. As a result, on \mathcal{F} , except for the intended duplicates, all instances of \mathbb{X} are distinct. Thus, for any $l \in S_p$, and any $2^{l} \leq t < 2^{l+1}$, we have $t \in \mathcal{T}^{p}$. Hence, on \mathcal{F} ,

$$\frac{1}{2^{l+1}} \sum_{2^{l} \le t < 2^{l+1}, t \in \mathcal{T}^p} \mathbb{1}_{A_p(l)}(X_t) \ge \frac{2^l}{2^{l+1}} = \frac{1}{2}.$$

In particular, this implies that

$$\mathbb{E}\left[\frac{1}{2^{l+1}}\sum_{2^l \le t < 2^{l+1}, t \in \mathcal{T}^p} \mathbb{1}_{A_p(l)}(X_t)\right] \ge \frac{1}{2}.$$

However, $\mathbb{E}[\hat{\mu}_{\mathbb{X}^p}(A_p(l))] = \mu(A_p(l)) = 2^{-p}$. Therefore, using the trivial stopping time $\tau = 2^l$, we showed $\delta^p(1/2; 2^l) \leq 2^{-p}$. Because this holds for all $l \in S_p$ which is infinite, we have $\delta^p(1/2) \leq 2^{-p}$. Thus, $\delta^p(1/2) \to 0$ as $p \to \infty$. This shows that $\mathbb{X} \notin C_6$ and ends the proof of the theorem.

A more natural condition on processes than C_6 would be one that does not involve these stopping times τ . In particular, for a process $X \in C_4$, we can define instead for any $\epsilon > 0$ and $T \ge 1$,

$$\begin{split} \bar{\delta}^p(\epsilon;T) &:= \sup\left\{ 0 \le \delta \le 1 : \forall A \in \mathcal{B} \text{ s.t. } \sup_l \mathbb{E}[\hat{\mu}_{\mathbb{X}^l}(A)] \le \delta, \\ & \mathbb{E}\left[\sup_{T' \ge T} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_A(X_t) \right] \le \epsilon \right\}. \end{split}$$

As before, $\bar{\delta}^p(\epsilon; T)$ is non-decreasing in T and $\bar{\delta}^p(\epsilon) := \lim_{T \to \infty} \delta^p(\epsilon; T) > 0$. We can then observe that $\bar{\delta}^p(\epsilon)$ is non-increasing. Similarly to C_6 , we can then define the following condition.

Condition 7. $X \in C_4$ and for any $\epsilon > 0$, we have $\lim_{p\to\infty} \overline{\delta}^p(\epsilon) > 0$. Denote by C_7 the set of all processes X satisfying this condition.

As a simple remark, we have the inclusion $C_7 \subset C_6$, since if for any given process $X \in C_4$, set $A \in \mathcal{B}$ and online stopping time $\tau \geq T$,

$$\mathbb{E}\left[\frac{1}{2\tau}\sum_{\tau\leq t<2\tau,t\in\mathcal{T}^p}\mathbb{1}_A(X_t)\right]\leq \mathbb{E}\left[\sup_{T'\geq T}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}^p}\mathbb{1}_A(X_t)\right].$$

Unfortunately, for oblivious rewards, we were unable to prove that C_7 is a necessary condition. Indeed, for a process $X \in C_4$, time $T \ge 1$ and $\epsilon > 0$, if

$$\mathbb{E}\left[\sup_{T' \ge T} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^p} \mathbb{1}_A(X_t)\right] > \epsilon,\tag{5}$$

it is in general not true that there exists an online stopping time $\tau \geq T$ such that

$$\mathbb{E}\left[\frac{1}{2\tau}\sum_{\tau\leq t<2\tau,t\in\mathcal{T}^p}\mathbb{1}_A(X_t)\right] > \eta\epsilon,\tag{6}$$

even for a fixed multiplicative tolerance $0 < \eta < 1$, which should be independent of $\epsilon > 0$. Thus, it seems unlikely that $C_6 = C_7$ in general for spaces \mathcal{X} admitting a non-atomic probability measure.

However, if one considers a stronger type of adversary, we can show that C_7 becomes necessary for universal learning. Precisely, one can introduce *prescient* rewards, that are stronger than oblivious rewards in that rewards are allowed to depend on the complete sequence X instead of the revealed contexts to the learner $X_{< t}$ at step t. Formally, these are defined as follows.

Definition 24 (Reward models). The reward mechanism is said to be prescient if there are conditional distributions $(P_{r|a, \boldsymbol{x}_{t'\geq 1}})_{t\geq 1}$ such that r_t given the selected action a_t and the sequence of contexts \mathbb{X} , follows $P_{r|a, \boldsymbol{x}_{t'>1}}$.

In this model, given a process $\mathbb{X} \in C_4$, a time $T \ge 1$ and $\epsilon > 0$ satisfying Eq (5), finding a time $\tau \ge T$ (measurable with respect to the sigma-algebra $\sigma(\mathbb{X})$, i.e., conditionally on \mathbb{X}) such that Eq (6) is satisfied becomes trivial even with $\eta = 1$. Therefore, the same proof as for Theorem 22 shows that the last condition on stochastic processes is necessary for prescient rewards.

Theorem 25. Let \mathcal{X} be a metrizable separable Borel space, and a finite action space \mathcal{A} with $|\mathcal{A}| \geq 2$. Then, $C_{prescient} \subset C_7$.

5.1.3 Condition 5 is necessary for universal learning with online rewards

In this section, we show that condition C_5 is necessary for universal learning with online rewards, tightening the result on the necessity of condition C_6 from the previous section. In fact, in Section 5.2 we show that C_5 is also sufficient, which together with the result from this section shows that C_5 exactly characterizes universally learnable processes for online rewards. We recall that this is the strongest reward model that we consider in this paper and allows the reward adversary to also take into account the past actions selected by the learner. We first briefly recall the definition of condition C_5 .

Condition 5. There exists an increasing sequence of integers $(T_i)_{i>0}$ such that letting

$$\mathcal{T} = \bigcup_{i \ge 0} \mathcal{T}^i \cap \{t \ge T_i\},$$

we have $\tilde{\mathbb{X}} = (X_t)_{t \in \mathcal{T}} \in \mathcal{C}'_1$. Denote by \mathcal{C}_5 the set of all processes \mathbb{X} satisfying this condition.

Before proving our main result, we need the following lemma that gives an equivalent formulation of the class of processes C_5 . Intuitively, it shows that if $X \notin C_5$, for any tentative rate to add duplicates—yielding the extended process \tilde{X} —we can uniformly lower-bound the proportion of failure for the C'_1 condition.

Lemma 26. Let X be a metrizable separable Borel space and X a stochastic process on X. The following are equivalent.

- $\mathbb{X} \in \mathcal{C}_5$,
- For any $\epsilon > 0$, there exists an increasing sequence of integers $(T_i)_{i\geq 0}$ such that letting $\mathcal{T} = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq T_i\}$, for any sequence $\{A_k\}_{k\geq 1}$ of measurable sets of \mathcal{X} with $A_k \downarrow \emptyset$,

$$\lim_{k \to \infty} \mathbb{E}[\hat{\mu}_{(X_t)_{t \in \mathcal{T}}}(A_k)] \le \epsilon$$

Proof. By definition of the condition C_5 , it is immediate that $\mathbb{X} \in C_5$ implies the second proposition. It remains to prove the converse. We then suppose that \mathbb{X} satisfies the second proposition. Denote by $(T_i(l))_{i\geq 0}$ the sequence obtained from the proposition by setting $\epsilon = 2^{-l}$. Now defining

$$T_i = \max_{j \le i} T_i(j),$$

it then suffices to argue that the sequence $(T_i)_{i\geq 0}$ satisfies the requirements for the C_5 condition. We write $\mathcal{T} = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq T_i\}$ and $\mathcal{T}(l) = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq T_i(l)\}$ for any $l \geq 0$. Now fix $l \geq 0$, and note that for any $i \geq l$, one has $T_i \geq T_i(l)$. As a result,

$$\bigcup_{i\geq l} \mathcal{T}^i \cap \{t\geq T_i\} \subset \bigcup_{i\geq l} \mathcal{T}^i \cap \{t\geq T_i(l)\}$$

Next, note that because the sets \mathcal{T}^i are increasing in *i*, we have $\mathcal{T} \setminus \bigcup_{i \ge l} \mathcal{T}^i \cap \{t \ge T_i\} \subset \{t < T_l\}$. Therefore, for any measurable set $A \in \mathcal{B}$, one has

$$\hat{\mu}_{(X_t)_{t\in\mathcal{T}}}(A) = \limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}} \mathbb{1}_A(X_t) \leq \limsup_{T\to\infty} \frac{T_l}{T} + \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}(l)} \mathbb{1}_A(X_t) = \hat{\mu}_{(X_t)_{t\in\mathcal{T}(l)}}(A).$$

Thus, for any sequence of measurable sets $A_k \downarrow \emptyset$, one has

$$\lim_{k \to \infty} \mathbb{E}[\hat{\mu}_{(X_t)_{t \in \mathcal{T}}}(A_k)] \le \lim_{k \to \infty} \mathbb{E}[\hat{\mu}_{(X_t)_{t \in \mathcal{T}(l)}}(A_k)] \le 2^{-l}$$

Because this holds for all $l \ge 0$, we obtain $\lim_{k\to\infty} \mathbb{E}[\hat{\mu}_{(X_t)_{t\in\mathcal{T}}}(A_k)] = 0$ and the lemma is proved.

We are now ready to prove the following theorem.

Theorem 27. Let \mathcal{X} be a metrizable separable Borel space, and a finite action space \mathcal{A} with $|\mathcal{A}| \geq 2$. Then, $C_{online} \subset C_5$.

Proof. Fix $X \notin C_5$. If $X \notin C_4$, we already proved that (even for oblivious rewards) universal learning is not achievable. We therefore suppose that $X \in C_4$ and suppose by contradiction that there is a universally consistent learning rule f under X. We will construct by induction some online rewards on which the learning rule is not consistent. For convenience, we denote by \hat{a}_t the action selected by the learning rule at time t. Last, since $|\mathcal{A}| \ge 2$, we can fix $a_1 \ne a_2 \in \mathcal{A}$ two arbitrary actions. These will be the only used actions for our constructions, all other actions $a \in \mathcal{A} \setminus \{a_1, a_2\}$ will have zero reward at all times.

We start by constructing rewards that will depend on the actions of the learning rule. By Lemma 26, we can fix ϵ such that for any increasing sequence $(T_i)_{i\geq 0}$, letting $\mathcal{T} = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq T_i\}$, there exists a sequence of sets $A_k \downarrow \emptyset$ such that

$$\mathbb{E}[\hat{\mu}_{(X_t)_{t\in\mathcal{T}}}(A_k)] \ge \epsilon, \quad \forall k \ge 0.$$

Here we used that the sequence of sets is decreasing so that $\mathbb{E}[\hat{\mu}_{(X_t)_{t\in\mathcal{T}}}(A_k)]$ is decreasing in *i*.

The end rewards are constructed by induction: at the phase p of the construction, the rewards r_t^* have been constructed for all $t < T_p^*$ for some time $T_p^* = 2^{R_p^*}$. Further, we have defined some disjoint sets B_1, \ldots, B_p , increasing times T_1^*, \ldots, T_{p-1}^* , and a policy $\pi^{(p)}$ such that $\pi^{(p)}(x) = a_2$ for all $x \notin B_1 \cup \cdots B_p$, and for any $p' \leq p$,

$$\mathbb{E}\left[\max_{\substack{T_{p'-1}^{\star} \leq T < T_{p'}^{\star}}} \frac{1}{T} \sum_{t=1}^{T} r_t^{\star}(\pi^{(p)}(X_t)) - r_t^{\star}(\hat{a}_t)\right] \ge \frac{\epsilon}{16} + \frac{\epsilon}{2^{p+10}},\tag{7}$$

where we used the notation $T_0^{\star} = 0$. Last, at phase p we have also constructed a sequence of increasing indices $(Q_p(i))_{i\geq 0}$ with $Q_p(i) \geq 4i$ such that with $\mathcal{T}^{(p)} = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq 2^{Q_p(i)}\}$, one has

$$\mathbb{E}\left[\sup_{T\geq 1}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}^{(p)}}\mathbb{1}_{B_{p'}}(X_t)\right]\leq \frac{\epsilon}{2^{p'+10}}, \quad p'\leq p.$$
(8)

For instance, for p = 0 we can simply take $Q_0(i) = 2i$ for all $i \ge 0$. We then suppose that we completed phase $p \ge 0$ and proceed with the induction to construct the set B_{p+1} , time T_{p+1}^* and rewards r_t^* until time T_{p+1}^* .

Before doing so, we need to construct an auxiliary reward process. These rewards have the following behavior. Before $T_p^* = 2^{R_p^*}$, these are constructed identically as the rewards r^* . Then, at time $t \ge 2^{R_p^*}$, either the rewards are always zero and this is called an inactive time; or the time is active, in which case the "safe" action a_2 always receives a reward 3/4, and the "uncertain" action a_1 receives a reward that can either be 0 or 1 with equal probability. We say that the learning rule explores at an active time t if it selects action a_1 . At the high level, the rewards proceed by period and tentatively activate the times from \mathcal{T}^i for some $i \ge 0$. If the learning rule performs too many explorations, the trial fails and we instead aim to activate fewer times from \mathcal{T}^j for j < i. We construct the rewards inductively by period $[2^r, 2^{r+1})$ for $r \ge r_0$. Each of these periods will be associated with a level $i(r) \ge 0$, which roughly corresponds to the fact that the active times during period r were times in $\mathcal{T}^{i(r)}$. We also denote by \mathcal{S}_t the set of active times up until time t (included). The formal procedure to define the online rewards is given in Algorithm 1, where $r_t(a)$ denotes the reward for action a defined by the procedure at time t, for $t \ge 1$.

Let $S = \bigcup_{t \ge 1} S_t$ be the set of all active times. We first give some properties on the learning procedure starting from time T_p^* . As a first step, we show that the learner cannot make better predictions than the simple policy $\pi_0 : x \in \mathcal{X} \mapsto a_2 \in \mathcal{A}$. Precisely, we show that the quantities $r_t(\hat{a}_t) - r_t(a_2) + \mathbb{1}_{t \in S} \mathbb{1}_{\hat{a}_t \neq a_2}/4$ for $t \ge T_p^*$ form the increments of a super-martingale with respect to the filtration $\sigma(\mathbb{X}_{\le t}, \hat{a}_{\le t}, r_{\le t-1})$. First, note that whether t is active, i.e., $t \in S$ only requires the knowledge of $\mathbb{X}_{\le t}$ and the actions $\hat{a}_{\le t}$, hence is measurable with respect to the given filtration. Next, if t is inactive, all rewards are zero. We now consider active times. Denote by u(t) the time of the first occurrence of X_t starting from T_p^* , i.e., $u(t) = \min\{T_p^* \le u \le t : X_t = X_u\}$. Then, if t is active, $r_t(a_1) - r_t(a_2) = B_{u(t)} - 3/4$. Moreover, by construction, the learning rule has not queried a_1 for any previous active time u within the same period as t such that $X_t = X_u$. However, these are the only times when $B_{t'}$ affected the rewards. As a result, all rewards that the learning rule has received before time t are independent of $B_{u(t)}$ (whether t is active or not). This shows that $B_{u(t)}$ is independent from $\mathbb{X}_{\le t}$, $\hat{a}_{\le t}$ and $r_{\le t-1}$ together. As a result,

$$\mathbb{E}[r_t(\hat{a}_t) - r_t(a_2) + \mathbb{1}_{t \in \mathcal{S}} \mathbb{1}_{\hat{a}_t \neq a_2}/4 \mid \mathbb{X}_{\leq t}, \hat{a}_{\leq t}, r_{\leq t-1}] = \mathbb{1}_{t \in \mathcal{S}} (-1/2 \cdot \mathbb{1}_{\hat{a}_t \notin \{a_1, a_2\}} + \mathbb{1}_{\hat{a}_t = a_1} \mathbb{E}[B_{u(t)} - 1/2 \mid \mathbb{X}_{\leq t}, \hat{a}_{\leq t}, r_{\leq t-1}]) = -1/2 \cdot \mathbb{1}_{t \in \mathcal{S}} \mathbb{1}_{\hat{a}_t \notin \{a_1, a_2\}} \leq 0.$$

This ends the proof that $(r_t(\hat{a}_t) - r_t(a_2) + \mathbb{1}_{t \in S} \mathbb{1}_{\hat{a}_t \neq a_2}/4)_{t \geq T_p^{\star}}$ form the increments of a super-martingale, and these are bounded in absolute value by one. Azuma-Hoeffding's inequality then implies for any $T \geq T_p^{\star}$,

$$\mathbb{P}\left[\sum_{t=T_p^{\star}}^{T} r_t(\hat{a}_t) - r_t(a_2) \ge 2T^{3/4} - \frac{1}{4} \sum_{t=T_p^{\star}}^{T} \mathbb{1}_{t \in \mathcal{S}} \mathbb{1}_{\hat{a}_t \neq a_2}\right] \le e^{-2\sqrt{T}}.$$

Let $(B_t)_{t\geq 1}$ be an i.i.d. $\mathcal{B}(\frac{1}{2})$ sequence for $t = 1, ..., T_p^{\star} - 1$ do Observe context X_t Define $r_t(a) = r_t^{\star}(a)$ for all $a \in \mathcal{A}$ Observe action selected by learner \hat{a}_t end Initialize $i(R_p^{\star}) = 0$ and let $S_{T_p^{\star}-1} = \emptyset$ for $r \geq R_p^{\star}$ do for $t = 2^r, \dots, 2^{r+1} - 1$ do Observe context X_t if $t \notin \mathcal{T}^{i(r)}$ then | Define $r_t(a) = 0$ for all $a \in \mathcal{A}$ and $\mathcal{S}_t = \mathcal{S}_{t-1}$ else if $\forall T_p^* \leq t' < t, X_{t'} \neq X_t$ then Define $r_t(a) = \begin{cases} B_t & a = a_1 \\ \frac{3}{4} & a = a_2, \\ 0 & a \notin \{a_1, a_2\} \end{cases}$ for $a \in \mathcal{A}$ $\mathcal{S}_t = \mathcal{S}_{t-1} \cup \{t\}$ else if $\exists T_p^* \leq t' < t$ such that $X_t = X_{t'}, t' \in S_{t-1}$ and $\hat{a}_{t'} = a_1$ then | Define $r_t(a) = 0$ for all $a \in \mathcal{A}$ and $\mathcal{S}_t = \mathcal{S}_{t-1}$ else Define $r_t(a) = r_{t'}(a)$ for all $a \in \mathcal{A}$ where t' < t, $X_t = X_{t'}$ and $t' \in \mathcal{S}_{t-1}$ $\mathcal{S}_t \leftarrow \mathcal{S}_{t-1} \cup \{t\}$ end Observe action selected by learner \hat{a}_t while $\frac{1}{t} \sum_{u=T_p^{\star}}^{t} \mathbb{1}_{u \in S_t} \mathbb{1}_{\hat{a}_u \neq a_2} \geq \frac{1}{2^{2i(r)}(i(r)+1)}$ do $i(r) \leftarrow \max(0, i(r) - 1)$; end Define $i(r+1) = \min\{i(r)+1, k\}$ where k is such that $Q_p(k) \le r+1 < Q_p(k+1)$ end

Algorithm 1: Procedure to define the online rewards

Borel-Cantelli's lemma then implies that on an event \mathcal{E} of probability one, there exists $\hat{T} \geq T_p^*$ such that for any $T \geq \hat{T}$,

$$\sum_{t=T_p^{\star}}^T r_t(\hat{a}_t) - r_t(a_2) < 2T^{3/4} - \frac{1}{4} \sum_{t=T_p^{\star}}^T \mathbbm{1}_{t \in \mathcal{S}} \mathbbm{1}_{\hat{a}_t \neq a_2}$$

We now focus on the level i(r) at each period. Note that this quantity is updated by the procedure along the learning process: it starts at i(r-1)+1 (or 0 if $r = r_0$) at the beginning of the period $[2^r, 2^{r+1})$, then can only decrease during the period. Starting from the end of the period 2^{r+1} , the level i(r) is never updated again. To avoid any confusions, we denote by I(r) this final value of i(r) once the period is completed. We aim to prove that the level at each period i(r) eventually diverges to infinity. Fix $j \ge 0$. Because f is universally consistent under \mathbb{X} , it has in particular vanishing excess error compared to π_0 . Hence, we have

$$\mathbb{P}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(a_2) - r_t(\hat{a}_t) \ge \frac{1}{2^{2j+4}(j+1)}\right] = 0.$$

As a result, by the dominated convergence theorem there exists $t_j \ge 1$ such that

$$\mathbb{P}\left[\sup_{T \ge t_j} \frac{1}{T} \sum_{t=1}^T r_t(a_2) - r_t(\hat{a}_t) \ge \frac{1}{2^{2j+4}(j+1)}\right] \le \frac{\epsilon}{2^{j+10}}.$$

We denote by \mathcal{F}_j the complement event. Next, because \mathcal{E} has full probability, there exists t'_j such that

$$\mathbb{P}\left[\sum_{t=T_p^{\star}}^{T} r_t(\hat{a}_t) - r_t(a_2) < 2T^{3/4} - \frac{1}{4} \sum_{t=T_p^{\star}}^{T} \mathbb{1}_{t \in \mathcal{S}} \mathbb{1}_{\hat{a}_t \neq a_2}, \, \forall T \ge t_j'\right] \le \frac{\epsilon}{2^{j+10}}$$

We denote by \mathcal{E}_j the complement event. Now, we define an integer $R_j \geq R_p^*$ such that $2^{R_j-j} \geq \max(t_j, t'_j, 2^{8j+16}(j+1)^4, 2^{2j+4}(j+1)T_p^*, 2^{Q_p(j)})$. Using the previous two equations shows that on $\mathcal{E}_j \cap \mathcal{F}_j$ of probability at most $1 - \frac{\epsilon}{2^{j+9}}$, for all $T \geq 2^{R_j-j}$,

$$\frac{1}{T} \sum_{t=T_p^{\star}}^{T} \mathbb{1}_{t \in \mathcal{S}} \mathbb{1}_{t \neq a_2} < \frac{4}{T} \sum_{t < T_p^{\star}} (r_t(\hat{a}_t) - r_t(a_2)) + \frac{8}{T^{1/4}} + \frac{1}{2^{2j+2}(j+1)} \le \frac{4T_p^{\star}}{T} + \frac{8}{T^{1/4}} + \frac{1}{2^{2j+2}(j+1)} \le \frac{1}{2^{2j}(j+1)}.$$

Also, for any $r \ge R_j - j$, one has $r \ge Q_p(j)$ so that the quantities I(r) can freely increase until they reach j from when the quantities i(r) are always lower bounded by j. In particular, by the union bound, this shows that

$$\mathbb{P}\left[\forall j \ge 0, \inf_{r \ge R_j} I(r) \ge j\right] \ge \mathbb{P}\left[\bigcap_{j \ge 0} \mathcal{E}_j \cap \mathcal{F}_j\right] \ge 1 - \frac{\epsilon}{2^8}$$

We denote by $\mathcal{F} = \{ \forall j \ge 0, \inf_{r \ge R_j} I(r) \ge j \}$ the corresponding event.

We are now ready to show that f is not universally consistent. Because $\mathbb{X} \notin C_5$, with $\mathcal{T} = \bigcup_{i \ge 0} \mathcal{T}^i \cap \{t \ge 2^{R_j}\}$, there exists a measurable sets $A_k \downarrow \emptyset$ such that for all $k \ge 1$ we have $\mathbb{E}[\hat{\mu}_{(X_t)_{t \in \mathcal{T}}}(A_k)] \ge \epsilon$. Now because $A_k \downarrow \emptyset$, we have

$$0 \le \lim_{k \to \infty} \mathbb{P}\left(\exists t < T_p^{\star} : X_t \in A_k\right) \le \sum_{t < T_p^{\star}} \lim_{k \to \infty} \mathbb{P}(X_t \in A_k) = 0.$$

Also, because $X \in C_4$, by Lemma 17 we have

$$\lim_{k \to \infty} \mathbb{E} \left[\sup_{i \ge 0} \hat{\mu}_{(X_t)_{t \in \mathcal{T}^i}}(A_k) \right] = 0.$$

As a result, there exists $l \ge 1$ such that

$$\mathbb{E}\left[\sup_{i\geq 0}\hat{\mu}_{(X_t)_{t\in\mathcal{T}^i}}(A_l)\right] \leq \frac{\epsilon}{2^{p+11}} \quad \text{and} \quad \mathbb{P}\left(\exists t < T_p^\star : X_t \in A_k\right) \leq \frac{\epsilon}{2^{p+11}}.$$
(9)

We fix this index l in the rest of the proof. Let L_p^{\star} be an integer such that $L_p^{\star} \geq \max(R_p^{\star} + 10 - \log_2 \epsilon, R_{10-\log_2 \epsilon}, 4(\log_2(C_{\epsilon}) + 10 - \log_2 \epsilon))$, where $C_{\epsilon} = \sqrt{2 \ln \frac{8}{\epsilon}}$. Now by construction, since we have $\mathbb{E}[\hat{\mu}_{(X_t)_{t\in\mathcal{T}}}(A_l)] \geq \epsilon$, we have in particular

$$\mathbb{E}\left[\sup_{T\geq 2^{L_p^{\star}}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}}\mathbb{1}_{A_l}(X_t)\right]\geq\epsilon.$$

Thus, by the dominated convergence theorem, there exists an integer $R_{p+1}^{\star} > 2^{L_p^{\star}}$ such that

$$\mathbb{E}\left[\max_{2^{L_p^{\star}} \leq T < 2^{R_{p+1}^{\star}}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} \mathbb{1}_{A_l}(X_t)\right] \geq \frac{\epsilon}{2}.$$
(10)

We define $T_{p+1}^{\star} = 2^{R_{p+1}^{\star}}$. As a second step, we show that when during the learning process until time T_{p+1}^{\star} , for a large proportion of active times t for which $X_t \in A_l$, the optimal arm in hindsight is a_1 . Precisely, we aim to show that

$$\mathbb{E}\left[\max_{2^{L_p^{\star}} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{S}} \mathbb{1}_{A_l}(X_t) \cdot B_{u(t)}\right] \geq \frac{\epsilon}{8}$$

To prove this, we reason conditionally on X. Define

$$\hat{T} = \operatorname*{argmax}_{2^{L_p^{\star}} \le T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_{A_l}(X_t).$$

Also, let $Exp = \{T_p^* \le t \le \hat{T} : t \in \mathcal{S}, X_t \in A_l, \hat{a}_t = a_1\}$ the set of "exploration" times on A_l when the learning rule selected action a_1 without prior knowledge on the value $B_{u(t)}$ for active time t. For any exploration time $t \in Exp$, we also define $N(t) = |\{T_p^* \leq t' \leq t : t \in S, X_{t'} = X_t\}|$ the number of active occurrences of X_t before the exploration at t. Note that after the exploration, new duplicates of X_t will never be active anymore. Last, denote by $Unexp = (A_l \cap \{X_t, T_p^* \le t \le T\}) \setminus \{X_t, t \in Exp\}$ the set of points in A_l that were left unexplored until horizon \hat{T} . As above, for $x \in Unexp$, we denote by $N(x) = |\{T_p^* \le t \le \hat{T} : t \in \mathcal{S}, X_{t'} = x|$ the number of active occurrences of x until \hat{T} . Also, by abuse of notation, for any $x \in Unexp$, we denote $u(x) = \min\{T_p^* \le t \le \hat{T} : X_t = x\}$ the first occurrence of X_t . Conditionally on the realization of X (which as a result makes \hat{T} deterministic), the sequence $(\mathbb{1}_{t \in Exp} N(t)(B_{u(t)} - \frac{1}{2}))_{T_p^{\star} \leq t \leq \hat{T}}$ followed by the sequence $(N(x)(B_{u(x)} - \frac{1}{2}))_{x \in Unexp}$ form the increments of a martingale with filtration given by the σ -algebras $\sigma(\mathbb{X}, \hat{a}_{\leq t}, r_{\leq t-1})$. Indeed, conditionally on X, the past history $\hat{a}_{< t-1}, r_{< t-1}$ and the selected action \hat{a}_t , at an exploration time $t \in Exp$, the value $B_{u(t)}$ is independent from X and has never been revealed yet, hence is independent from the history as well. Similarly, for unrevealed points $x \in Unexp$, the variables $B_{u(x)}$ are together independent and also independent from X and the history $\hat{a}_{<\hat{T}}, r_{<\hat{T}}$. The final term of the described martingale writes

$$\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t \in Exp} N(t) \left(B_{u(t)} - \frac{1}{2} \right) + \sum_{x \in Unexp} N(x) \left(B_{u(x)} - \frac{1}{2} \right) = \sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t \in \mathcal{S}} \mathbb{1}_{A_l}(X_t) \left(B_{u(t)} - \frac{1}{2} \right).$$

We now bound these increments. For any $R_p^* \leq r < R_{p+1}^*$, during the period $[2^r, 2^{r+1})$, one has $S \cap [2^r, 2^{r+1}) \subset \mathcal{T}^k$, where k is such that $Q_p(k) \leq r < Q_p(k+1)$. Now recall that $Q_p(k) \geq 4k$ so that the number of active duplicates for a given point x during period r is at most $2^k \leq 2^{r/4}$. Hence, if $\hat{T} \in [2^{\hat{r}}, 2^{\hat{r}+1})$, the number of active duplicates of any point until \hat{T} satisfies

$$\max_{t \in Exp} N(t), \max_{x \in Unexp} N(x) \le \sum_{r=r_0}^{\hat{r}} 2^{r/4} \le \frac{2^{r/4}}{1 - 2^{-1/4}} \le \frac{\hat{T}^{1/4}}{2^{1/4} - 1} \le 6\hat{T}^{1/4}.$$

In particular, all increments of the constructed martingale have elements norm bounded by the above value. Azuma-Hoeffding's inequality then yields

$$\mathbb{P}\left[\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in\mathcal{S}} \mathbb{1}_{A_l}(X_t) \left(B_{u(t)} - \frac{1}{2}\right) \leq -C_{\epsilon} \hat{T}^{3/4} \mid \mathbb{X}\right] \leq \frac{\epsilon}{8}$$

Let \mathcal{G} be the complement event, i.e., the event when $\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in\mathcal{S}} \mathbb{1}_{A_l}(X_t) \left(B_{u(t)} - \frac{1}{2}\right) > -C_{\epsilon} \hat{T}^{3/4}$. Then, using Eq (10) we obtain

$$\mathbb{E}\left[\frac{\mathbb{1}_{\mathcal{F}\cap\mathcal{G}}}{\hat{T}}\sum_{t\leq\hat{T},t\in\mathcal{T}}\mathbb{1}_{A_l}(X_t)\right] \geq \mathbb{E}\left[\frac{1}{\hat{T}}\sum_{t\leq\hat{T},t\in\mathcal{T}}\mathbb{1}_{A_l}(X_t)\right] - \mathbb{P}[\mathcal{F}] - \mathbb{P}[\mathcal{G}] \geq \frac{\epsilon}{2} - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{\epsilon}{4}.$$
 (11)

As a last step, we show that under $\mathcal{F} \cap \mathcal{G}$, the learning rule incurs significant regret compared to the best action in hindsight for times with contexts falling in A_l . On $\mathcal{F} \cap \mathcal{G}$,

$$\frac{1}{\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in\mathcal{S}}\mathbb{1}_{A_l}(X_t)B_{u(t)} \geq \frac{1}{2\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in\mathcal{S}}\mathbb{1}_{A_l}(X_t) - \frac{C_{\epsilon}}{\hat{T}^{1/4}} \geq \frac{1}{2\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}}\mathbb{1}_{t\in\mathcal{S}}\mathbb{1}_{A_l}(X_t) - \frac{\epsilon}{2^{10}}$$

We used $\hat{T} \geq 2^{L_p^*}$ in the last inequality. We now aim to compare the right-hand side of the last inequality to $\frac{1}{\hat{T}} \sum_{T_p^* \leq t \leq \hat{T}, t \in \mathcal{T}} \mathbb{1}_{A_l}(X_t)$. Because \mathcal{F} is satisfied, $\mathcal{T} \setminus \mathcal{S}$ the set of inactive times that are counted within \mathcal{T} only contains times t such that there exists t' < t with $t' \in \mathcal{S}$ when the learning rule performed an exploration (see Algorithm 1). Thus,

$$\sum_{t=T_p^{\star}}^T \mathbb{1}_{t\in\mathcal{S}}\mathbb{1}_{A_l}(X_t) \ge \sum_{t\leq\hat{T},t\in\mathcal{T}}\mathbb{1}_{A_l}(X_t) - T_p^{\star} - \sum_{t\in Exp} |\{t < t' \le \hat{T}, t' \in \mathcal{T} \setminus \mathcal{S}, X_{t'} = X_t\}|.$$

Letting \hat{j} be the integer such that $R_{\hat{j}} \leq \hat{R} < R_{\hat{j}+1}$, i.e., $2^{R_{\hat{j}}} \leq \hat{T} < 2^{R_{\hat{j}+1}}$, we observe that

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$$\sum_{t \in Exp} |\{t < t' \le \hat{T}, t' \in \mathcal{T} \setminus \mathcal{S}, X_{t'} = X_t\}| \le 2^{\hat{R} - \hat{j}} + \sum_{t \in Exp} |\{2^{\hat{R} - \hat{j}}, t < t' \le \hat{T}, t' \in \mathcal{T}^{\hat{j}}, X_{t'} = X_t\}|$$
$$\le 2^{\hat{R} - \hat{j}} + |Exp|2^{\hat{j}}(\hat{j} + 1)$$
$$\le \frac{\hat{T}}{2^{\hat{j} - 1}} + |Exp|2^{\hat{j}}(\hat{j} + 1).$$

where we used the fact that because $(R_j)_{j\geq 1}$ is increasing, each distinct point is duplicated at most $2^{\hat{j}}$ times in any period $\mathcal{T} \cap [2^r, 2^{r+1})$ with $r < R_{\hat{j}+1}$. Next, because \mathcal{F} is satisfied we have in particular $I(\hat{R}) \geq \hat{j}$, implying that at time \hat{T} , we had the guarantee

$$\frac{|Exp|}{\hat{T}} \le \frac{1}{\hat{T}} \sum_{u=T_p^{\star}}^{T} \mathbb{1}_{u \in \mathcal{S}} \mathbb{1}_{\hat{a}_u \neq a_2} < \frac{1}{2^{2I(\hat{R})}(I(\hat{R})+1)} \le \frac{1}{2^{2\hat{j}}(\hat{j}+1)}$$

Combining the previous four equations and the fact that $\hat{T} \geq 2^{L_p^{\star}}$ shows that on $\mathcal{F} \cap \mathcal{G}$ one has

$$\begin{aligned} \frac{1}{\hat{T}} \sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in\mathcal{S}} \mathbb{1}_{A_l}(X_t) B_{u(t)} &\geq \frac{1}{2\hat{T}} \sum_{t\leq\hat{T},t\in\mathcal{T}} \mathbb{1}_{A_l}(X_t) - \frac{\epsilon}{2^{10}} - \frac{T_p^{\star}}{2\hat{T}} - \frac{1}{2\hat{j}} - \frac{1}{2\hat{j}^{+1}} \\ &\geq \frac{1}{2\hat{T}} \sum_{t\leq\hat{T},t\in\mathcal{T}} \mathbb{1}_{A_l}(X_t) - \frac{\epsilon}{2^8}. \end{aligned}$$

In the last inequality, we used $\hat{j} \ge 10 - \log_2 \epsilon$, a consequence of $\hat{T} \ge 2^{L_p^*}$. We are now ready to compare the reward of the learning rule to the best action in hindsight for times t such that $X_t \in A_l$. Precisely, consider the following actions a_t^* : at an active time $t \in S$ and $X_t \in A_l$, we pose $a_t^* = a_1$ if $B_{u(t)} = 1$ and $a_t^* = a_2$ otherwise. For any other active time $t \in S$ and $X_t \notin A_l$, we pose $a_t^* = a_2$ (which is in that case not necessarily the best action in hindsight). First note that

$$\begin{split} \frac{1}{\hat{T}} \sum_{t=T_p^{\star}}^{\hat{T}} \mathbbm{1}_{A_l}(X_t) (r_t(a_t^{\star}) - r_t(\hat{a}_t)) &= \frac{1}{\hat{T}} \sum_{t=T_p^{\star}}^{\hat{T}} \mathbbm{1}_{t \in \mathcal{S}} \mathbbm{1}_{A_l}(X_t) \left(\frac{3 + B_{u(t)}}{4} - r_t(\hat{a}_t) \right) \\ &\geq \frac{1}{4\hat{T}} \sum_{t=T_p^{\star}}^{\hat{T}} \mathbbm{1}_{t \in \mathcal{S}} \mathbbm{1}_{A_l}(X_t) \mathbbm{1}_{t \notin Exp} B_{u(t)} \\ &\geq \frac{1}{4\hat{T}} \sum_{t=T_p^{\star}}^{\hat{T}} \mathbbm{1}_{t \in \mathcal{S}} \mathbbm{1}_{A_l}(X_t) B_{u(t)} - \frac{1}{4\hat{T}} \sum_{t=T_p^{\star}}^{\hat{T}} \mathbbm{1}_{t \in Exp} \mathbbm{1}_{A_l}(X_t). \end{split}$$

Also, note that

$$\frac{1}{\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{A_l^c}(X_t)(r_t(a_t^{\star}) - r_t(\hat{a}_t)) \ge \frac{1}{\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in\mathcal{S}}\mathbb{1}_{A_l^c}(X_t)\left(\frac{3}{4} - r_t(\hat{a}_t)\right) \ge -\frac{1}{4\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in Exp}\mathbb{1}_{A_l^c}(X_t).$$

Combining the two previous equations shows that on $\mathcal{F} \cap \mathcal{G}$,

$$\frac{1}{\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}} r_t(a_t^{\star}) - r_t(\hat{a}_t) \ge \frac{1}{4\hat{T}}\sum_{t=T_p^{\star}}^{\hat{T}} \mathbb{1}_{t\in\mathcal{S}} \mathbb{1}_{A_l}(X_t) B_{u(t)} - \frac{|Exp|}{4\hat{T}} \ge \frac{1}{2\hat{T}}\sum_{t\leq\hat{T},t\in\mathcal{T}} \mathbb{1}_{A_l}(X_t) - \frac{\epsilon}{2^7}$$

Combining this with Eq (11) shows that

$$\mathbb{E}\left[\max_{T_{p}^{\star} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_{t}(a_{t}^{\star}) - r_{t}(\hat{a}_{t})\right] \geq \mathbb{E}\left[\frac{1}{\hat{T}} \sum_{t=T_{p}^{\star}}^{\hat{T}} r_{t}(a_{t}^{\star}) - r_{t}(\hat{a}_{t}) - \frac{T_{p}^{\star}}{\hat{T}}\right] \geq \frac{\epsilon}{8} - \frac{\epsilon}{2^{10}} - \frac{\epsilon}{2^{7}}.$$
 (12)

As a last step before defining new rewards, we introduce the scale $\delta_l > 0$ such that

$$\mathbb{P}\left[\min_{1 \le t, t' < 2^{R_{p+1}^{\star}}, X_t \neq X_{t'}} \rho(X_t, X_{t'}) \le \delta_l\right] \le \frac{\epsilon}{2^{10}}$$

We denote by \mathcal{H} the complement event.

We are now ready to introduce the new online rewards. To do so, we first need to introduce some notations for partitions of the space \mathcal{X} . Let $(x^u)_{u\geq 1}$ be a dense sequence in \mathcal{X} . We define the sets $P_u = (A_l \cap B(x^u, \delta_l)) \setminus \bigcup_{v < u} B(x^u, \delta_l)$ for $u \geq 1$. We can easily check that the sequence of measurable sets $(P_u)_{u\geq 1}$ forms a partition of A_l , and that each set P_u has diameter at most δ_l . For any binary sequence $\mathbf{b} = (b_u)_{u\geq 1}$, we define online rewards that follow the same structure as defined with the procedure from Algorithm 1, with the difference that rewards r_t^b , at any active time $t \in S$ with $X_t \in P_u$ for some $u \geq 1$, are constructed using the binary value b_u instead of the random binary variable $B_{u(t)}$ where $u(t) = \min\{T_p^* \leq u \leq t : X_t = X_u\}$. The procedure to construct the rewards r^b until time T_{p+1}^* is given in Algorithm 2.

Consider the case when the binary sequence b is sampled as an i.i.d. $\mathcal{B}(\frac{1}{2})$ process. We argue that under the event \mathcal{H} , these rewards r^b from Algorithm 2 are not distinguishable from the rewards r from Algorithm 1. First, observe that they share the same overall structure, the only difference is that when needed to define rewards r_t^b at an active time $t \in S$, one may use b_u instead of B_t , where u is such that $X_t \in P_u$. Recall that b_u is by hypothesis sampled as $b_u \sim \mathcal{B}(\frac{1}{2})$ as B_t and further, under the event \mathcal{H} , all distinct points from $\mathbb{X}_{< T_{p+1}^*}$ falling within A_l are at distance at least δ_l . We only use b_u for r_t^b when $X_t \in P_u$. Therefore, under \mathcal{H} , one has $\{t' < t : X_t \in P_u\} = \emptyset$. This shows that the variable b_u was never observed prior to time t and as a result, is not distinguishable from a true random binary variable $B_t \sim \mathcal{B}(\frac{1}{2})$. In particular, under \mathcal{H} , the rewards r^b when $b \stackrel{i.i.d.}{\sim} \mathcal{B}(\frac{1}{2})$, yield the same selected actions as the rewards r from Algorithm 1. Now for any binary sequence b, we define the policy

$$\pi^{\boldsymbol{b}}(x) = \begin{cases} a_1 & \text{if } b_u^k = 1, x \in P_u, \\ a_2 & \text{if } b_u^k = 0, x \in P_u, \\ a_2 & \text{if } x \notin A_l. \end{cases}$$

By construction, these are constructed exactly similarly to the best action in hindsight a_t^* for contexts falling in A_l as defined previously. Therefore,

$$\begin{split} \mathbb{E}_{\boldsymbol{b}^{i,i,d}\cdot\mathcal{B}(\frac{1}{2})} \left[\mathbb{E}_{\mathbb{X},\boldsymbol{a}} \left(\max_{T_{p}^{\star} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_{t}^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_{t})) - r_{t}^{\boldsymbol{b}}(\hat{a}_{t}) \right) \right] \\ &\geq \mathbb{P}[\mathcal{H}] \cdot \mathbb{E}_{\mathbb{X}|\mathcal{G}} \left[\mathbb{E}_{\boldsymbol{b}^{i,i,d}\cdot\mathcal{B}(\frac{1}{2}),\boldsymbol{a}} \left(\max_{T_{p}^{\star} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_{t}^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_{t})) - r_{t}^{\boldsymbol{b}}(\hat{a}_{t}) \right) \mid \mathbb{X},\mathcal{G} \right] \\ &= \mathbb{P}[\mathcal{H}] \cdot \mathbb{E}_{\mathbb{X}|\mathcal{G}} \left[\mathbb{E}_{\boldsymbol{a}} \left(\max_{T_{p}^{\star} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_{t}(a_{t}^{\star}) - r_{t}(\hat{a}_{t}) \right) \mid \mathbb{X},\mathcal{G} \right] \\ &\geq \mathbb{E}_{\mathbb{X},\boldsymbol{a}} \left[\max_{T_{p}^{\star} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_{t}(a_{t}^{\star}) - r_{t}(\hat{a}_{t}) \right] - \mathbb{P}[\mathcal{H}^{c}] \geq \frac{\epsilon}{8} - \frac{\epsilon}{2^{6}}. \end{split}$$

In particular, there exists a realization **b** such that

$$\mathbb{E}\left[\max_{\substack{T_p^{\star} \leq T < T_{p+1}^{\star}}} \frac{1}{T} \sum_{t=T_p^{\star}}^{T} r_t^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_t)) - r_t^{\boldsymbol{b}}(\hat{a}_t)\right] \geq \frac{\epsilon}{8} - \frac{\epsilon}{2^6}.$$
(13)

We fix this realization of **b** in the rest of the proof. We are now ready to close the induction by letting $B_{p+1} := A_l \setminus (B_1 \cup \ldots \cup B_p)$ and defining the policy $\pi^{(p+1)}$ so as to be consistent with the selected

Input: Binary sequence *b* Let $(B_t)_{t\geq 1}$ be an i.i.d. $\mathcal{B}(\frac{1}{2})$ sequence for $t = 1, ..., T_p^{\star} - 1$ do Observe context X_t Define $r_t(a) = r_t^{\star}(a)$ for all $a \in \mathcal{A}$ Observe action selected by learner \hat{a}_t end Initialize $i(R_p^{\star}) = 0$ and let $S_{T_p^{\star}-1} = \emptyset$ $\begin{array}{l} \text{for } r=R_p^\star,\ldots,R_{p+1}^\star-1 \text{ do} \\ \mid \quad \text{for } t=2^r,\ldots,2^{r+1}-1 \text{ do} \end{array}$ Observe context X_t if $t \notin \mathcal{T}^{i(r)}$ then Let $r_t^{\boldsymbol{b}}(a) = 0$ for all $a \in \mathcal{A}$ and $\mathcal{S}_t = \mathcal{S}_{t-1}$ else if $\forall T_p \star \leq t' < t, X_{t'} \neq X_t; X_t \in P_u$ for some $u \geq 1$ then $\begin{array}{c} b_u & a = a_1 \\ a_1 & a = a_2, \\ 0 & a \notin \{a_1, a_2\} \end{array}$ for $a \in \mathcal{A}$ $S_t = S_{t-1} \cup \{t\}$ else if $\forall T_p \star \leq t' < t, X_{t'} \neq X_t$ then $Let r_t^b(a) = \begin{cases} B_t & a = a_1 \\ \frac{3}{4} & a = a_2, \\ 0 & a \notin \{a_1, a_2\} \end{cases}$ for $a \in \mathcal{A}$ $\mathcal{S}_t = \mathcal{S}_{t-1} \cup \{t\}$ else if $\exists T_p \star \leq t' < t$ such that $X_t = X_{t'}, t' \in S_{t-1}$ and $\hat{a}_{t'} = a_1$ then Let $r_t^{\dot{\boldsymbol{b}}}(a) = 0$ for all $a \in \mathcal{A}$ and $\mathcal{S}_t = \mathcal{S}_{t-1}$ else $\begin{vmatrix} \text{ Define } r_t^{\mathbf{b}}(a) = r_{t'}(a) \text{ for all } a \in \mathcal{A} \text{ where } t' < t, X_t = X_{t'} \text{ and } t' \in \mathcal{S}_{t-1} \\ \mathcal{S}_t \leftarrow \mathcal{S}_{t-1} \cup \{t\} \end{aligned}$ end Observe action selected by learner \hat{a}_t while $\frac{1}{t} \sum_{u=T_p^*}^t \mathbb{1}_{u \in \mathcal{S}_t} \mathbb{1}_{\hat{a}_u \neq a_2} \ge 2^{-2i(r)} \text{ do } i(r) \leftarrow \max(0, i(r) - 1);$ end Define $i(r+1) = \min\{i(r)+1, k\}$ where k is such that $Q_p(k) \le r < Q_p(k+1)$ end

Algorithm 2: Procedure to define the online rewards $r^b_{< T^{\star}_{n-1}}$

actions of $\pi^{(p)}$ on B_1, \ldots, B_p . We pose

$$\pi^{(p+1)}(x) = \begin{cases} \pi^{(p)} & \text{if } x \in B_1 \cup \ldots \cup B_p, \\ \pi^{\boldsymbol{b}} & \text{otherwise.} \end{cases}$$

Observe that by construction, $\pi^{(p+1)}(x) = a_2$ for all $x \notin B_1 \cup \ldots \cup B_{p+1}$. Next, we define the rewards r_t^{\star} to be exactly r_t^{b} for any $t < T_{p+1}^{\star}$. Note that by the construction given in Algorithm 2, these rewards are consistent with the rewards r_t^{\star} that had already been constructed for $t < T_p^{\star}$. In the rest of the proof,

we show that these satisfy the induction requirements.

We first check that the fact that $\pi^{(p+1)}$ differs from $\pi^{(p)}$ on A_l does not affect significantly the guarantees of the constructed rewards until time T_p^* . Indeed, for any $T < T_p^*$,

$$\left|\sum_{t=1}^{T} r_t^{\star}(\pi^{(p+1)}) - r_t^{\star}(\pi^{(p)})\right| \le |\{t \le T : X_t \in A_l\}| \le T \mathbb{1}_{\exists t \le T : X_t \in A_l}\}|$$

so that, using Eq (7) and Eq (9), for any $p' \leq p$,

$$\mathbb{E}\left[\max_{\substack{T_{p'-1}^{\star} \leq T < T_{p'}^{\star}}} \frac{1}{T} \sum_{t=1}^{T} r_{t}^{\star}(\pi^{(p+1)}(X_{t})) - r_{t}^{\star}(\hat{a}_{t})\right]$$

$$\geq \mathbb{E}\left[\max_{\substack{T_{p'-1}^{\star} \leq T < T_{p'}^{\star}}} \frac{1}{T} \sum_{t=1}^{T} r_{t}^{\star}(\pi^{(p+1)}(X_{t})) - r_{t}^{\star}(\hat{a}_{t})\right] - \mathbb{P}(\exists t < T_{p}^{\star} : X_{t} \in A_{l})$$

$$\geq \frac{\epsilon}{16} + \frac{\epsilon}{2^{p+10}} - \frac{\epsilon}{2^{p+11}} \geq \frac{\epsilon}{16} + \frac{\epsilon}{2^{p+11}}.$$

Now we check that the guarantee also holds for p' = p+1. First, recall that by construction of Algorithm 2, for any $r \ge R_p^*$, one has that $i(r) \le k$ where k is such that $Q_p(k) \le r < Q_p(k+1)$. In particular, the active times during the corresponding period satisfy $S \cap [2^r, 2^{r+1}) \subset \mathcal{T}^k$. As a result, we obtain $S \subset \mathcal{T}^{(p)}$, where we recall that $\mathcal{T}^{(p)} := \bigcup_{i\ge 0} \mathcal{T}^i \cap \{t \ge 2^{Q_p(i)}\}$. Then, because $\pi^{(p+1)}$ only differs from π^b on $B_1 \cup \ldots \cup B_p$, for any $T_p^* \le T < T_{p+1}^*$,

$$\frac{1}{T} \sum_{t=1}^{T} r_t^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_t)) - r_t^{\boldsymbol{b}}(\pi^{(p+1)}(X_t)) \leq \frac{1}{T} \sum_{t \leq T, t \in \mathcal{S}} (r_t^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_t)) - r_t^{\boldsymbol{b}}(\pi^{(p+1)}(X_t)))$$
$$\leq \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^{(p)}} \sum_{p'=1}^{p} \mathbb{1}_{B_{p'}}(X_t)$$
$$\leq \sum_{p'=1}^{p} \sup_{T \geq 1} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^{(p)}} \mathbb{1}_{B_{p'}}(X_t).$$

Therefore, combining Eq (13) and the induction hypothesis Eq (8), we obtain

$$\mathbb{E}\left[\max_{T_p^{\star} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_t^{\boldsymbol{b}}(\pi^{(p+1)}(X_t)) - r_t^{\boldsymbol{b}}(\hat{a}_t)\right]$$

$$\geq \mathbb{E}\left[\max_{T_p^{\star} \leq T < T_{p+1}^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_t^{\boldsymbol{b}}(\pi^{\boldsymbol{b}}(X_t)) - r_t^{\boldsymbol{b}}(\hat{a}_t)\right] - \sum_{p'=1}^{p} \mathbb{E}\left[\sup_{T \geq 1} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}^{(p)}} \mathbb{1}_{B_{p'}}(X_t)\right]$$

$$\geq \frac{\epsilon}{8} - \frac{\epsilon}{26} - \frac{\epsilon}{2^{10}} \geq \frac{\epsilon}{16} + \frac{\epsilon}{2^{p+10}}.$$

The last step consists in constructing the increasing indices $Q_{p+1}(i)$ for $i \ge 0$. By the dominated convergence theorem, for any $i \ge 0$, there exists $\tilde{T}_i \ge 1$ such that

$$\mathbb{E}\left[\sup_{T\geq \tilde{T}_i}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}^i}\mathbb{1}_{B_{p+1}}(X_t)-\hat{\mu}_{(X_t)_{t\in\mathcal{T}^i}}(B_{p+1})\right]\leq \frac{\epsilon}{2^{p+12+i}}$$

We then define by induction the sequence of integers $Q_{p+1}(i)$ such that $Q_{p+1}(0) \ge \max(Q_p(0), \log_2 \tilde{T}_0)$ and for all $i \ge 1$, $Q_{p+1}(i) \ge \max(Q_p(i), \log_2 \tilde{T}_i, Q_{p+1}(i-1))$. In particular, the sequence is increasing and the above equation shows that

$$\mathbb{E}\left[\sup_{T \ge 2^{Q_{p+1}(i)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^i} \mathbb{1}_{B_{p+1}}(X_t) - \hat{\mu}_{(X_t)_{t \in \mathcal{T}^i}}(B_{p+1})\right] \le \frac{\epsilon}{2^{p+12+i}}.$$
(14)

Now letting $\mathcal{T}^{(p+1)} = \bigcup_{i \ge 0} \mathcal{T}^i \cap \{t \ge 2^{Q_{p+1}(i)}\}$, we note that

$$\sup_{T \ge 1} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{(p+1)}} \mathbb{1}_{B_{p+1}}(X_t) = \sup_{i \ge 0} \sup_{2^{Q_{p+1}(i)} \le T < 2^{Q_{p+1}(i+1)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{(p+1)}} \mathbb{1}_{B_{p+1}}(X_t)$$
$$\leq \sup_{i \ge 0} \sup_{2^{Q_{p+1}(i)} \le T < 2^{Q_{p+1}(i+1)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^i} \mathbb{1}_{B_{p+1}}(X_t).$$

As a result,

$$\mathbb{E}\left[\sup_{T \ge 1} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{(p+1)}} \mathbb{1}_{B_{p+1}}(X_t)\right]$$

$$\leq \mathbb{E}\left[\sup_{i \ge 0} \hat{\mu}_{(X_t)_{t \in \mathcal{T}^i}}(B_{p+1})\right] + \sum_{i \ge 0} \mathbb{E}\left[\sup_{T \ge 2^{Q_{p+1}(i)}} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^i} \mathbb{1}_{B_{p+1}}(X_t) - \hat{\mu}_{(X_t)_{t \in \mathcal{T}^i}}(B_{p+1})\right]$$

$$\leq \sup_{i \ge 0} \mathbb{E}\left[\hat{\mu}_{(X_t)_{t \in \mathcal{T}^i}}(B_{p+1})\right] + \frac{\epsilon}{2^{p+11}} \le \frac{\epsilon}{2^{p+10}}.$$

In the second inequality we used Eq (14), and in the third inequality, we used Eq (9). Finally, because for all $i \ge 0$, one has $Q_{p+1}(i) \ge Q_p(i)$, we have directly $\mathcal{T}^{(p)} \subset \mathcal{T}^{(p+1)}$, which shows that for all $p' \le p$, we still have

$$\mathbb{E}\left[\sup_{T\geq 1}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}^{(p+1)}}\mathbb{1}_{B_{p'}}(X_t)\right]\leq \frac{\epsilon}{2^{p'+10}}, \quad p'\leq p.$$

This ends the inductive construction of the rewards r^{\star} .

The last step of the proof is to show that f is not universally consistent under X for these online rewards r^* . Having constructed the sequence of sets $(B_p)_{p\geq 1}$, we let π^* be the policy defined by

$$\pi^{\star}(x) = \begin{cases} \pi^{(p)}(x) & \text{if } x \in B_p, \\ a_2 & \text{otherwise.} \end{cases}$$

Recall that the sequence of policies $\pi^{(p)}$ for $p \ge 1$ was constructed so that they are consistent: $\pi^{(p')}$ for $p' \ge p \ge 1$ all coincide on A_p . Further, all $\pi^{(p)}$ coincide on $(\bigcup_{p\ge 1} B_p)^c$ on which they select a_2 . Now fix $p \ge 1$. Because the rewards are also constructed to be consistent over time, if \hat{a}_t denotes the selected action at time t for rewards r^* , the induction implies that for all $p' \ge p$ one has

$$\mathbb{E}\left[\max_{\substack{T_{p-1}^{\star} \leq T < T_{p}^{\star}}} \frac{1}{T} \sum_{t=1}^{T} r_{t}^{\star}(\pi^{(p')}(X_{t})) - r_{t}^{\star}(\hat{a}_{t})\right] \geq \frac{\epsilon}{16}.$$
(15)

As a result, because $\pi^{(p')}$ and π^* coincide everywhere except on $\bigcup_{q > p'} B_q$, we have for any $T_{p-1}^* \leq T < T_p^*$,

$$\frac{1}{T}\sum_{t=1}^{T}r_{t}^{\star}(\pi^{\star}(X_{t})) - r_{t}^{\star}(\hat{a}_{t}) \geq \frac{1}{T}\sum_{t=1}^{T}r_{t}^{\star}(\pi^{(p')}(X_{t})) - r_{t}^{\star}(\hat{a}_{t}) - \mathbb{1}\left(\exists t < t_{p}^{\star} : X_{t} \in \bigcup_{q > p'} B_{q}\right).$$

Because the sets $(B_p)_{p\geq 1}$ are all disjoint, we have $\mathbb{P}\left(\exists t < t_p^* : X_t \in \bigcup_{q>p'} B_q\right) \to 0$ as $p' \to \infty$. Thus, using Eq (15) yields

$$\mathbb{E}\left[\max_{T_{p-1}^{\star} \leq T < T_p^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_t^{\star}(\pi^{\star}(X_t)) - r_t^{\star}(\hat{a}_t)\right] \geq \frac{\epsilon}{16}$$

Because this holds for all $p \ge 1$, Fatou's lemma implies

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t^{\star}(\pi^{\star}(X_t)) - r_t^{\star}(\hat{a}_t)\right] \ge \limsup_{p \to \infty} \mathbb{E}\left[\max_{T_{p-1}^{\star} \le T < T_p^{\star}} \frac{1}{T} \sum_{t=1}^{T} r_t^{\star}(\pi^{\star}(X_t)) - r_t^{\star}(\hat{a}_t)\right] \ge \frac{\epsilon}{16}.$$

As a result, the learning rule is not universally consistent under X, which ends the proof of the theorem.

5.2 A sufficient condition on learnable processes

In this section, we show that C_5 is sufficient universal learning for all reward models. We recall that the condition C_5 asks that there exists an increasing sequence $(T_i)_{i\geq 0}$ such that $\tilde{\mathbb{X}} = (X_t)_{t\in\mathcal{T}} \in \mathcal{C}'_1$ where $\mathcal{T} = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq T_i\}$ is obtained by adding the times \mathcal{T}^i according to the rate given by $(T_i)_{i\geq 0}$.

It is straightforward to see $C_1 \subset C_5$ since for any $\mathbb{X} \in C_1$, one can take any arbitrary sequence, for instance $T_i = i$ for $i \ge 0$, and satisfy property C_5 . Before showing that C_5 is a sufficient condition for universal learning with online rewards, we state a known result showing that for C'_1 processes, there is a countable sequence of policies that is empirically dense within all measurable policies.

Lemma 28 ([2] Lemma 24). Let \mathcal{A} be a finite action space and \mathcal{X} a separable metrizable Borel space. There exists a countable sequence of measurable policies $(\pi^l)_{l\geq 1}$ from \mathcal{X} to \mathcal{A} such that for extended process $\tilde{\mathbb{X}} = (X_t)_{t\in\mathcal{T}} \in \mathcal{C}'_1$, and any measurable policy $\pi : \mathcal{X} \to \mathcal{A}$,

$$\inf_{l \ge 1} \mathbb{E} \left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}[\pi^l(X_t) \neq \pi(X_t)] \right] = 0.$$

We are now ready to prove the sufficiency of C_5 .

Theorem 29. Let \mathcal{X} be a metrizable separable Borel space and \mathcal{A} a finite action space. Then, $C_5 \subset C_{online}$.

Proof. Let $\mathbb{X} \in C_5$, and $(T_i)_{i\geq 0}$ such that letting $\mathcal{T} = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq T_i\}$ we have $\tilde{\mathbb{X}} = (X_t)_{t\in\mathcal{T}} \in \mathcal{C}'_1$. We suppose that $T_i = 2^{u(i)}$ for some indices u(i) increasing in *i*. This is without loss of generality, because one could take $\tilde{T}_i = \min\{2^s, 2^s \geq T_i\}$ and still have a \mathcal{C}'_1 process in the definition of $\tilde{\mathbb{X}}$ (a slower sequence $(T_i)_i$ only reduces considered points, hence does not impact the \mathcal{C}'_1 property). We may also suppose that $u(i) \geq 2i$. Also, letting $\eta_i = \sqrt{\frac{8\ln(i+1)}{2^i}}$ for $i \geq 0$, we suppose that $u(i) \geq \eta_i 2^{i+5}$. Last, we suppose that u(0) = 0 which again can be done without loss of generality since the C'_1 property is not affected by the behavior of the process on the first T_0 times. Hence, $T_0 = 1$.

Similarly to the algorithm that was proposed for stationary rewards in [1], the learning rule associates a category p to each time t and acts separately on each category. To do so, the algorithm first computes the phase of t as follows: PHASE(t) is the unique integer i such that $T_i \leq t < T_{i+1}$. Then, we define the stage STAGE $(t) := \lfloor \log_2 t \rfloor = l$ so that $t \in [2^l, 2^{l+1})$, and the period k = PERIOD(t) as the unique integer k such that $T_i^{l2^i+k} \leq t < T_i^{l2^i+k+1}$ where i = PHASE(t). (Recall that $T_i^{l2^i} = 2^l$). We will refer to $[T_i^{l2^i+k}, T_i^{l2^i+k+1})$ as period k of stage l of phase i. The category of t is then defined in terms of number of occurrences of X_t within its period.

$$\mathsf{CATEGORY}(t, \mathbb{X}_{\leq t}) := \left\lfloor \log_4 \sum_{t'=T_i^{l2^i+k}}^t \mathbb{1}[X_{t'} = X_t] \right\rfloor,$$

where i = PHASE(t), l = STAGE(t), k = PERIOD(t). For conciseness, we will omit the argument $\mathbb{X}_{\leq t}$ of the function in the rest of the proof. In words, category p contains duplicates with indices in $[4^p, 4^{p+1})$ within the periods defined by \mathcal{T} . Now using Lemma 28, let $(\pi^l)_{l\geq 1}$ be a sequence of dense functions from \mathcal{X} to \mathcal{A} within measurable functions under \mathcal{C}'_1 processes. The learning rule acts separately on times from different categories. We now fix a category p and only consider points from this category. Essentially, between times T_i and T_{i+1} , the learning rule performs the Hedge algorithm for learning with experts to select between the strategies j for $1 \leq j \leq i$, which apply π^j and a strategy 0 which assigns a different EXP3.IX learner to each new instance within each period at scale i.

Precisely, during an initial phase $[1, 2^{u(16p)})$, the learning rule only applies strategy 0. Then, let $l \ge u(16p)$ and $u(i) \le l < u(i+1)$, we define the learning rule on stage $[2^l, 2^{l+1})$ as follows. For $0 \le k < 2^i$, before period k of stage l, we construct probabilities $P_p(l,k;j)$ for $j = 0, \ldots, i$. These will be probabilities of exploration for each strategy. At the first phase k = 0 we initialize at the uniform distribution $P_p(l,0;j) = \frac{1}{i+1}$. During period k, each new time of category p is assigned a strategy $\hat{j}(t)$ sampled independently from the past according to probabilities $P_p(l,k;\cdot)$. Duplicates of X_t within the same category and period are also assigned the same strategy $\hat{j}(t)$. The learning rule then performs the assigned strategy: for $\hat{j} = 0$, it performs an EXP3.IX algorithm and for $1 \le \hat{j} \le i$, it applies the policy $\pi^{\hat{j}}$. At the end of the phase, the learning rule computes the average reward obtained by each strategy,

$$\tilde{r}_p(l,k;j) := \frac{1}{2^{l-i}} \sum_{\substack{T_i^{l2^i}+k \le t < T_i^{l2^i}+k+1 \\ P_p(l,k;j)}} \frac{\mathbbm{1}[\mathsf{CATEGORY}(t) = p, \ \hat{j}(t) = j]}{P_p(l,k;j)} r_t$$

and $\hat{r}_p(l, k+1; j) = \sum_{0 \le k' \le k} \tilde{r}_p(l, k'; j)$ the cumulative average reward of strategy j. These rewards are then used to define the probabilities for the next phase $P_p(l, k+1; \cdot)$ using the exponentially weighted averages.

$$P_p(l, k+1; j) = \frac{\exp(\eta_i \hat{r}_p(l, k+1; j))}{\sum_{j'=0}^{i} \exp(\eta_i \hat{r}_p(l, k+1; j'))},$$

where $\eta_i = \sqrt{\frac{8 \ln(i+1)}{2^i}}$ is the parameter of the Hedge algorithm for 2^i steps. The detailed algorithm is given in Algorithm 3.

We now show that this is a universally consistent algorithm for \mathbb{X} . We first introduce some notations.

$$\begin{split} \eta_i &= \sqrt{\frac{8\ln(i+1)}{2^i}}, i \geq 0 & // \text{ learning rates for Hedge} \\ \hat{r}_p^j(l,0) &= 0, P_p(l,0;j) = \frac{1}{i+1}, \quad p,l,j \geq 0 & // \text{ initialization} \\ \text{for } t \geq 1 \text{ do} & // \text{ initialization} \\ \text{for } t \geq 1 \text{ do} & // \text{ initialization} \\ \text{Observe context } X_t & i = \text{PHASE}(t), l = \text{STAGE}(t), k = \text{PERIOD}(t), p = \text{CATEGORY}(t), \\ S_t &= \{t' \in [T_t^{12^i+k}, t): \text{CATEGORY}(t') = p, X_{t'} = X_t\} \\ \text{if } t < 2^{u(16p)} \text{ then} & // \text{ initially play strategy } 0 \\ & | \hat{a}_t = \text{EXP3.IX}_{\mathcal{A}}(\hat{a}_{S_t}, \mathbf{r}_{S_t}) \\ \text{else} & \\ \text{if } S_t = \emptyset \text{ then } \hat{j}(t) \sim P_p(l, k; \cdot) & // \text{ select strategy } \hat{j}(t) \\ \text{else } \hat{j}(t) &= \hat{j}(\min S_t) \\ \text{if } \hat{j}(t) &= 0 \text{ then } \hat{a}_t = \text{EXP3.IX}_{\mathcal{A}}(\hat{a}_{S_t}, \mathbf{r}_{S_t}) & // \text{ play strategy } \hat{j}(t) \\ \text{else } \hat{a}_t &= \pi^{\hat{j}(t)}(X_t) \\ \text{end} \\ \text{Receive reward } r_t \\ \text{if } l \geq u(16p), t &= T_t^{l2^i+k+1} - 1 \text{ then} & // \text{ update probabilities} \\ \hat{r}_p(l, k+1; j) &= \hat{r}(l, k; j) + \frac{1}{2^{l-i}} \sum_{t \in [T_t^{l2^i+k}, T_t^{l2^i+k+1}]} \frac{1[\text{CATEGORY}(t) = p, \hat{j}(t) = j]}{P_p(l, k+1; j)} = \frac{\exp(\eta_i \hat{r}_p(l, k+1; j))}{\sum_{j'=0}^i \exp(\eta_i \hat{r}_p(l, k+1; j'))}, \quad 0 \leq j \leq i \\ \text{end} \\ \text{end} \\ \text{end} \end{array}$$

Algorithm 3: Learning rule for C_5 processes on times T_p

For $p \ge 0$,

$$\mathcal{T}_p := \bigcup_{i \ge 1} [T_i, T_{i+1}) \cap \left\{ t \ge 1 : T_i^k \le t < T_i^{k+1}, 4^p \le \sum_{t'=T_i^k}^t \mathbb{1}[X_{t'} = X_t] < 4^{p+1} \right\},$$

is the set of times in category p. We will also denote $\mathbb{X}^p := (X_t)_{t \in \mathcal{T}^p}$. In this setting, the rewards are independent from the selected actions of the learner. First, note that the constructed rewards $\hat{r}_p(l,k;j)$ are estimates of the average reward that would have been obtained by strategy j during period k of stage l. For convenience, we denote $\mathcal{T}_p(k,l) = [T^{l2^i+k}, T^{l2^i+k+1}) \cap \mathcal{T}_p$. We denote by $R_p(l,k;j)$ the reward that would have been obtained had we selected always $\hat{j} = j$ on this period, and $r_p(l,k;j) = \frac{R_p(l,k;j)}{2^{l-i}}$ the average reward of strategy j for $0 \le j \le i$. For example, for strategy $1 \le j \le i$ we have $R_p(l,k;j) =$ $\sum_{t \in \mathcal{T}_p(l,k)} r_t(\pi^j(X_t))$. Let $\mathcal{X}_p(l,k) = \{X_t, t \in \mathcal{T}_p(k,l)\}$ the set of visited instances during this period. For $x \in \mathcal{X}_p(l,k)$ we denote $t_p(l,k;x) = \min\{t \in \mathcal{T}_p(k,l) : X_t = x\}$ the first time of occurrence of x during this period, and $N_p(l,k;x) = |\{t \in \mathcal{T}_p(l,k) : X_t = x\}|$ its number of occurrences. Let $0 \le j \le i$. We use Hoeffding's inequality conditionally on X and $P_p(l,k;j)$, to obtain

$$\mathbb{P}\left[\left|\sum_{x\in\mathcal{X}_{p}(l,k)}\mathbb{1}[\hat{j}(t)=j]\sum_{t\in\mathcal{T}_{p}(l,k),X_{t}=x}r_{t}-P_{p}(l,k;j)R_{p}(l,k;j)\right| \\ \geq P_{p}(l,k;j)4^{p+1}2^{\frac{3}{4}(l-i)} \mid \mathbb{X}, P_{p}(l,k;j)\right] \\ \leq 2\exp\left(-2\frac{P_{p}(l,k;j)^{2}2^{3/2(l-i)}}{|\mathcal{X}_{p}(l,k)|}\right) \leq 2\exp\left(-2\frac{2^{3/2(l-i)}}{(i+1)^{2}e^{\eta_{i}2^{i+1}}|\mathcal{X}_{p}(l,k)|}\right).$$

Now by construction of $\mathcal{T}_p(l,k)$, each instance of $\mathcal{X}_p(l,k)$ has at least 4^p duplicates within the same period. Hence $|\mathcal{X}_p(l,k)| \leq \frac{2^{l-i}}{4^p}$. As a result, dividing the inner inequality by $P_p(l,k;j)2^{l-i}$, we obtain for $l \geq u(16p)$, with probability at least $1 - 2\exp\left(-\frac{2^{2p+(l-i)/2}}{(i+1)^2e^{\eta_i2^{i+1}}}\right) := 1 - p_1(l,k;p)$,

$$\left|\hat{r}_{p}(l,k;j) - r_{p}(l,k;j)\right| < \frac{4^{p+1}}{2^{(l-i)/4}} \le \frac{4}{2^{l/16}},\tag{16}$$

where in the last inequality we used $l \ge u(i) \ge 2i$ and $l \ge u(16p) \ge 32p$. We now focus on the rewards for strategy 0. For any $t \in \mathcal{T}_p(l,k)$ we denote by \tilde{r}_t the reward that would have been obtained had we selected strategy 0 for time t, i.e. $\hat{j}(t_p(l,k;X_t)) = 0$. In particular, we have $R_p(l,k;0) = \sum_{t \in \mathcal{T}_p(l,k)} \tilde{r}_t$. Let $\pi^* : \mathcal{X} \to \mathcal{A}$ be a measurable policy, we now compare $R_p(l,k;0)$ to the rewards obtained by the policy π^* on $\mathcal{T}_p(l,k)$. Intuitively, we wish to apply Theorem 5 independently for each EXP3.IX algorithm corresponding to elements of $\mathcal{X}_p(l,k)$. However, these runs are not independent for general adaptive adversaries. Therefore, we will need to go back to the standard analysis of EXP3.IX. Using the same notations as in this analysis, for $t \in \mathcal{T}_p(l,k)$, denote $u(t) = |\{t' \le t : t' \in \mathcal{T}_p(l,k), X_{t'} = X_t\}|$ the index of t for its corresponding EXP3.IX learner. Let $\eta_u = 2\gamma_u = \sqrt{\frac{\ln|\mathcal{A}|}{u|\mathcal{A}|}}$ be the parameters used by the learner at step u. Also, denote by $p_{t,a}$ the probability that the EXP3.IX learner chose $a \in \mathcal{A}$ at time t. Further, for $a \in \mathcal{A}$ denote by $\ell_{t,a} = 1 - r_t(a)$ and $\tilde{\ell}_{t,a} = \frac{\ell_{t,a}}{p_{t,a} + \gamma_{u(t)}} \mathbb{1}[a \text{ selected}]$. We keep in mind that the term "selected" refers to the selection of the EXP3.IX algorithm, but not necessarily the selection of our learning rule, which potentially did not apply strategy 0 at that time. To avoid confusion, for $t \in \mathcal{T}_p(l,k)$, denote \tilde{a}_t the action that would be selected by the EXP3.IX learner at time t. Last, we define

$$A_p(l,k) = \sum_{t \in \mathcal{T}_p(l,k)} \tilde{\ell}_{t,\pi^*(X_t)} - \ell_{t,\pi^*(X_t)} \quad \text{and} \quad B_p(l,k) = \sum_{t \in \mathcal{T}_p(l,k)} \sum_{a \in \mathcal{A}} \eta_{u(t)}(\tilde{\ell}_{t,a} - \ell_{t,a}).$$

Then, the same arguments as in Proposition 13 give

$$\sum_{t \in \mathcal{T}_p(l,k)} r_t(\pi^*(X_t)) - r_t(\tilde{a}_t) \le A_p(l,k) + B_p(l,k) + \sum_{x \in \mathcal{X}_p(l,k)} 3\sqrt{|\mathcal{A}| \ln |\mathcal{A}| N_p(l,k;x)}$$

$$\le A_p(l,k) + B_p(l,k) + 3\sqrt{|\mathcal{A}| \ln |\mathcal{A}| 4^{p+1}} |\mathcal{X}_p(l,k)|$$

$$\le A_p(l,k) + B_p(l,k) + 6\sqrt{|\mathcal{A}| \ln |\mathcal{A}|} 2^{-p} 2^{l-i},$$

where in the last inequality, we used the fact that $|\mathcal{X}_p(l,k)| \leq \frac{2^{l-i}}{4^p}$. Now similarly to Proposition 13, note that conditionally on X, the increments of $A_p(l,k)$ and $B_p(l,k)$ form a super-martingale with increments

upper bounded by $2\sqrt{\frac{|\mathcal{A}|4^{p+1}}{\ln |\mathcal{A}|}}$ and $2|\mathcal{A}|\sqrt{\frac{|\mathcal{A}|4^{p+1}}{\ln |\mathcal{A}|}}$ respectively. Thus, Azuma's inequality implies

$$\mathbb{P}[A_p(l,k) \le 8p|\mathcal{A}|2^{p+\frac{3}{4}(l-i)} \mid \mathbb{X}] \ge 1 - e^{-2p^2 2^{(l-i)/2}},$$

$$\mathbb{P}[B_p(l,k) \le 8p|\mathcal{A}|^2 2^{p+\frac{3}{4}(l-i)} \mid \mathbb{X}] \ge 1 - e^{-2p^2 2^{(l-i)/2}}.$$

Thus, denoting $\delta_p = 6 \frac{\sqrt{|\mathcal{A}| \ln |\mathcal{A}|}}{2^p}$, for any $l \ge 2i, u(16p)$, with probability at least $1 - 2e^{-2p^2 2^{(l-i)/2}} := 1 - p_2(l,k;p)$, we have

$$R_p(l,k;0) \ge \sum_{t \in \mathcal{T}_p(l,k)} r_t(\pi^*(X_t)) - 16|\mathcal{A}|^2 2^{-i} 2^{15l/16} - \delta_p 2^{l-i}.$$
(17)

In the first phase where l < u(16p), we will need to proceed differently. Let $\mathcal{T}^{init} = \bigcup_{p \ge 0} \{t \in \mathcal{T}_p : t < 2^{u(16p)}\}$. Observe that on these times, the learning uses a distinct EXP3.IX learner for each new instance within each category and period. In Proposition 13 we showed that this learning rule is universally consistent under processes visiting a sublinear number of distinct instances almost surely. We now show that this is the case for the process $(X_t)_{t\in\mathcal{T}^{init}}$ where for any $t, t' \in \mathcal{T}^{init}$, we view X_t and $X_{t'}$ as duplicates if and only if $X_t = X_{t'}$ and they have same category and period. For $l \ge 1$, let p(l) denote the index p such that $u(16p) \le l < u(16(p+1))$ and i(l) be the index i such that $u(i) \le l < u(i+1)$. Fix $T \ge 1$ and let $l \ge 0$ such that $2^l \le T < 2^{l+1}$. We now count the number of distinct instances N(T) of $(X_t)_{t\in\mathcal{T}^{init}}$ before time T. To do so, we distinguish whether $t \le 2^{l/2}$ or $t > 2^{l/2}$ as follows,

$$\begin{split} N(T) &\leq \sum_{p \geq 0} \sum_{l' \leq u(16p), l} \sum_{k} |\mathcal{X}_{p}(l', k)| \leq 2^{l/2} + \sum_{p \geq p(\frac{l}{2})} \sum_{\frac{l}{2} \leq l' \leq l} \sum_{k} |\mathcal{X}_{p}(l', k)| \\ &\leq 2^{l/2} + \sum_{p \geq p(\frac{l}{2})} \sum_{\frac{l}{2} \leq l' \leq l} \sum_{k} \frac{2^{l' - i(l')}}{4^{p}} \\ &\leq 2^{l/2} + \sum_{p \geq p(\frac{l}{2})} \frac{2^{l+1}}{4^{p}} \\ &\leq 2^{l/2} + \frac{2^{l+1}}{4^{p(l/2) - 1}} \\ &\leq \sqrt{T} + \frac{8T}{4^{p(\log_{4}(T))}} = o(T). \end{split}$$

Now let $\pi^* : \mathcal{X} \to \mathcal{A}$ a measurable policy. Because of the above estimate, Proposition 13 implies that on an event \mathcal{E} of probability one,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{init}} r_t(\pi^*(X_t)) - r_t \le 0.$$

Now recall that $l \ge u(i) \ge 2i$, $\eta_i 2^{i+5}$, hence $\frac{2^{(l-i)/2}}{e^{\eta_i 2^{i+1}}} \ge 2^{l/4 - \eta_i 2^{i+2}} \ge 2^{l/8}$. As a result,

$$\sum_{p\geq 0} \sum_{l\geq 32p} \sum_{k} (i+1)p_1(l,k;p) + p_2(l,k;p) < \infty.$$

Then, the Borel-Cantelli lemma implies that on an event \mathcal{F} of probability one, there exists \hat{l} such that for all $p \ge 0$, $l \ge \max(\hat{l}, u(16p))$ Eq (16) holds, for all $p \ge 0$ and $l \ge \hat{l}$, Eq (17) holds, and \mathcal{E} is satisfied. We suppose that this event is met in the rest of the proof.

The probabilities $P_p(l, k; j)$ are chosen according to the Hedge algorithm. As a result, we have that for any $l \ge \max(\hat{l}, u(16p)), 0 \le k < 2^i$,

$$\max_{0 \le j \le i} \sum_{k' \le k} \hat{r}_p(l,k;j) - \sum_{k' \le k} \sum_{j=0}^i P_p(l,k;j) \hat{r}_p(l,k;j) \le \frac{\ln(i+1)}{\eta_i} + \frac{(k+1)\eta_i}{8}.$$

We then use Eq (16) and $k + 1 \le 2^i$ to obtain

$$\max_{0 \le j \le i} \sum_{k' \le k} r_p(l,k;j) - \sum_{k' \le k} \sum_{j=0}^i P_p(l,k;j) \hat{r}_p(l,k;j) \le 2^i \frac{4}{2^{l/16}} + \frac{\eta_i}{4} 2^i$$

As a result,

$$\max_{0 \le j \le i} \sum_{k' \le k} R_p(l,k;j) - \sum_{k' \le k} \sum_{t \in \mathcal{T}_p(l,k)} r_t \le 4 \cdot 2^{15l/16} + \frac{\eta_i}{4} 2^l.$$
(18)

Now because $l \ge u(16p)$, we have $i \ge 16p$, we have

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$$\sum_{0 \le k' \le k} \sum_{t \in \mathcal{T}_p(l,k')} r_t \ge \sum_{0 \le k' \le k} R_p(l,k';0) - 4 \cdot 2^{15l/16} - \frac{\eta_{16p}}{4} 2^l$$
$$\ge \sum_{0 \le k' \le k} \sum_{t \in \mathcal{T}_p(l,k)} r_t(\pi^*(X_t)) - 20|\mathcal{A}|^2 2^{15l/16} - \left(\delta_p + \frac{\eta_{16p}}{4}\right) 2^l$$

where in the second inequality we used Eq (17). Therefore, summing these equations, for any $T \ge 2^{\hat{l}}, 2^{u(16p)}$,

$$\sum_{u^{(16p)} < t \le T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t \le 2^{\hat{l}} + c|\mathcal{A}|^2 T^{15/16} + 2\left(\delta_p + \frac{\eta_{16p}}{4}\right) T,\tag{19}$$

where $c = \frac{20}{1-2^{-15/16}}$. An important remark is that $\sum_{p\geq 0} (\delta_p + \frac{\eta_{16p}}{4}) < \infty$, which will allow us to consider only a finite number of $p \geq 0$ when comparing the performance of the learning rule compared to π^* .

Before doing so, we show that for all $p \ge 0$, we have $\mathbb{X}^p = (X_t)_{t \in \mathcal{T}_p} \in \mathcal{C}'_1$. By definition, letting $\mathcal{T} = \bigcup_{i\ge 0} \mathcal{T}^i \cap \{t \ge T_i\}$, we have that $\tilde{\mathbb{X}} = (X_t)_{t\in\mathcal{T}} \in \mathcal{C}'_1$. Then note that each instance of $[T_i^k, T_i^{k+1}) \cap \mathcal{T}_p$ has at least one duplicate in $[T_i^k, T_i^{k+1}) \cap \mathcal{T}$ and to each instance of $[T_i^k, T_i^{k+1}) \cap \mathcal{T}$ corresponds at most 4^{p+1} duplicates in $[T_i^k, T_i^{k+1}) \cap \mathcal{T}_p$. As a result, for any set $A \in \mathcal{B}$, we have $\hat{\mu}_{\mathbb{X}^p}(A) \le 4^{p+1}\hat{\mu}_{\tilde{\mathbb{X}}}(A)$, which yields $\mathbb{E}[\hat{\mu}_{\mathbb{X}^p}(A)] \le 4^{p+1}\mathbb{E}[\hat{\mu}_{\tilde{\mathbb{X}}}(A)]$. Using the definition of \mathcal{C}'_1 processes ends the proof that $\mathbb{X}^p \in \mathcal{C}'_1$ for all $p \ge 0$.

Now let $\epsilon > 0$ and p_0 such that $\sum_{p \ge p_0} (\delta_p + \frac{\eta_{16p}}{4}) < \epsilon$. Recall that if $t \in \mathcal{T}_p$, we have $t \ge 4^p$. Therefore, summing Eq (19) gives

$$\sum_{p \ge p_0} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t \le \sum_{p_0 \le p \le \log_4 T} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} r_t(\pi(X_t)) - r_t \le 2^{\hat{l}} \log_4 T + c|\mathcal{A}|^2 T^{15/16} \log_4 T + \epsilon T.$$

We now treat the case of $p < p_0$. Because $\mathbb{X}^p \in \mathcal{C}'_1$, by Lemma 28, there exists $r^p \ge 1$ such that

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}_p} \mathbb{1}[\pi^*(X_t) \neq \pi^{r_p}(X_t)]\right] \le \frac{\epsilon^2}{2p_0^2}$$

By dominated convergence theorem, let l^p such that

$$\mathbb{E}\left[\sup_{T\geq 2^{l^p}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}_p}\mathbb{1}[\pi^*(X_t)\neq\pi^{r_p}(X_t)]\right]\leq\frac{\epsilon^2}{p_0^2}.$$

Using the Markov inequality, we have

$$\mathbb{P}\left[\sup_{T\geq 2^{t^p}}\frac{1}{T}\sum_{t\leq T,t\in\mathcal{T}_p}\mathbb{1}[\pi^*(X_t)\neq\pi^{r_p}(X_t)]\geq\frac{\epsilon}{p_0}\right]\leq\frac{\epsilon}{p_0}.$$

By union bound, on an event \mathcal{G} of probability at least $1 - \epsilon$, for all $p < p_0$ and $T \ge 2^{l^p}$, we have $\sum_{t \le T, t \in \mathcal{T}_p} \mathbb{1}[\pi(X_t) \neq \pi^{r_p}(X_t)] < \frac{\epsilon}{p_0} T$. Next, let $l_0 = \max(u(r^p), l^p, p < p_0)$. Thus, any phase $l \ge l_0$, has $r^p \le i$ for all $p < p_0$. Last, let i_0 such that $\eta_{i_0} \le 2\frac{\epsilon}{p_0}$. On the event $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$, for $p < p_0$, for any $l \ge \hat{l}_1 := \max(l_0, 32p_0, u(i_0), \hat{l})$ and $0 \le k < 2^i$, Eq (18) yields

$$\sum_{\substack{2^l \le t < T_i^{l2^i + k + 1}, t \in \mathcal{T}_p}} r_t(\pi^{r^p}(X_t)) - r_t \le 4 \cdot 2^{15l/16} + \frac{\eta_i}{4} 2^l \le 4 \cdot 2^{15l/16} + \frac{\epsilon}{2p_0} 2^l$$

As a result, for $T \ge 1$, letting i(T), l(T) the indices i, l such that $2^{u(i)} \le T < 2^{u(i+1)}$ and $2^l \le T < 2^{l+1}$, on $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$,

$$\sum_{p < p_0} \sum_{2^{u(16p)} \le t \le T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t \le 2^{\hat{l}_1} + 2^{-i(T)}T + \sum_{p < p_0} \sum_{t < T, t \in \mathcal{T}_p} \mathbb{1}[\pi^*(X_t) \neq \pi^{r_p}(X_t)] + \sum_{p < p_0} \sum_{\hat{l}_1 \le l' \le l} \left(4 \cdot 2^{15l'/16} + \frac{\epsilon}{2p_0} 2^{l'} \right) \le 2^{\hat{l}_1} + 2^{-i(T)}T + \epsilon T + cp_0 T^{15/16} + \epsilon T.$$

Finally, putting everything together, for T sufficiently large, we have

$$\sum_{t \leq T} r_t(\pi^*(X_t)) - r_t \leq \sum_{t \in \mathcal{T}^{init}, t \leq T} \bar{r}_t(\pi^*(X_t)) - r_t + \sum_{p \geq 0} \sum_{2^{u(16p)} \leq t \leq T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t$$
$$\leq 2^{\hat{l}_1 + 1} \log_4 T + 2^{-i(T)} T + c(|\mathcal{A}|^2 + p_0) T^{15/16} \log_4 T + 3\epsilon T + \sum_{t \in \mathcal{T}^{init}, t \leq T} r_t(\pi^*(X_t)) - r_t,$$

which shows that on $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t)) - r_t \le 3\epsilon.$$

We denote by $(x)_{+} = \max(0, x)$ the positive part. Recall that $\mathbb{P}[\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}] \ge 1 - \epsilon$. Thus,

$$\mathbb{E}\left[\left(\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^T r_t(\pi(X_t)) - r_t\right)_+\right] \le 4\epsilon.$$

Because this holds for any $\epsilon > 0$, this shows that almost surely, $\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi(X_t)) - r_t \le 0$. As a result, the learning rule is universally consistent on X. This ends the proof of the theorem.

To the best of our knowledge, while we believe that for general spaces \mathcal{X} with non-atomic probability measures, one may have a gap $C_5 \subsetneq C_6$, it seems plausible that $C_5 = C_7$. As a consequence, this would imply that we have an exact characterization for processes admitting universal learning with prescient rewards $C_{prescient} = C_5 = C_7$.

Comparison to a more natural condition C_8 . In the rest of this section, we compare condition C_5 to another potentially more natural sufficient condition. [1] showed that given any $\mathbb{X} \in C_2$ process, only allowing for a finite number of duplicates in \mathbb{X} yields a C'_1 process. Precisely, for any M, letting

$$\mathcal{T}^{\leq M} = \left\{ t \geq 1 : \sum_{t' \leq t} \mathbb{1}[X_{t'} = X_t] \leq M \right\},\$$

the set of times when contexts are duplicates of index at most M, one has $(X_t)_{t \in \mathcal{T}^{\leq M}} \in \mathcal{C}'_1$. However, if one does not restrict the maximum number of duplicates, one loses the \mathcal{C}'_1 property. A natural condition on stochastic processes would therefore be that for some increasing rate of maximum number of duplicates, the \mathcal{C}'_1 property is conserved. For any process \mathbb{X} , we denote the occurrence count as $N_t(x) = \sum_{i=1}^t \mathbb{1}[X_t = x]$ for all $x \in \mathcal{X}$. Then, the condition on stochastic processes can be formally defined as follows.

Condition 8. There exists an increasing function $\Psi : \mathbb{N} \to \mathbb{N}$ with $\Psi(T) \to \infty$ as $T \to \infty$ such that for any sequence of measurable sets $A_i \in \mathcal{B}$ for $i \ge 1$ with $A_i \downarrow \emptyset$,

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{1}_{A_i}(X_t) \mathbb{1}_{N_t(X_t) \le \Psi(T)}\right] \to 0.$$

Although this condition is indeed sufficient for universal learning, we show that the more involved C_5 class of processes is larger, and strictly larger whenever \mathcal{X} admits a non-atomic probability measure.

Proposition 30. Let \mathcal{X} be a metrizable separable Borel space, then $C_8 \subset C_5$. Further, if there exists a non-atomic probability measure on \mathcal{X} , then $C_8 \subsetneq C_5$.

Proof. We first show $C_8 \subset C_5$. Indeed, suppose that $\mathbb{X} \in C_8$, then there exists $\Psi : \mathbb{N} \to \mathbb{N}$ increasing to infinity such that for any measurable sets $A_k \downarrow \emptyset$, we have

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, N_t(X_t) \le \Psi(T)} \mathbb{1}_{A_k}(X_t)\right] \xrightarrow[k \to \infty]{} 0.$$

Now let $T_i \ge 1$ such that $\Psi(T_i) \ge 1 + i2^i$. We now show that $(T_i)_i$ satisfies the condition of condition C_5 . Let $\mathcal{T} = \bigcup_{i\ge 0} \mathcal{T}^i \cap \{t \ge T_i\}$, and $A_k \downarrow \emptyset$. For any $T \ge 1$, we denote $\mathcal{X}(T) = \{X_t, t \le T\}$ the set of visited instances. Now fix $k \ge 0$. Then, for $T \ge T_k$, let $i \ge k$ such that $T_i \le T < T_{i+1}$,

$$\frac{1}{T} \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_{A_k}(X_t) \le \frac{1}{2^k} + \frac{1}{T} \sum_{2^{-k}T < t \le T, t \in \mathcal{T}} \mathbb{1}_{A_k}(X_t) \\
= \frac{1}{2^k} + \frac{1}{T} \sum_{x \in \mathcal{X}(T) \cap A_k} |\{2^{-k}T < t \le T, t \in \mathcal{T} : X_t = x\}|.$$

In \mathcal{T} , we accept at most one duplicate per phase. Because $T_i \leq T < T_{i+1}$, the interval $[2^{-k}T, T]$ intersects at most $1 + k2^i$ phases. Thus, for any $x \in \mathcal{X}(T)$, $|\{2^{-k}T < t \leq T, t \in \mathcal{T} : X_t = x\}| \leq 1 + k2^i \leq 1 + i2^i \leq \Psi(T)$. Thus, for any $T \geq T_k$,

$$\frac{1}{T} \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_{A_k}(X_t) \le \frac{1}{2^k} + \frac{1}{T} \sum_{x \in \mathcal{X}(T) \cap A_k} \min(|\{t \le T : X_t = x\}|, \Psi(T))$$
$$= \frac{1}{2^k} + \frac{1}{T} \sum_{t \le T, N_t(X_t) \le \Psi(T)} \mathbb{1}_{A_k}(X_t).$$

Using the hypothesis on Ψ applied to $A_k \downarrow \emptyset$ yields $\mathbb{E}\left[\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}} \mathbb{1}_{A_k}(X_t)\right] \xrightarrow[k\to\infty]{} 0.$ Hence, this shows that $\tilde{\mathbb{X}} = (X_t)_{t\in\mathcal{T}} \in \mathcal{C}'_1$ and $\mathbb{X} \in \mathcal{C}_5.$

Next, suppose that there exists a non-atomic probability measure on \mathcal{X} . We will construct explicitly a process $\mathbb{X} \in C_5 \setminus C_8$. By Lemma 21, there exists a sequence of disjoint measurable sets $(A_i)_{i\geq 0}$ together with non-atomic probability measures $(\nu_i)_{i\geq 0}$ such that $\nu_i(A_i) = 1$. We now fix $x_0 \in A_0$ an arbitrary instance (we will not use the set A_0 any further) and define subsets of indices as follows, $S_i = \{k \geq 1 : k \equiv 2^{i-1} \mod 2^i\}$. Note that the sets $(S_i)_{i\geq 1}$ form a partition of \mathbb{N} . We now introduce independent processes \mathbb{Z}^i for $i \geq 1$ such that $\mathbb{Z}^i = (Z_t^i)_{t\geq 1}$ is an i.i.d. process with distribution ν_i . Last, for all $i \geq 1$ we denote $n_i = 2^{\lfloor \log_2 i \rfloor}$. Now consider the following process \mathbb{X} where $X_1 = x_0$ and for any $t \geq 1$,

$$X_t = Z^i_{\lfloor \frac{t}{n_i} \rfloor}, \quad 2^k \le t < 2^{k+1}, k \equiv 2^{i-1} \bmod 2^i.$$

When the process is in phase *i*, it corresponds to an i.i.d. process on A_i which is duplicated n_i times. Note that we used n_i duplicates instead of *i* so that each point is duplicated exactly n_i times (we do not have boundary issues at the end of the phase). We now show that $\mathbb{X} \notin C_8$. Let $\Psi : \mathbb{N} \to \mathbb{N}$ an increasing function with $\Psi(T) \to \infty$ as $T \to \infty$. For $i \ge 1$, we first construct an increasing sequence of times T_i such that $\Psi(T_i) > n_i$. Then, for any $k \ge 1$, consider times $T^k = k2^i + 2^{i-1}$ which belong to S_i . Then, consider the event \mathcal{F}_i such that the process \mathbb{Z}^i only takes distinct values in A_i . Note that $\mathbb{P}[\mathcal{F}_i] = 1$ because the ν_i is non-atomic and $\nu_i(A_i) = 1$. Then, on \mathcal{F}_i , by construction, we have for any $k \ge 0$, with $T^k \ge T_i$,

$$\frac{1}{2T^{k}-1} \sum_{t=1}^{2T^{k}-1} \mathbb{1}_{A_{i}}(X_{t}) \mathbb{1}_{N_{t}(X_{t}) \leq \Psi(2T^{k}-1)} \geq \frac{1}{2T^{k}} \sum_{t=T^{k}}^{2T^{k}-1} \mathbb{1}_{A_{i}}(X_{t}) \mathbb{1}_{N_{t}(X_{t}) \leq n_{t}}$$
$$= \frac{1}{2T^{k}} \sum_{t=T^{k}}^{2T^{k}-1} \mathbb{1}_{A_{i}}(X_{t})$$
$$\geq \frac{T^{k}}{2T^{k}-1}.$$

Hence, on the event \mathcal{F}_i , we have $\limsup_{T\to\infty} \frac{1}{T} \sum_{i=1}^T \mathbb{1}_{A_i}(X_t) \mathbb{1}_{N_t(X_t) \leq \Psi(T)} \geq \frac{1}{2}$. Because $\mathbb{P}[\mathcal{F}_i] = 1$, we obtain

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} \mathbb{1}_{A_i}(X_t) \mathbb{1}_{N_t(X_t) \le \Psi(T)}\right] \ge \frac{1}{2}.$$

Now consider $B_i = \bigcup_{j \ge i} A_i$. Then, we have $B_i \downarrow \emptyset$ and for any $i \ge 1$,

$$\mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{i=1}^T \mathbb{1}_{B_i}(X_t) \mathbb{1}_{N_t(X_t) \le \Psi(T)}\right] \ge \mathbb{E}\left[\limsup_{T \to \infty} \frac{1}{T} \sum_{i=1}^T \mathbb{1}_{A_i}(X_t) \mathbb{1}_{N_t(X_t) \le \Psi(T)}\right] \ge \frac{1}{2}.$$

As a result, $\mathbb{X} \notin C_8$.

We now show that $\mathbb{X} \in C_5$. To do so, we first prove that $\mathbb{X} \in C_2$. Let $(B_l)_{l \ge 1}$ be a sequence of disjoint measurable sets. Because \mathbb{Z}^i are i.i.d. processes, we have $\mathbb{Z}^i \in C_2$. In particular, on an event \mathcal{E}_i of probability one, we have

$$|\{l: \mathbb{Z}_{\leq T}^i \cap B_l \neq \emptyset\}| = o(T).$$

Now consider the event $\mathcal{E} = \bigcap_{i \ge 1} \mathcal{E}_i$. This has probability one by the union bound. Let $\epsilon > 0$ and $i^* = \lceil \frac{2}{\epsilon} \rceil$. In particular, we have $\frac{1}{n_{i^*}} \le \epsilon$. On the event \mathcal{E} , for any $i \le i^*$, there exist T_i such that for all $T \ge T_i$,

$$|\{l: \mathbb{Z}^i_{\leq T} \cap B_l \neq \emptyset\}| \leq \frac{\epsilon}{2^i} T.$$

Now consider $T^0 = \max_{i < i^*} T_i n_i$. Then, for any $T \ge T^0$, we have

$$\begin{split} |\{l: \mathbb{X}_{\leq T} \cap B_l \neq \emptyset\}| &\leq \sum_{i=1}^{i^*} |\{l: \mathbb{Z}_{\leq \lfloor T/n_i \rfloor}^i \cap B_l \neq \emptyset\}| \\ &+ |\{l: \exists t \leq T: X_t \in B_l, 2^k \leq t < 2^{k+1}, k \equiv 0 \bmod 2^{i^*}\}| \\ &\leq \epsilon T + |\{X_t, \quad t \leq T, 2^k \leq t < 2^{k+1}, k \equiv 0 \bmod 2^{i^*}\}| \\ &\leq \epsilon T + 2\frac{T}{n_{i^*}}, \end{split}$$

where in the last inequality we used the fact that in a phase $i > i^*$, each point is duplicated $n_i \ge n_{i^*}$ times. As a result, on the event \mathcal{E} , we have

$$\operatorname{limsup} \frac{|\{l : \mathbb{X}_{\leq T} \cap B_l \neq \emptyset\}|}{T} \leq 3\epsilon.$$

Because this holds for all $\epsilon > 0$, we obtain that on \mathcal{E} , $|\{l : \mathbb{X}_{\leq T} \cap B_l \neq \emptyset\}| = o(T)$. Because \mathcal{E} has probability one, this ends the proof that $\mathbb{X} \in \mathcal{C}_2$. Now consider the following times $T_j = 4^j$ for $j \ge 0$ and define $\mathcal{T} = \bigcup_{j\ge 0} \mathcal{T}^j \cap \{t \ge T_i\}$. We aim to show $\tilde{\mathbb{X}} = (X_t)_{t\in\mathcal{T}} \in \mathcal{C}'_1$. First, note that for any $j \ge 0$, the phases $[2^k, 2^{k+1})$ contained in $[T_j, T_{j+1})$ satisfy $k \le 2j + 1$. Let $i(j) = 1 + \log_2(2j + 1)$. We have $k \in \bigcup_{i\le i(j)} S_i$, which implies that each instance X_t is duplicated consecutively at most $n_{i(j)}$ times in \mathbb{X} within $[T_j, t_{j+1})$. However, the sections defined by \mathcal{T} have length at least $2^{-j}T_j = 2^j$. Further, all the phases were constructed so that there are no boundary issues: if $n_{i(j)} \le 2^j$, then \mathcal{T} does not contain any duplicates during the period $[T_j, T_{j+1})$. Because $n_{i(j)} \le i(j) = o(2^j)$, there exists $j_0 \ge 0$ such that \mathcal{T} does not contain any duplicate on $[T_{j_0}, \infty)$. Let $\mathcal{T}(0) = \{t \ge 1 : N_t(X_t) = 1\}$ the set of first appearances. Then, for any $A \in \mathcal{B}$ and $T \ge 1$,

$$\sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_A(X_t) \le T_{j_0} + \sum_{t \le T, t \in \mathcal{T}(0)} \mathbb{1}_A(X_t).$$

Now because $\mathbb{X} \in \mathcal{C}_2$, we have $(X_t)_{t \in \mathcal{T}(0)} \in \mathcal{C}'_1$ which implies $(X_t)_{t \in \mathcal{T}} \in \mathcal{C}'_1$ by the above inequality. This ends the proof of the proposition.

5.3 Universal learning with fixed excess error tolerance

In this section, we show that as an application of the methods developed in [1] and in this paper, achieving a fixed excess regret $\epsilon > 0$ is always possible for C_2 processes. This is stated in Proposition 8. We first need to state a result from [1] showing that C_2 processes without duplicates are C'_1 extended processes.

Lemma 31 ([1]). Let X be a stochastic process on X, and define for any $M \ge 1$,

$$\mathcal{T}^{\leq M} = \left\{ t \geq 1 : \sum_{t' \leq t} \mathbb{1}[X_{t'} = X_t] \leq M \right\},\$$

the set of times which are duplicates of index at most M. In particular, $\mathcal{T}^{\leq 1}$ is the set of times where we delete all duplicates. The following are equivalent.

- 1. $\mathbb{X} \in \mathcal{C}_2$.
- 2. For all $M \geq 1$, $(X_t)_{t \in \mathcal{T} \leq M} \in \mathcal{C}'_1$.

We are now ready to prove Proposition 8.

Proof of Proposition 8. We first describe the algorithm that depends on a parameter $M \ge 1$ which we will fix later. We use the notation $\mathcal{T}^{\le M}$ from Lemma 31 for the set of times that are duplicates of index at most M. Note that whether $t \in \mathcal{T}^M$ or $t \notin \mathcal{T}^M$ can be decided in an online manner. Next we fix a sequence $\Pi = (\pi^l)_{l\ge 1}$ of policies that are dense within \mathcal{C}'_1 processes from Lemma 28. The learning rule f. simply performs the EXPINF strategy on the sequence Π for times in $\mathcal{T}^{\le M}$ and for other times performs independent copies of the EXP3.IX algorithm in parallel for each distinct instance. Formally, for any $t \ge 1$, instances $\mathbf{x}_{\le t}$ and observed rewards $\mathbf{r}_{\le t-1}$, we define

$$f_t(\boldsymbol{x}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}, x_t) = \begin{cases} \text{EXPINF}(\boldsymbol{x}_{U_t}, \boldsymbol{\hat{a}}_{U_t}, \boldsymbol{r}_{U_t}, x_t) & \text{if } t \in \mathcal{T}^M\\ \text{EXP3.IX}_{\mathcal{A}}(\boldsymbol{\hat{a}}_{S_t}, \boldsymbol{r}_{S_t}) & \text{o.w.} \end{cases}$$

where $U_t = \{t' \leq t - 1 : t \in \mathcal{T}^M\}$ and $S_t = \{t' < t : x_t = x_{t'}, t' \in \mathcal{T}^M\}$ and $\hat{a}_{t'}$ denotes the action selected at time $t' \leq t - 1$.

Let $X \in C_2$. We now prove that this learning rule achieves low excess error compared to a fixed measurable policy $\pi^* : \mathcal{X} \to \mathcal{A}$. We denote by $\hat{a}_t(M)$ its selected action at time t. First, by Lemma 31, $\tilde{X} = (X_t)_{t \in \mathcal{T}^M} \in C'_1$. Further, as discussed in Section 5.2, the same proof of universal consistence of EXPINF under C_1 processes for stationary rewards given in [1] shows that EXPINF is universally consistent under C'_1 extended processes for adversarial rewards. This is a consequence from the fact that the regret guarantee of EXP3.IX—Theorem 5—holds for adversarial rewards as well. Thus, on an event \mathcal{E} of probability one,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^M} r_t(\pi^*(X_t)) - r_t(\hat{a}_t(M)) \le 0.$$

Next, similarly to the proof of Proposition 13, let $\epsilon(T) = \frac{1}{T} |\{X_t : t \leq T, t \notin T^M\}|$. The same proof as in Proposition 13 shows that on an event \mathcal{F} of probability one, for all $T \geq 1$,

$$\frac{1}{T} \sum_{t \leq T, t \notin \mathcal{T}^M} r_t(\pi^*(X_t)) - r_t(\hat{a}_t(M)) \\
\leq 8|\mathcal{A}| \frac{\ln T}{T^{1/4}} + 3c\sqrt{|\mathcal{A}|\ln|\mathcal{A}|} \frac{1}{\ln T} + \sqrt{\epsilon(T)} + 3\sqrt{|\mathcal{A}|\ln|\mathcal{A}|}\epsilon(T)^{1/4}.$$

Note that to each element of $\{X_t : t \leq T, t \notin \mathcal{T}^M\}$ correspond least M duplicates in \mathcal{T}^M so that $\epsilon(T) \leq \frac{1}{M}$. As a result, combining the two previous equations yields on $\mathcal{E} \cap \mathcal{F}$ of probability one,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^M} r_t(\pi^*(X_t)) - r_t(\hat{a}_t(M)) \le 4 \frac{\sqrt{|\mathcal{A}| \ln |\mathcal{A}|}}{M^{1/4}}.$$

Thus, taking $M \ge 4^4 |\mathcal{A}|^2 \ln^2 |\mathcal{A}| \epsilon^{-4}$ gives a learning rule with the desired ϵ excess error almost surely. This ends the proof of the proposition.

6 Model extensions

6.1 Infinite action spaces

The previous sections focused on the case of finite action spaces. For infinite action spaces, we argue that as a direct consequence from the analysis of the stationary case in [1], one can obtain a characterization of learnable processes and same optimistically universal learning rules.

For countably infinite action spaces, they showed that EXPINF performed with the countable sequence of dense policies given by Lemma 28 is universally consistent under C_1 processes with stationary rewards, and that C_1 is necessary. As discussed in Sections 5.2 and 5.3, the same arguments as in [1] show that EXPINF is universally consistent under C_1 processes for adversarial rewards as well. Further, since adversarial rewards generalize stationary rewards, C_1 is still necessary for universal learning. Thus, $C_{online} = C_{prescient} = C_{oblivious} = C_{memoryless} = C_{stat} = C_1$ and EXPINF is optimistically universal in all reward settings.

For uncountable separable metrizable Borel action spaces \mathcal{A} , even for stationary rewards, universal learning is impossible [1]. Hence, $\mathcal{C}_{online} = \mathcal{C}_{prescient} = \mathcal{C}_{oblivious} = \mathcal{C}_{memoryless} = \mathcal{C}_{stat} = \emptyset$.

6.2 Unbounded rewards

We now turn to the case of unbounded rewards $\mathcal{R} = [0, \infty)$. We further suppose that for any $t \ge 1$, and history $\mathbf{x} \in \mathcal{X}^{\infty}$, $\mathbf{a}_{\le t} \in \mathcal{A}^t$, $\mathbf{r}_{\le t-1} \in \mathcal{R}^{t-1}$, the random variable $r_t(a_t \mid \mathbb{X} = \mathbf{x}, \hat{\mathbf{a}}_{\le t-1} = \mathbf{a}_{\le t-1}, \mathbf{r}(\hat{a})_{\le t-1} = \mathbf{r}_{\le t-1})$ is integrable so that the immediate expected reward is well defined. Again, in this case, adversarial rewards yield the same results as stationary rewards. Clearly, for uncountable separable metrizable Borel action spaces, under unbounded rewards, universal learning is still impossible $C_{online} = C_{prescient} = C_{oblivious} = C_{memoryless} = C_{stat} = \emptyset$, because this was alreay the case for bounded rewards.

For countable action spaces A, condition C_3 is necessary even under the full-feedback noiseless setting [2, 16], hence necessary for contextual bandits as well. Also, [1] proposed the algorithm which runs an independent EXPINF learner on each distinct context instance, which is universally consistent under C_3 processes. As in the previous section, this guarantee still holds for adversarial rewards, and C_3 is still necessary for universal learning. Therefore, $C_{online} = C_{prescient} = C_{oblivious} = C_{memoryless} = C_{stat} = C_3$.

6.3 Uniformly-continuous rewards

We assume that the rewards are bounded again. In the previous sections, we showed that for finite action sets, universal learning is possibly under large classes of processes, namely at least on C_5 processes. However, for countable action sets, this is reduced to C_1 and for uncountable action sets, universal learning is not achievable. Therefore, imposing no constraints on the rewards is too restrictive for universal learning in the last cases. Here, we investigate the case when A is a separable metric space given with a metric d, and the rewards are uniformly-continuous. Crucially, modulus of continuity should be uniformly bounded over time as well. We recall the definition of uniformly-continuous rewards.

Let (\mathcal{A}, d) be a separable metric space. The reward mechanism $(r_t)_{t\geq 1}$ is uniformly-continuous if for any $\epsilon > 0$, there exists $\Delta(\epsilon) > 0$ such that

$$\begin{aligned} \forall t \geq 1, \forall (\boldsymbol{x}_{\leq t}, \boldsymbol{a}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}) \in \mathcal{X}^{t} \times \mathcal{A}^{t-1} \times \mathcal{R}^{t-1}, \forall a, a' \in \mathcal{A}, \\ d(a, a') \leq \Delta(\epsilon) \Rightarrow \left| \mathbb{E}[r_t(a) - r_t(a') \mid \mathbb{X}_{\leq t} = \boldsymbol{x}_{\leq t}, \boldsymbol{a}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}] \right| \leq \epsilon, \end{aligned}$$

In the definition, the expectation is taken over the rewards' randomness, in the event when the context sequence until t is exactly $x_{\leq t}$, the learner selected actions $a_{\leq t-1}$ and received rewards $r_{\leq t-1}$ in the first t-1 steps. For instance, for stationary rewards, only x_t is relevant in this expectation, while for online rewards, $x_{\leq t}$, $a_{\leq t-1}$, $r_{\leq t-1}$ may be relevant. The above definition is not written for prescient rewards for simplicity. For these, we need to condition on the complete sequence X:

$$\begin{aligned} \forall t \geq 1, \forall (\boldsymbol{x}, \boldsymbol{a}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}) \in \mathcal{X}^{\infty} \times \mathcal{A}^{t-1} \times \mathcal{R}^{t-1}, \forall a, a' \in \mathcal{A}, \\ d(a, a') \leq \Delta(\epsilon) \Rightarrow \left| \mathbb{E}[r_t(a) - r_t(a') \mid \mathbb{X} = \boldsymbol{x}, \boldsymbol{a}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}] \right| \leq \epsilon. \end{aligned}$$

As in the unrestricted rewards case, we consider the set of processes $C_{setting}^{uc}$ admitting universal learning for uniformly-continuous rewards under any chosen reward setting. The uniform-continuity assumption defined above generalizes the corresponding assumption proposed in [1] for stationary rewards. They also proposed a weaker continuity assumption on the immediate expected rewards, however, similarly as in Section 6.1 one can easily check that with this reward assumption, adversarial settings give the same results as the stationary case.

The goal of this section is to show that under the mild uniform-continuity assumption on the rewards, one can recover all the results from the finite action space case, when the action space is totally-bounded. We first start by showing that the derived necessary conditions still hold. To do so, we will use the following reduction lemma.

Lemma 32. Let \mathcal{X} be a metrizable separable Borel space and let (\mathcal{A}, d) be a separable metric space. Let $S \subset \mathcal{A}$ such that $\min_{a,a' \in S} d(a,a') > 0$. Then, we have $\mathcal{C}^{uc}_{setting}(\mathcal{A}) \subset \mathcal{C}_{setting}(S)$ for any setting $\in \{stat, memoryless, oblivious, prescient, online\}.$

Further, if there is a learning rule for uniformly continuous rewards in A that is universally consistent under a set of processes \tilde{C} on X, there is also a learning rule for unrestricted rewards in S that is universally consistent under all \tilde{C} processes.

Proof. The first claim was proven in [1] for the specific case of stationary rewards. They show that the case of uniformly-continuous rewards on \mathcal{A} is at least harder than the unrestricted rewards on S through a simple reduction. Here, we show that the reduction can be extended to adversarial rewards as well.

Denote $\eta = \frac{1}{3} \min_{a,a' \in S} d(a,a')$. Any realization $r : S \to [0,1]$ can be extended to a $1/\eta$ -Lipschitz function $F(r) : \mathcal{X} \to \mathcal{A}$ by

$$F(r)(a) = \max\left(0, \max_{a' \in S} r(a') - d(a, a')\frac{\bar{r}}{\eta}\right), \quad a \in \mathcal{A}.$$

Then, a general reward mechanism $(r_t)_{t\geq 1}$ on S can be extended to a reward mechanism on \mathcal{A} such that for any realization, $r_t : a \in \mathcal{A} \to [0, 1]$ is $1/\eta$ -Lipschitz. Hence, the mechanism $(r_t)_{t\geq 1}$ is uniformlycontinuous. From now, the same arguments as in the proof of [1, Lemma 6.3] show that the reduction holds and that $\mathcal{C}_{setting}^{uc}(\mathcal{A}) \subset \mathcal{C}_{setting}$ for the considered setting. Intuitively, since for any realization, $r_t : a \in \mathcal{A} \to [0, 1]$ has zero value outside of the balls $B_d(a, \eta)$ for $a \in S$, that on the ball $B_d(a, \eta)$ for $a \in S$, the action a has maximum reward, and that these balls are disjoint, without loss of generality, one can assume that a universally consistent learning rule always selects actions in S under these rewards, in which case, the problem becomes equivalent to having unrestricted rewards on the action set S. The formal learning rule reduction is defined in the original proof, and one can check that the reduction is invariant in the process X. Hence, this also proves the second claim of the lemma.

This lemma allows to use the necessary conditions to the unrestricted reward setting by changing the terms "finite action set" (resp. "countably infinite action set") into "totally-bounded action set" (resp. "non-totally-bounded action set"). The second claim of Lemma 32 will be useful to show that no optimistically universal learning exists for adversarial uniformly-continuous rewards either. More precisely, the following result is a direct consequence from the first claim of Lemma 32.

Proposition 33. Let \mathcal{X} be a metrizable separable Borel space and let \mathcal{A} be a non-totally-bounded metric space. Then, for any reward setting, $\mathcal{C}^{uc} \subset \mathcal{C}_1$. Let \mathcal{A} be a totally-bounded metric space with $|\mathcal{A}| > 2$. Then, for any reward setting, $\mathcal{C}^{uc} \subset \mathcal{C}_2$. Further, if \mathcal{X} admits a non-atomic probability measure, $\mathcal{C}^{uc}_{memoryless} \subsetneq \mathcal{C}_2$, $\mathcal{C}^{uc}_{oblivious} \subset \mathcal{C}_6$ and $\mathcal{C}^{uc}_{prescient} \subset \mathcal{C}_7$.

We now show that we can recover the sufficient conditions from previous sections as well. For uniformly-continuous rewards, we can show that there exists a countable set of dense policies under C'_1 processes, as was the case for unrestricted rewards and countable action sets.

Lemma 34. Let \mathcal{A} be a separable metric space. There is a countable set of measurable policies Π such that for any extended process $\tilde{\mathbb{X}} = (X_t)_{t \in \mathcal{T}} \in \mathcal{C}'_1$, any measurable policy $\pi^* : \mathcal{X} \to \mathcal{A}$, and any uniformly-continuous possibly stochastic rewards $(r_t)_t$, with probability one over the rewards,

$$\begin{cases} \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} r_t(\pi^*(X_t)) - r_t(\pi(X_t)) \leq 0, \\ \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} \bar{r}_t(\pi^*(X_t)) - \bar{r}_t(\pi(X_t)) \leq 0, \end{cases}$$

where $\bar{r}_t = \mathbb{E}r_t$ is the immediate average reward.

Proof. For any $\epsilon > 0$, let $\Delta(\epsilon)$ be the ϵ -modulus of continuity of the sequence of rewards $(\bar{r}_t)_t$. By [1, Lemma 6.1] (and with a straightforward adaptation for extended processes), on an event \mathcal{E} of probability one, for any $i \ge 1$, there exists $\pi^i \in \Pi$ such that $\limsup_{T\to\infty} \frac{1}{T} \sum_{t\le T,t\in\mathcal{T}} \mathbb{1}[d(\pi^*(X_t),\pi^i(X_t)) \ge 2^{-i}] \le 2^{-i}$, for all $i \ge 1$, $\frac{1}{T} \sum_{t\le T,t\in\mathcal{T}} r_t(\pi^i(X_t)) - \bar{r}_t(\pi^i(X_t)) \to 0$ and similarly for π^* , where \bar{r}_t is the immediate expected reward at time t. We now suppose that this event is met. Let $\epsilon > 0$, let $i \ge 1$

such that $2^{-i} \leq \Delta(\epsilon)$. Then,

$$\begin{split} \sum_{t \le T, t \in \mathcal{T}} \bar{r}_t(\pi^*(X_t)) - \bar{r}_t(\pi^i(X_t)) \le \sum_{t \le T, t \in \mathcal{T}} (\bar{r}_t(\pi^*(X_t)) - \bar{r}_t(\pi^i(X_t))) \mathbb{1}_{d(\pi^i(x), \pi^*(x)) < \Delta(\epsilon)} \\ &+ \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_{d(\pi(x), \pi^*(x)) \ge 2^{-i}} \\ \le \epsilon T + \sum_{t \le T, t \in \mathcal{T}} \mathbb{1}_{d(\pi(x), \pi^*(x)) \ge 2^{-i}}. \end{split}$$

As a result, $\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}} \bar{r}_t(\pi^*(X_t)) - \bar{r}_t(\pi^i(X_t)) \leq \epsilon + \Delta(\epsilon)$. Further, because the event \mathcal{E} is satisfied, $\limsup_{T\to\infty} \frac{1}{T} \sum_{t\leq T,t\in\mathcal{T}} r_t(\pi^*(X_t)) - r_t(\pi^i(X_t)) \leq \epsilon + \Delta(\epsilon)$. This holds for any $\epsilon > 0$. Now because $\Delta(\epsilon) \to 0$ as $\epsilon \to 0$, we proved that on \mathcal{E} ,

$$\begin{cases} \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} r_t(\pi^*(X_t)) - r_t(\pi(X_t)) \leq 0, \\ \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t \leq T, t \in \mathcal{T}} \bar{r}_t(\pi^*(X_t)) - \bar{r}_t(\pi(X_t)) \leq 0. \end{cases}$$

This ends the proof of the lemma.

We are now ready to generalize our algorithms from previous sections, using Π as a countable set of functions that are dense within all policies in the uniformly-continuous rewards context. First, note that using EXPINF directly with the countable family described in Lemma 34 is universally consistent on all C_1 processes. This shows that we always have $C_1 \subset C^{uc}$ for all models. In particular, together with Proposition 33, this shows that for non-totally-bounded metric action spaces A, we have $C^{uc} = C_1$ for all reward models.

Next, we turn to the case of finite action spaces and context spaces \mathcal{X} that do not admit a nonatomic measure. In this case, we showed that the algorithm that simply uses different EXP3.IX for each distinct instance is optimistically universal. In the case of uniformly-continuous rewards, we can replace EXP3.IX with EXPINF over a countable set of actions. This yields an optimistically universal learning rule for any totally bounded action spaces \mathcal{A} .

Theorem 35. Let \mathcal{X} be a metrizable separable Borel space that does not admit a non-atomic probability measure. Let \mathcal{A} be a totally-bounded metric space. Then, there exists an optimistically universal learning rule for uniformly-continuous rewards (in any setting) and learnable processes are exactly $C_{stat}^{uc} = C_{online}^{uc} = C_2$.

Proof. We first describe the learning rule. For any $\epsilon > 0$, let $\mathcal{A}(\epsilon)$ be an ϵ -net of \mathcal{A} . By abuse of notation, for any $a \in \mathcal{A}$, we use the same notation a for the expert which selects action a at all time steps. Now consider the countable set of experts $\bigcup_{i\geq 1} \mathcal{A}(2^{-i}) = \{a_1, a_2, \ldots\}$, where the sets are concatenated by increasing order of index i. Now consider the learning rule that uses a distinct EXPINF over this set of experts, for each distinct instance. Formally, the learning rule is

$$f_t(\boldsymbol{x}_{\leq t-1}, \boldsymbol{r}_{\leq t-1}, x_t) = \text{EXPINF}(\hat{\boldsymbol{a}}_{S_t}, \boldsymbol{r}_{S_t})$$

where $S_t = \{t' < t : x_{t'} = x_t\}$ is the set of times that x_t was visited previously and $\hat{a}_{t'}$ denotes the action selected at time t' for t' < t. We now show that this learning rule is universally consistent on all C_2 processes for uniformly bounded rewards. In the proof of Theorem 16 we showed that for spaces \mathcal{X} that do not admit a non-atomic probability measure, any C_2 process visits a sublinear number of distinct instances almost surely. Therefore, for $\mathbb{X} \in C_2$, on an event \mathcal{E} of probability one, we have
$|\{x \in \mathcal{X} : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\}| = o(T)$. It now suffices to adapt the proof of Proposition 13. Let $(r_t)_t$ be an uniformly continuous reward mechanism. For $\epsilon > 0$, let $\Delta(\epsilon) > 0$ its ϵ -modulus of continuity. We keep the same notations as in the proof of Proposition 13. Let $S_T = \{x : \{x\} \cap \mathbb{X}_{\leq T} \neq \emptyset\}, \epsilon(T) = \frac{|S_T|}{T}$ and for $x \in S_T$, let $\mathcal{T}_T(x) = \{t \leq T : X_t = x\}$. Further, for any $x \in S_T$ we pose $\mathcal{T}_T(x) = \{t \leq T : X_t = x\}$. Let $\mathcal{H}_0(T) = \{x \in S_T : |\mathcal{T}_T(x)| < \frac{1}{\sqrt{\epsilon(T)}}\}, \mathcal{H}_1(T) = \{x \in S_T : \frac{1}{\sqrt{\epsilon(T)}} \leq |\mathcal{T}_T(x)| < \ln^8 T\}$ and $\mathcal{H}_2(T) = \{x \in S_T : |\mathcal{T}_T(x)| \geq \ln^8 T\}$. Now let $\pi : \mathcal{X} \to \mathcal{A}$ be a measurable policy. We still have

$$\frac{1}{T}\sum_{x\in\mathcal{H}_0(T)}|\mathcal{T}_T(x)|\leq\sqrt{\epsilon(T)}.$$

Next, we turn to points $x \in \mathcal{H}_2(T)$. By Theorem 6, conditionally on the realization \mathbb{X} , for any $x \in \mathcal{H}_2(T)$, with probability at least $1 - \frac{1}{T^3}$,

$$\max_{i \le \ln T} \sum_{t \in \mathcal{T}_T(x)} r_t(a_i) - r_t(\hat{a}_t) \le 4c |\mathcal{T}_T(x)|^{3/4} (\ln T)^{3/2} \le 4c \frac{|\mathcal{T}_T(x)|}{\sqrt{\ln T}}.$$

Therefore, since $|\mathcal{H}_2(T)| \leq T$, by union bound, with probability at least $1 - \frac{1}{T^2} := 1 - p_2(T)$,

$$\sum_{x \in \mathcal{H}_2(T)} \max_{i \le \ln T} \sum_{t \in \mathcal{T}_T(x)} r_t(a_i) - r_t(\hat{a}_t) \le 4c \frac{T}{\sqrt{\ln T}}$$

We then treat points in $\mathcal{H}_1(T)$ for which we will need to go back to the proof of the regret bounds for EXPINF and the underlying EXP3.IX algorithm which is used as subroutine. First we recall the structure of EXPINF. Let $i(k) = \sum_{r < k} r^3$. It works by periods $[i(k) + 1, i(k) + k^3)$ on which a new EXP3.IX learner to find the best expert within the first k experts in the sequence provided to EXPINF. We will refer to this as period k. As useful inequalities, we have $\frac{k^4}{4} \le i(k) \le \frac{(k+1)^4}{4}$. Let $k_0 = \lceil \epsilon(T)^{-1/8} \rceil$ and focus on a period k for $k \ge k_0$ of an EXPINF run. We denote by \hat{a}_u the action selected at horizon u by EXPINF. Following the same arguments as in Proposition 13 and the analysis of EXP3.IX in [43], for any $j \le k_0$

$$\sum_{u=1}^{k^3} (\ell_{u,\hat{a}_{i(k)+u}} - \tilde{\ell}_{u,a_j}) \le \frac{\ln k}{\eta_{k^3}} + \sum_{u=1}^{k^3} \eta_u \sum_{i=1}^k \tilde{\ell}_{u,a_i}.$$

As a result,

$$\sum_{u=1}^{k^3} \ell_{u,\hat{a}_{i(k)+u}} - \ell_{u,a_j} \le 3\sqrt{k\ln k \cdot k^3} + \sum_{u=1}^{k^3} (\tilde{\ell}_{u,a_j} - \ell_{u,a_j}) + \sum_{u=1}^{k^3} \sum_{i=1}^k \eta_u (\tilde{\ell}_{u,a_j} - \ell_{u,a_j})$$

Now for any $a \in \mathcal{A}$, let $a^{(k_0)} = \operatorname{argmin}_{1 \le i \le k_0} d(a, a_i)$ the nearest neighbor of a where ties are broken alphabetically. We will sum this inequality for all EXPINF runs for $x \in \mathcal{H}_1(T)$, and periods $k \ge k_0$ that were completed, i.e. $|\mathcal{T}_T(x)| \ge i(k+1)$, taking $a_j = \pi(x)^{(k_0)}$. Before doing so, note that $\sum_{k' \le k} \sqrt{3(k')^4 \ln k'} \le (k+1)^3 \sqrt{\ln k} \le 4i(k+1)^{3/4} \sqrt{\ln i(k+1)}$. Further, for simplicity, denote by A(T) (resp. B(T)) the sum that is obtained after summing all the terms $\sum_{u=1}^{k^3} (\tilde{\ell}_{u,a_j} - \ell_{u,a_j})$ (resp. $\sum_{u=1}^{k^3} \sum_{i=1}^k \eta_u(\tilde{\ell}_{u,a_j} - \ell_{u,a_j})$). Using these notations, we obtain

$$\sum_{x \in \mathcal{H}_1(T)} \sum_{t \in \mathcal{T}_T(x)} r_t(\pi(X_t)^{(k_0)}) - r_t(\hat{a}_t) \le \sum_{x \in \mathcal{H}_1(T)} \left(\frac{k_0^4}{4} + 4|\mathcal{T}_T(x)|^{3/4} + 4|\mathcal{T}_T(x)|^{3/4} \sqrt{3\ln|\mathcal{T}_T(x)|} \right) + A(T) + B(T).$$

where in the first inequality, $\frac{k_0^4}{4}$ accounts for the first k_0 initial periods and $4|\mathcal{T}_T(x)|^{3/4}$ accounts for the last phase which potentially was not completed. Now recall that for each $x \in \mathcal{H}_1(T)$, $\epsilon(T)^{-1/2} \leq |\mathcal{T}_T(x)| < \ln^8 T$. Let $n_0 \geq 1$ such that for any $n \geq n_0$, $8n^{3/4}\sqrt{3\ln n} \leq n^{7/8}$. Since on the event \mathcal{E} , we have $\epsilon(T) \to 0$, there exists an index \hat{T} such that for $T \geq \hat{T}$, $\epsilon(T)^{-1/2} \geq n_0$. Therefore, on \mathcal{E} , for $T \geq \hat{T}$ we have

$$\sum_{x \in \mathcal{H}_1(T)} \left(\frac{k_0^4}{4} + 20 |\mathcal{T}_T(x)|^{3/4} + |\mathcal{T}_T(x)|^3 \sqrt{3\ln|\mathcal{T}_T(x)|} \right) \le 2\sqrt{\epsilon(T)}T + \sum_{x \in \mathcal{H}_1(T)} |\mathcal{T}_T(x)|^{7/8} \le (2\sqrt{\epsilon(T)} + \epsilon(T)^{1/16})T.$$

Next, using the same arguments as in the proof of Proposition 13, observe that conditionally on \mathbb{X} , $(A(T'))_{T' \leq T}$ is a super-martingale, with increments bounded in absolute value by $2\sqrt{\frac{k \cdot k^3}{\ln k}} \leq 2k^2 \leq 4\sqrt{i(k+1)} \leq 4\ln^4 T$. Therefore, Azuma's inequality implies that

$$\mathbb{P}[A(T) \le 8T^{3/4} \ln^4 T \mid \mathbb{X}] \ge 1 - e^{-2\sqrt{T}}$$

Simialrly, $(B(T'))_{T' \leq T}$ is a super-martingale, with increments bounded in absolute value by $2k \sqrt{\frac{k \cdot k^3}{\ln k}} \leq 8i(k+1) \leq 8 \ln^8 T$. Therefore,

$$\mathbb{P}[B(T) \le 16T^{3/4} \ln^8 T \mid \mathbb{X}] \ge 1 - e^{-2\sqrt{T}}$$

Therefore, by the Borel-Cantelli lemma, on an event \mathcal{G} of probability one, $\limsup_{T\to\infty} \frac{1}{T}(A(T) + B(T)) \leq 0$. Finally, let $j(T) = \min(\epsilon(T)^{-1/8}, \ln T)$. Putting everything together, we proved that on $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$, for $T \geq \hat{T}$,

$$\frac{1}{T}\sum_{t\leq T} r_t(\pi(X_t)^{(j(T))}) - r_t(\hat{a}_t) \leq 3\sqrt{\epsilon(T)} + \epsilon(T)^{1/16} + \frac{4c}{\sqrt{\ln T}} + \frac{1}{T}(A(T) + B(T)).$$

In particular, this hows that on $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T} r_t(\pi(X_t)^{(j(T))}) - r_t(\hat{a}_t) \le 0.$$

Now using Hoeffding's bound, with probability at least $1 - 2e^{-2\sqrt{T}}$, we have

$$\left|\sum_{t=1}^{T} r_t(\pi(X_t)^{(j(T))}) - \bar{r}_t(\pi(X_t)^{(j(T))})\right| \le 2T^{3/4}.$$

We have the same bound for π . Therefore, the Borel-Cantelli lemma implies that on an event \mathcal{H} of probability one, $\frac{1}{T} \sum_{t=1}^{T} r_t(\pi(X_t)^{(j(T))}) - \bar{r}_t(\pi(X_t)^{(j(T))}) \to 0$ and $\frac{1}{T} \sum_{t=1}^{T} r_t(\pi(X_t)) - \bar{r}_t(\pi(X_t)) \to 0$. We now suppose that $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G} \cap \mathcal{H}$ is met.

Now fix $\epsilon > 0$. Let k_0 such that $2^{-k_0} \leq \Delta(\epsilon)$. Because \mathcal{E} is met, $\epsilon(T) \to 0$ and $j(T) \to \infty$. Thus, there exists $\tilde{T} \geq \hat{T}$ such that for any $T \geq \tilde{T}$, $\epsilon(T) \leq n_0^{-2}$ and $\mathcal{A}(2^{-k_0}) \subset \{a_i, j \leq j(T)\}$. Now for $T \geq \tilde{T}$ and any $a \in \mathcal{A}$, we have $d(a, a^{(j(T))}) \leq \Delta(\epsilon)$. As a result, using \mathcal{H} ,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi(X_t)) - r_t(\hat{a}_t) \leq \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \bar{r}_t(\pi(X_t)) - \bar{r}_t(\pi(X_t)^{(j(T))})$$
$$\leq \limsup_{T \to \infty} \frac{\tilde{T}}{T} + \epsilon$$
$$\leq \epsilon.$$

In the second inquality, we used the uniform-continuity assumption on the rewards and the fact that for $T \geq \tilde{T}$, $d(\pi(X_t), \pi(X_t)^{(j(T))}) \leq \min_{a \in \mathcal{A}(2^{-k_0})} d(a, \pi(X_t)) \leq 2^{-k_0} \leq \Delta(\epsilon)$. Because this holds for any $\epsilon > 0$ and $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G} \cap \mathcal{H}$ has probability one, this proves that the learning rule is universally consistent under \mathbb{X} . Then, the learning rule is universally consistent under any C_2 process. By Proposition 33, this shows that the learnable processes are exactly C_2 and that this is an optimistically universal learning rule. This ends the proof of the theorem.

The last algorithms needed to be adapted to the uniformly-continuous rewards setting are the algorithms for C_5 processes in finite action spaces. Precisely, we will show that we for totally-bounded metric action spaces A, the set of learnable processes for uniformly-continuous adversarial rewards contains C_5 processes. Recall that the class of constructed algorithms in Theorem 29 proceed separately on different categories of times. The category of t is defined based on the number of duplicates of X_t within its associated period. For each category of times, the learning rule performs a form of Hedge algorithm to perform the best strategy among strategy 0 which simply assigns a different EXP3.IX learner to distinct instances from the period; and strategy j for $j \ge 1$ which selected actions according to a fixed policy π^j , where $\tilde{\Pi} = {\pi^l, l \ge 1}$ was a dense of policies within C'_1 processes.

We make the following modifications to these learning rules. First, we replace Π with the countable set Π of measurable policies that are dense in the uniformly-continuous rewards setting, as given by Lemma 34. Second, for every category p, strategy 0 will use EXP3.IX learners from $\mathcal{A}(\gamma_p)$, a γ_p -nets of \mathcal{A} , where γ_p is to be defined. With these modifications, we obtain the following result.

Theorem 36. Let \mathcal{X} be a metrizable separable Borel space and let \mathcal{A} be a totally-bounded metric space. Then, $C_5 \subset C_{online}^{uc}$.

Proof. Fix $\mathbb{X} \in \mathcal{C}_5$ and let $(T_i)_{i\geq 0}$ such that with $\mathcal{T} = \bigcup_{i\geq 0} \mathcal{T}^i \cap \{t \geq T_i\}$, we have $(X_t)_{t\in\mathcal{T}} \in \mathcal{C}'_1$. We first define how we modify the learning rule from Theorem 29 for this process. The functions PHASE, STAGE, PERIOD, CATEGORY are left unchanged. In the initial phase when $t < 2^{u(16p)}$, we replace EXP3.IX_A with EXPINF run with the dense sequence of \mathcal{A} with the specific order described in the previous Theorem 35. We briefly recap the procedure. Let $\mathcal{A}(\epsilon)$ be an ϵ -net of \mathcal{A} . We consider the sequence of experts $\bigcup_{i\geq 1} \mathcal{A}(2^{-i})$ where we confuse $a \in \mathcal{A}$ with the constant policy equal to a and we concatenate the nets by increasing order of index i. EXPINF is then run with this sequence of experts. Next, we enumerate $\Pi = \{\pi^l, l \geq 1\}$ and use these policies as well for the learning rule (strategies $j \geq 1$). Last, when playing strategy 0 after the initial phase, we replace EXP3.IX_A with EXP3.IX_{A(\gamma_p)}, where γ_p will be defined shortly. In the original proof, we defined $\delta_p := 6\frac{\sqrt{|\mathcal{A}|\ln|\mathcal{A}|}{2^p}}$, $\eta_i := \sqrt{\frac{8\ln(i+1)}{2^i}}$ and showed that the average error of the learning rule on \mathcal{T}_p outside of the initial phase is $\mathcal{O}(\delta_p + \frac{\eta_{16p}}{4}) < \infty$ allowed the learner to converge separately on each \mathcal{T}_p . We now replace δ_p with $\delta_p := 4\sqrt{\frac{|\mathcal{A}(\gamma_p)|\ln|\mathcal{A}(\gamma_p)|}{2^p}}$ and choose γ_p such that $\sum_p \delta_p < \infty$. We pose

$$\gamma_p = \min\{2^{-i} : |\mathcal{A}(2^{-i})| \ln |\mathcal{A}(2^{-i})| \le 2^{p/4}\}.$$

Thus, we still have $\sum_{p} \delta_{p} < \infty$ and $\gamma_{p} \to 0$. We now show that the modified learning rule is universally consistent under online uniformly-continuous rewards on \mathcal{A} . Fix $(r_{t})_{t}$ such a reward mechanism and for $\epsilon > 0$, let $\Delta(\epsilon)$ the ϵ -modulus of continuity of the sequence of immediate rewards. As in the original proof of Theorem 29, let $\mathcal{T}^{init} = \bigcup_{p\geq 0} \{t \in \mathcal{T}_{p} : t < 2^{u(16p)}\}$ be the initial phase. The process $(X_{t})_{t\in\mathcal{T}^{init}}$ still visits a sublinear number of distinct instances almost surely, where we say that two instances $t, t' \in \mathcal{T}^{init}$ are duplicates if and only if they have same category, period and $X_{t} = X_{t'}$. As a

result, in the proof of Theorem 35, we showed that for any $\pi^* : \mathcal{X} \to \mathcal{A}$, on an event \mathcal{E} of probability one,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}^{init}} r_t(\pi^*(X_t)) - r_t \le 0.$$

We then turn to non-initial phases and adapt the original proof of Theorem 29. For any $a \in A$, we denote $a^{(\gamma)} = \operatorname{argmin}_{a' \in A(\gamma)} d(a, a')$, the nearest neighbor of a within the γ -net where ties are broken alphabetically. Keeping the same event \mathcal{F} , Eq (16) is unchanged and Eq (17) becomes

$$R_p(l,k;0) \ge \sum_{t \in \mathcal{T}_p(l,k)} r_t(\pi^*(X_t)^{(\gamma_p)}) - 16|\mathcal{A}(\gamma_p)|^2 2^{-i} 2^{15l/16} - \delta_p 2^{l-i}.$$

Eq (18) is left unchanged. For $p \ge 0$, let $\epsilon(p) = \min\{2^{-i} : \gamma_p \le \Delta(2^{-i})\}$. Note that because $\gamma_p \to 0$, we have $\epsilon(p) \to 0$ as $p \to \infty$. Following the same arguments as in the original proof and noting that $|\mathcal{A}(\gamma_p)| \le 2^{p/4}$, Eq (19) is replaced by

$$\sum_{\substack{2^{u(16p)} < t \le T, t \in \mathcal{T}_p}} r_t(\pi^*(X_t)^{(\gamma_p)}) - r_t \le 2^{\hat{l}} + c2^{p/2}T^{15/16} + \left(\delta_p + \frac{\eta_{16p}}{4}\right)T$$
$$\le 2^{\hat{l}} + cT^{31/32} + \left(\delta_p + \frac{\eta_{16p}}{4}\right).$$

Now fix $\epsilon > 0$, and let p_0 such that $\sum_{p \ge p_0} (\delta_p + \frac{\eta_{16p}}{4}) < \epsilon$ and $\epsilon(p_0) < \epsilon$. Following the original arguments,

$$\sum_{p \ge p_0} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)^{(\gamma_p)}) - r_t \le 2^{\hat{\ell}} \log_4 T + cT^{31/32} \log_4 T + \epsilon T.$$

Now using Azuma's inequality, with probability at least $1 - 4e^{-2\sqrt{T}}$, we have

$$\left| \sum_{p \ge p_0} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)^{(\gamma_p)}) - \bar{r}_t(\pi^*(X_t)^{(\gamma_p)}) \right| \le 2T^{3/4}$$
$$\left| \sum_{p \ge p_0} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - \bar{r}_t(\pi^*(X_t)) \right| \le 2T^{3/4}.$$

Therefore, using Borel-Cantelli, on an event \mathcal{G} of probability one, there exists \hat{T}_1 such that for $T \geq \hat{T}_1$, the above two equations hold. Then, on $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$, for T sufficiently large,

$$\sum_{p \ge p_0} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t \le 2^{\hat{l}} \log_4 T + cT^{31/32} \log_4 T + \epsilon T + 4T^{3/4} + \sum_{p \ge p_0} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} \bar{r}_t(\pi^*(X_t)) - \bar{r}_t(\pi^*(X_t)^{(\gamma_p)}) \le 2^{\hat{l}} \log_4 T + 4T^{3/4} + cT^{31/32} \log_4 T + 2\epsilon T,$$

where in the last inequality we used the uniform continuity of the immediate expected rewards since for $p \ge p_0$, one has $\gamma_p \le \gamma_{p_0} \le \Delta(\epsilon(p_0)) \le \Delta(\epsilon)$. This implies that on the event $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{p \ge p_0} \sum_{2^{u(16p)} \le t < T, t \in \mathcal{T}_p} r_t(\pi(X_t)) - r_t \le 2\epsilon$$

Now for $p < p_0$, by Lemma 34, on an event \mathcal{H}_p of probability one, there exists l^p such that

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t \le T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t(\pi^{l_p}(X_t)) \le \frac{\epsilon}{p_0}.$$

Following the arguments in the proof of Theorem 29, on the event $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G} \cap \bigcap_{p < p_0} \mathcal{H}_p$ of probability one, for T large enough,

$$\sum_{p < p_0} \sum_{2^{u(16p)} \le t \le T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t \le \sum_{p < p_0} \sum_{2^{u(16p)} \le t \le T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t(\pi^{l_p}(X_t)) + \sum_{p < p_0} \sum_{2^{u(16p)} \le t \le T, t \in \mathcal{T}_p} r_t(\pi^{l_p}(X_t)) - r_t \le \sum_{p < p_0} \sum_{2^{u(16p)} \le t \le T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t(\pi^{l_p}(X_t)) + 2^{\hat{l}_1} + 2^{-i(T)}T + cp_0T^{15/16} + \epsilon T.$$

As a result,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{p < p_0} \sum_{2^{u(16p)} \le t \le T, t \in \mathcal{T}_p} r_t(\pi^*(X_t)) - r_t \le 2\epsilon.$$

Combining all the estimates together, we proved that on $\mathcal{E} \cap \mathcal{F} \cap \mathcal{G} \cap \bigcap_{p < p_0} \mathcal{H}_p$ of probability one,

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi^*(X_t)) - r_t \le 4\epsilon.$$

This holds for all $\epsilon > 0$. The same arguments as in the original proof conclude that the learning rule is universally consistent under X. This ends the proof of the theorem.

As a summary, we generalized all results from the case of the unrestricted reward to uniformlycontinuous rewards with the corresponding assumptions on action spaces.

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