Internet Appendix for

"Macroeconomic Conditions and the Puzzles of Credit Spreads and Capital Structure"

Hui Chen[*]

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This internet appendix contains supplemental materials for the published article. It is organized as follows. Section A uses a simple example to illustrate how comovements among risk prices, default probabilities, and default losses raise the present value of expected default losses. Section B illustrates the scaling property. Section C provides detailed proofs of several propositions and formula for the risk premium of debt and equity in the model. Section D discusses the calibration of the Markov chain and examines its performance. Section E investigates the state-dependent default losses.

A. Simple Two-Period Example

In this section, I use a simple two-period example to illustrate how comovements among risk prices, default probabilities, and default losses can raise the present value of expected default losses.

Suppose that at $t = 1$ the economy can either be in a good state (G) or bad state (B) with equal probability (see Figure 1). The prices of one-period Arrow-Debreu securities that pay \$1 in one of the two states are Q_G and Q_B, respectively. Since marginal utility is high in the bad state, agents will pay more for consumption in that state: $Q_B > Q_G$. A firm issues a one-period bond with face value \$1 at $t = 0$. The probabilities of default in the two states at $t = 1$ are p_G and p_B. Given default, the default losses in the two states are L_G and L_B.

[*]MIT Sloan School of Management. Email: huichen@mit.edu. Tel. 617-324-3896.
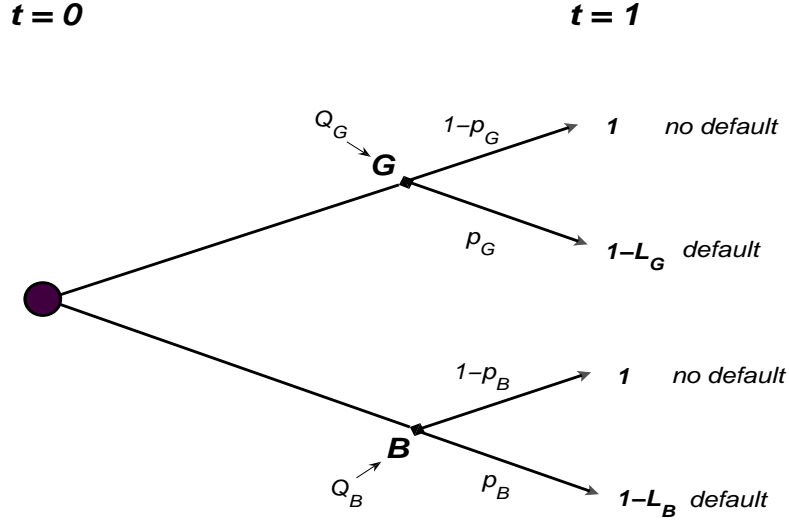
$t = 0$ $t = 1$

Q_G
  G $1-p_G$ **1**   *no default*

p_G **1−L$_G$** *default*

$1-p_B$ **1**   *no default*

B

Q_B p_B **1−L$_B$** *default*

Figure 1: Payoff Diagram of a Defaultable Zero Coupon Bond in a Two-period Example.

The price of the zero-coupon bond at $t = 0$ is:

$$B = Q_G\left[(1 - p_G) \cdot 1 + p_G \cdot (1 - L_G)\right] + Q_B\left[(1 - p_B) \cdot 1 + p_B \cdot (1 - L_B)\right]$$
$$= Q_G + Q_B - \left[Q_G p_G L_G + Q_B p_B L_B\right].$$

This equation says that the price of a defaultable bond is equal to the price of a default-free bond minus the present value of expected losses at default.

As a benchmark, we first assume that the default probabilities and default losses are constant across the two states, equal to their unconditional means: $\bar{p} = (p_G + p_B)/2$ and $\bar{L} = (L_G + L_B)/2$. Then, the bond price becomes

$$B = Q_G + Q_B - \left[Q_G \bar{p}\bar{L} + Q_B \bar{p}\bar{L}\right].$$

Next, raise the default probability and default losses in the bad state, but lower their values in the good state, so that the average default probabilities and default losses are unchanged. Such a change shifts the losses to the state with a higher Arrow-Debreu price, which raises the present value of expected default losses (the rise in $Q_B p_B L_B$ will be more than the drop in $Q_G p_G L_G$). Then, the bond price at $t = 0$ will be lower relative to the benchmark case. This (convexity) effect is stronger the bigger the spread in Q, p, L between the two states.
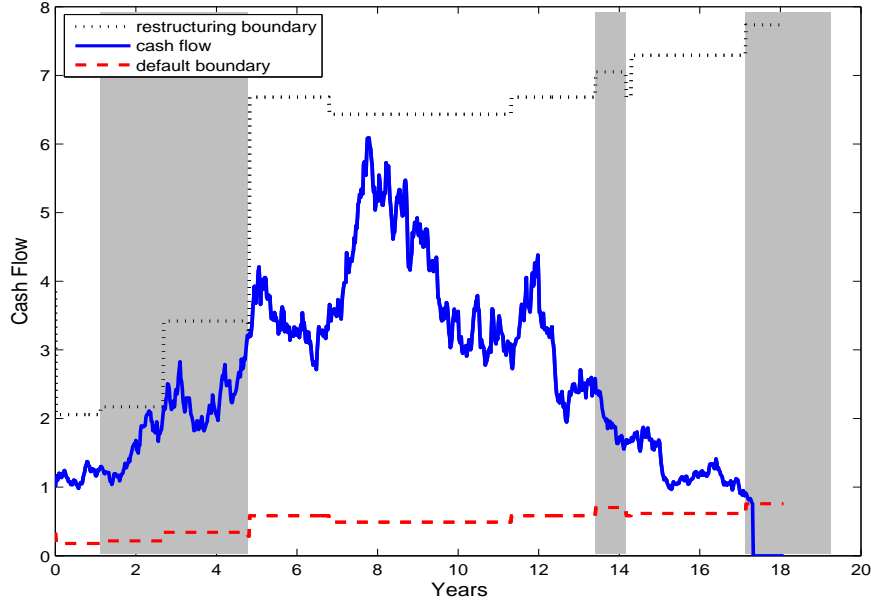
Figure 2: Illustration of the Scaling Property. The blue line is a cash flow sample path for the firm. The black dotted line and the red dash line, respectively, are the restructuring and default boundaries for the current state at each point in time. Shaded regions denote times when the economy is in a bad state.

This simple example treats the Arrow-Debreu prices, default probabilities, and default losses as exogenous. Empirically, the comovements among these variables are difficult to measure directly. Theoretically, it is not clear whether such relations should hold at all, because firms could adjust their capital structure over the business cycle to avoid defaulting in those bad states. In the published paper, I study the dynamic capital structure decisions in a model with business cycle risks, deriving Arrow-Debreu prices from consumption dynamics and preferences, connecting default probabilities to the business cycle through firms' endogenous decisions, and estimating default losses from the data of recovery rates. I then show that the comovements among these quantities can account for a large part of the puzzles of credit spreads and leverage ratios.

B. Illustration of the Scaling Property

Figure 2 plots a sample path of the firm's cash flows to illustrate the scaling property. The firm enjoys strong growth in early periods. Cash flow rises and hit the restructuring boundaries twice, causing the firm to raise more debt. When the firm restructures, both the default and restructuring

boundaries scale up proportionally. In several occasions, the economy moves into a bad state (shaded area), causing both default and restructuring boundaries to jump. Later on, the firm's cash flow declines until hitting the default boundary, and the firm defaults.

C. Proofs

C.1 Proof of Proposition 1

To obtain the stochastic discount factor, we first solve for the value function of the representative household. In equilibrium, the representative household consumes aggregate output Y_t,

$$\frac{dY_t}{Y_t} = \theta_m\left(s_t\right) dt + \sigma_m\left(s_t\right) dW_t^m. \tag{1}$$

Thus, I directly define the value function of the representative agent as:

$$J\left(Y_t, s_t\right) = E_t\left[\int_0^\infty f\left(Y_{t+s}, J_{t+s}\right) ds\right]. \tag{2}$$

The Hamilton-Jacoby-Bellman equation in state i is:

$$0 = f\left(Y, J\left(Y, i\right)\right) + J_c\left(Y, i\right) Y \theta_m\left(i\right) + \frac{1}{2} J_{cc}\left(Y, i\right) Y^2 \sigma_m^2\left(i\right) + \sum_{j \neq i} \lambda_{ij}\left(J\left(Y, j\right) - J\left(Y, i\right)\right). \tag{3}$$

There are n such differential equations for the n states. Thus, by using a Markov chain to model the expected growth rate and volatility, we replace a high-dimensional partial differential equation (expected growth rate and conditional volatility of Y_t will both be state variables) with a system of ordinary differential equations.

I conjecture that the solution for J is:

$$J\left(Y, s\right) = \frac{\left(h\left(s\right) Y\right)^{1-\gamma}}{1 - \gamma}, \tag{4}$$

where h is a function of the state variable s. Substituting J into the differential equations above, we get a system of nonlinear equations for h:

$$
\begin{aligned}
0 =\ & \rho \frac{1-\gamma}{1-\delta} h(i)^{\delta-\gamma} + \left[\left(1-\gamma\right)\theta_m\left(i\right) - \frac{1}{2}\gamma\left(1-\gamma\right)\sigma_m^2\left(i\right) - \rho\frac{1-\gamma}{1-\delta}\right] h\left(i\right)^{1-\gamma} \\
& + \sum_{j \neq i} \lambda_{ij}\left(h\left(j\right)^{1-\gamma} - h\left(i\right)^{1-\gamma}\right), \qquad i = 1, \cdots, n
\end{aligned} \tag{5}
$$

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where $\delta = 1/\psi$, the inverse of the intertemporal elasticity of substitution. These equations can be solved quickly using a nonlinear equation solver, even when the number of states is large (say $n = 100$).

Duffie and Epstein (1992) and Duffie and Skiadas (1994) show that the stochastic discount factor is equal to:

$$m_t = e^{\int_0^t f_v(c_u, J_u) du} f_c(c_t, J_t). \tag{6}$$

Plugging J and Y into (6) gives:

$$m_t = \exp\left(\int_0^t \frac{\rho(1-\gamma)}{1-\delta}\left[\left(\frac{\delta-\gamma}{1-\gamma}\right) h(s_u)^{\delta-1} - 1\right] du\right) \rho h(s_t)^{\delta-\gamma} Y_t^{-\gamma}. \tag{7}$$

Applying Ito's formula with jumps (see, e.g., Appendix F Duffie (2001)) to m, we get:

$$\frac{dm_t}{m_t} = -r(s_t) dt - \eta(s_t) dB_t + \sum_{s_t \neq s_{t-}} \left(e^{\kappa(s_{t-}, s_t)} - 1\right) dM_t^{(s_{t-}, s_t)}, \tag{8}$$

where

$$
\begin{aligned}
r(i) &= -\frac{\rho(1-\gamma)}{1-\delta}\left[\left(\frac{\delta-\gamma}{1-\gamma}\right) h(i)^{\delta-1} - 1\right] + \gamma\theta_m(i) \\
&\quad -\frac{1}{2}\gamma(1+\gamma)\sigma_m^2(i) - \sum_{j\neq i} \lambda_{ij}\left(e^{\kappa(i,j)} - 1\right), \\
\eta(i) &= \gamma\sigma_m(i), \\
\kappa(i,j) &= (\delta-\gamma)\log\left(\frac{h(j)}{h(i)}\right).
\end{aligned}
$$

$$\text{(9a)}$$
$$\text{(9b)}$$
$$\text{(9c)}$$

Consider two special cases. In the first case, $\delta = 1/\psi = \gamma$. In this case, the normalized aggregator reduces to the standard CRRA utility. As equation (9c) shows, the stochastic discount factor does not jump in this case, so that large shocks are no longer priced. The risk free rate in this case simplifies to:

$$r(i) = \rho + \gamma\theta_m(i) - \frac{1}{2}\gamma(1+\gamma)\sigma_m^2(i).$$

Moreover, the nonlinear equation (5) simplifies to a linear equation, and $h(i)$ can be solved analytically.

Another special case is when $\psi = 1$. In this case, the aggregator takes the form

$$f(c, v) = \rho(1-\gamma) v \left[\log(c) - \frac{1}{1-\gamma}\log((1-\gamma)v)\right]. \tag{10}$$

The value function is still given by (4), but the system of nonlinear equations for h becomes:

$$
\begin{aligned}
0 &= -\rho(1-\gamma)h(i)^{1-\gamma}\log(h(s)) + (1-\gamma)\left(\theta_m(i) - \frac{1}{2}\gamma\sigma_m^2(i)\right)h(i)^{1-\gamma} \\
&\quad + \sum_{j\neq i}\lambda_{ij}\left(h(j)^{1-\gamma} - h(i)^{1-\gamma}\right), \qquad i = 1, \cdots, n.
\end{aligned} \tag{11}
$$

Similarly, the risk free rate becomes:

$$
r(i) = \rho + \rho(1-\gamma)\log(h(i)) + \gamma\theta_m(i) - \frac{1}{2}\gamma(1+\gamma)\sigma_m^2(i) - \sum_{j\neq i}\lambda_{ij}\left(e^{\kappa(i,j)} - 1\right), \tag{12}
$$

and the relative jump size of the discount factor becomes:

$$
\kappa(i,j) = (1-\gamma)\log\left(\frac{h(j)}{h(i)}\right). \tag{13}
$$

C.2   The Risk-Neutral Measure

Let $(\Omega, \mathfrak{F}, \mathcal{P})$ be the probability space on which the Brownian motions and Poisson processes in the model are defined. Let the corresponding information filtration be $(\mathfrak{F}_t)$.

The nominal stochastic discount factor is:

$$
n_t = \frac{m_t}{P_t}. \tag{14}
$$

Applying Ito's formula to n_t, we get the dynamics of the nominal stochastic discount factor n_t,

$$
\frac{dn_t}{n_t} = -r^n(s_t)\,dt - \eta^m(s_t)\,dW_t^m - \eta^P\,dW_t^P + \sum_{s_t \neq s_{t-}}\left(e^{\kappa\left(s_{t-},s_t\right)} - 1\right)dM_t^{\left(s_{t-},s_t\right)}, \tag{15}
$$

where the nominal risk-free rate is

$$
r^n(s_t) = r(s_t) + \pi - \sigma_{P,1}\eta(s_t) - \sigma_P^2, \tag{16}
$$

and the risk prices for the two Brownian motions are

$$
\begin{aligned}
\eta^m(s_t) &= \eta(s_t) + \sigma_{P,1}, \tag{17} \\
\eta^P &= \sigma_{P,2}. \tag{18}
\end{aligned}
$$

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We can define the risk-neutral measure $\mathcal{Q}$ associated with the nominal stochastic discount factor n_t (equation (15)) by specifying the density process ξ_t,

$$\xi_t = E_t \left[\frac{dQ}{dP} \right],$$

which evolves according to the following process:

$$\frac{d\xi_t}{\xi_t} = -\eta^m (s_t) \, dW_t^m - \eta^P dW_t^P + \sum_{s_t \neq s_{t-}} \left(e^{\kappa \left(s_{t-}, s_t \right)} - 1 \right) dM_t^{\left(s_{t-}, s_t \right)}. \tag{19}$$

Applying the Girsanov theorem, we get the new standard Brownian motions under $\mathcal{Q}$, $\tilde{W}^m$ and $\tilde{W}^P$, which solve:

$$d\tilde{W}_t^m = dW_t^m + \eta^m (s_t) \, dt, \tag{20}$$

$$d\tilde{W}_t^P = dW_t^P + \eta^P dt. \tag{21}$$

The Girsanov theorem for point processes (see Elliott (1982)) gives the new jump intensity of the Poisson process under $\mathcal{Q}$:

$$\tilde{\lambda}_{jk} = E \left[e^{\kappa(j,k)} \right] \lambda_{jk} = e^{\kappa(j,k)} \lambda_{jk}, \quad j \neq k \tag{22}$$

which adjusts the intensity of the Poisson processes under measure $\mathcal{P}$ by the expected jump size of the density ξ_t. Finally, the diagonal elements of the generator has to be reset to make each row sum up to zero,

$$\tilde{\lambda}_{jj} = -\sum_{k \neq j} \tilde{\lambda}_{jk}. \tag{23}$$

These two equations characterize the new generator matrix $\tilde{\mathbf{\Lambda}}$ under $\mathcal{Q}$.

C.3   Value of Unlevered Firm

The risk-neutral dynamics of the log nominal cash flow $x_t = \ln(X_t)$ of the firm is:

$$dx_t = \left(\tilde{\theta}_X (s_t) - \frac{1}{2} \sigma_X (s_t)^2 \right) dt + \sigma_X (s_t) \, d\tilde{W}_t^f, \tag{24}$$

where $\tilde{\theta}_X$ is the risk-neutral growth rate,

$$\tilde{\theta}_X(s_t) = \theta_X(s_t) - \sigma_{X,m}(s_{t-}) \eta^m(s_{t-}) - \sigma_{P,2}\eta^P, \tag{25}$$

$\sigma_X(s_t)$ is the total volatility of cash flow,

$$\sigma_X(s_t) = \sqrt{\sigma_{X,m}^2(s_t) + \sigma_{P,2}^2 + \sigma_f^2}, \tag{26}$$

and $\tilde{W}_t^f$ is a standard Brownian motion under $\mathcal{Q}$, defined by

$$d\tilde{W}_t^f = \frac{\sigma_{X,m}(s_t)}{\sigma_X(s_t)}d\tilde{W}_t^m + \frac{\sigma_{P,2}}{\sigma_X(s_t)}d\tilde{W}_t^P + \frac{\sigma_f}{\sigma_X(s_t)}dW_t^f. \tag{27}$$

The total present value of the firm's cash-flows before taxes is:

$$V(x_t, s_t) = \mathbb{E}_t^Q\left[\int_t^\infty \exp\left(-\int_t^\tau r^n(s_u)\,du\right)\exp(x_\tau)d\tau\right]. \tag{28}$$

I compute the value of a cash flow stream by solving a system of ordinary differential equations.[1] Let $\mathbf{V}(x) = [V(x,1),...,V(x,n)]'$ be a vector of the firm's asset value in n states. The Feynman-Kac formula implies that $\mathbf{V}$ satisfies the following system of ODEs:

$$\mathbf{r}^n\mathbf{V} = \left(\tilde{\theta}_X - \frac{1}{2}\Sigma_X\right)\mathbf{V}_x + \frac{1}{2}\Sigma_X\mathbf{V}_{xx} + \tilde{\Lambda}\mathbf{V} + e^x \cdot \mathbf{1}, \tag{29}$$

where $\mathbf{r}^n = \mathbf{diag}\left([r^n(1),\cdots,r^n(n)]'\right)$, $\tilde{\theta}_X = \mathbf{diag}\left(\left[\tilde{\theta}_X(1),\cdots,\tilde{\theta}_X(n)\right]'\right)$, $\mathbf{1}+$ is an $n \times 1$ vector of ones, and $\Sigma_X = \mathbf{diag}\left(\left[\sigma_X^2(1),\cdots,\sigma_X^2(n)\right]'\right)$.

The boundary conditions are:

$$\lim_{x\downarrow-\infty}\mathbf{V}^i(x) = \mathbf{0} \tag{30}$$

$$\lim_{x\uparrow+\infty}\mathbf{V}(x)e^{-x} < \infty \tag{31}$$

The first condition specifies that the value of the firm goes to zero as cash flow goes to 0. The second condition rules out bubbles.

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[1]Veronesi (2000) provides an alternative proof, which exploits the right-continuity of the continuous-time Markov chain and obtains the same pricing formula with a limit argument.

Next, it is easy to verify that

$$\mathbf{V}(x) = e^x \mathbf{v},$$

with

$$\mathbf{v} = \left(\mathbf{r}^n - \tilde{\theta}_X - \tilde{\mathbf{\Lambda}} \right)^{-1} \mathbf{1}. \tag{32}$$

C.4   Proof of Proposition 2

Consider a corporate contingent claim $J(x_t, s_t)$, which pays dividend at rate $F(x_t, s_t)$ when the firm is solvent, a default payment $H(x_{\tau_D}, s_{\tau_D})$ when default occurs at time τ_D, and a restructuring payment $K(x_{\tau_U}, s_{\tau_U})$ when restructuring occurs at time τ_U. Let $\mathbf{F}(x)$, $\mathbf{H}(x)$, $\mathbf{K}(x)$, and $\mathbf{J}(x)$ be $n \times 1$ vectors with their ith elements equal to $F(x, i)$, $H(x, i)$, $K(x, i)$, and $J(x, i)$, respectively.

I also define an $n \times n$ diagonal matrix $\mathcal{A}$. Its ith diagonal element $\mathcal{A}^i$ is the infinitesimal generator for any C^2 function $\phi(x)$ in state i, where x is the log nominal cash flow specified in (26):

$$\mathcal{A}^i \phi(x) = \left(\tilde{\theta}_X(i) - \frac{1}{2}\sigma_X^2(i) \right) \frac{\partial}{\partial x} \phi(x) + \frac{1}{2}\sigma_X^2(i) \frac{\partial^2}{\partial x^2} \phi(x). \tag{33}$$

In the dynamic problem, we have the following default/restructuring boundaries, $\left(X_D^1, \cdots, X_D^n, X_U^{u(1)}, \cdots, X_U^{u(n)} \right)$. When cash flow X is in the region $\mathcal{D}_k = [X_D^k, X_D^{k+1})$ for $k = 1, \cdots, n-1$, the firm will already be in default in the states $s > k$. Thus, the security will only be "alive" in the first k states. Let the index set $\mathcal{I}_k = \{1, ..., k\}$ denote the states in which the firm would not have defaulted yet (given $X \in \mathcal{D}_k$), and its complement $\mathcal{I}_k^c = \{k+1, ..., n\}$ denote the states where the firm would have defaulted already. Similarly, when $X \in \mathcal{D}_{n+k} = \left(X_U^{u(k)}, X_U^{u(k+1)} \right]$ for $k = 1, \cdots, n-1$, I use index set $\mathcal{I}_{n+k} = \{u(k+1), \cdots, u(n)\}$ to denote states where the firm has not yet restructured, with its compliment $\mathcal{I}_{n+k}^c = \{u(1), \cdots, u(k)\}$ denoting the states where restructuring has occurred.

When $X \in \mathcal{D}_k$ $(k \le n-1)$, the claims that are not in default yet are $\mathbf{J}_{[\mathcal{I}_k]}$. The Feynman-Kac formula implies that $\mathbf{J}_{[\mathcal{I}_k]}$ satisfy the following system of ordinary differential equations:

$$\mathcal{A}_{[\mathcal{I}_k, \mathcal{I}_k]} \mathbf{J}_{[\mathcal{I}_k]} + \mathbf{F}_{[\mathcal{I}_k]} + \tilde{\mathbf{\Lambda}}_{[\mathcal{I}_k, \mathcal{I}_k]} \mathbf{J}_{[\mathcal{I}_k]} + \tilde{\mathbf{\Lambda}}_{[\mathcal{I}_k, \mathcal{I}_k^c]} \mathbf{H}_{[\mathcal{I}_k^c]} = \mathbf{r}_{[\mathcal{I}_k, \mathcal{I}_k]}^n \mathbf{J}_{[\mathcal{I}_k]}. \tag{34}$$

This equation states that, under the risk-neutral measure, the instantaneous expected return of a claim in any state should be equal to the riskfree rate in the corresponding state. A sudden change of the state can lead to abrupt changes in the value of the claim. It could also lead to immediate default, in which case the default payment is realized. These explain the last two terms on the LHS of the equation.

In regions $\mathcal{D}_n$, a sudden change of the state will not cause immediate default or restructuring. Thus, the ODE becomes:

$$\mathcal{A}\mathbf{J} + \mathbf{F} + \tilde{\mathbf{\Lambda}}\mathbf{J} = \mathbf{r}^n\mathbf{J}. \tag{35}$$

When $X \in \mathcal{D}_{n+k}$ $(k \le n-1)$, the firm has not restructured yet in those states in $\mathcal{I}_{n+k}$, where

$$\mathcal{A}_{[\mathcal{I}_{n+k}, \mathcal{I}_{n+k}]}\mathbf{J}_{[\mathcal{I}_{n+k}]} + \mathbf{F}_{[\mathcal{I}_{n+k}]} + \tilde{\mathbf{\Lambda}}_{[\mathcal{I}_{n+k}, \mathcal{I}_{n+k}]}\mathbf{J}_{[\mathcal{I}_{n+k}]} + \tilde{\mathbf{\Lambda}}_{[\mathcal{I}_{n+k}, \mathcal{I}_{n+k}^c]}\mathbf{K}_{[\mathcal{I}_{n+k}^c]} = \mathbf{r}^n_{[\mathcal{I}_{n+k}, \mathcal{I}_{n+k}]}\mathbf{J}_{[\mathcal{I}_{n+k}]}. \tag{36}$$

Notice that the restructuring payments K appear in the equation, which specify the value of the claim when a change of state triggers restructuring.

The homogeneous equation from (34 and 35) can be written as:

$$\mathcal{A}_{[\mathcal{I}_k, \mathcal{I}_k]}\mathbf{J}_{[\mathcal{I}_k]} + \left(\tilde{\mathbf{\Lambda}}_{[\mathcal{I}_k, \mathcal{I}_k]} - \mathbf{r}^n_{[\mathcal{I}_k, \mathcal{I}_k]}\right)\mathbf{J}_{[\mathcal{I}_k]} = \mathbf{0}, \tag{37}$$

which is a quadratic eigenvalue problem. Jobert and Rogers (2006) show its solution takes the following form:

$$\mathbf{J}(x)_{[\mathcal{I}_k]} = \sum_{j=1}^{2k} w_{k,j}\mathbf{g}_{k,j}\exp(\beta_{k,j}x), \tag{38}$$

Plugging this solution into the ODE gives

$$\left(\tilde{\theta}_X - \frac{1}{2}\mathbf{\Sigma}_{\mathbf{X}}\right)\beta g + \frac{1}{2}\sigma_X^2(i)\beta^2 g + \left(\tilde{\mathbf{\Lambda}} - \mathbf{r}^n\right)g = 0. \tag{39}$$

Define $h = \beta g$, then

$$\left(\tilde{\theta}_X - \frac{1}{2}\mathbf{\Sigma}_{\mathbf{X}}\right)\beta g + \frac{1}{2}\mathbf{\Sigma}_{\mathbf{X}}\beta h + \left(\tilde{\mathbf{\Lambda}} - \mathbf{r}^n\right)g = 0, \tag{40}$$

or

$$-2\mathbf{\Sigma}_{\mathbf{X}}^{-1}\left(\tilde{\theta}_X - \frac{1}{2}\mathbf{\Sigma}_{\mathbf{X}}\right)\beta g - 2\mathbf{\Sigma}_{\mathbf{X}}^{-1}\left(\tilde{\mathbf{\Lambda}} - \mathbf{r}^n\right)g = \beta h. \tag{41}$$

Thus, $\mathbf{g}_{k,j}$ and $\beta_{k,j}$ are solutions to the following standard eigenvalue problem:

$$\left[\begin{array}{cc} 0 & \mathbf{I} \\ -\left(2\mathbf{\Sigma}_X^{-1}\left(\tilde{\mathbf{\Lambda}} - \mathbf{r}^n\right)\right)_{[\mathcal{I}_k, \mathcal{I}_k]} & -\left(2\mathbf{\Sigma}_X^{-1}\tilde{\theta}_X - \mathbf{I}\right)_{[\mathcal{I}_k, \mathcal{I}_k]} \end{array}\right]\left[\begin{array}{c} \mathbf{g}_k \\ \mathbf{h}_k \end{array}\right] = \beta_k\left[\begin{array}{c} \mathbf{g}_k \\ \mathbf{h}_k \end{array}\right], \tag{42}$$

where $\mathbf{I}$ is an $n \times n$ identity matrix, $\mathbf{r}^n$, $\tilde{\theta}_X$ and $\mathbf{\Sigma}_X$ are defined in (29). The coefficients $w_{k,j}$ will be different for different securities. Barlow, Rogers, and Williams (1980) show that there are exactly n eigenvalues with negative real parts, and n with positive real parts.

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Similarly, the homogeneous equation from (36) can be written as:

$$\mathcal{A}_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}]}\mathbf{J}_{[\mathcal{I}_{n+k}]} + \left(\tilde{\mathbf{\Lambda}}_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}]} - \mathbf{r}^n_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}]}\right)\mathbf{J}_{[\mathcal{I}_{n+k}]} = \mathbf{0}. \tag{43}$$

Its solution is

$$\mathbf{J}(x)_{[\mathcal{I}_{n+k}]} = \sum_{j=1}^{2(n-k+1)} w_{n+k,j}\mathbf{g}_{n+k,j}\exp\left(\beta_{n+k,j}x\right), \tag{44}$$

where $\mathbf{g}_{n+k,j}$ and $\beta_{n+k,j}$ are solutions to the following standard eigenvalue problem:

$$\left[\begin{matrix} 0 & \mathbf{I} \\ -\left(2\mathbf{\Sigma}_X^{-1}\left(\tilde{\mathbf{\Lambda}}-\mathbf{r}^n\right)\right)_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}]} & -\left(2\mathbf{\Sigma}_X^{-1}\tilde{\theta}_X - \mathbf{I}\right)_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}]} \end{matrix}\right]\left[\begin{matrix}\mathbf{g}_{n+k}\\\mathbf{h}_{n+k}\end{matrix}\right] = \beta_{n+k}\left[\begin{matrix}\mathbf{g}_{n+k}\\\mathbf{h}_{n+k}\end{matrix}\right], \tag{45}$$

$\mathbf{I}$ is an $n \times n$ identity matrix, $\mathbf{r}^n$, $\tilde{\theta}_X$ and $\mathbf{\Sigma}_X$ are defined in (29). Again, the coefficients $w_{n+k,j}$ will be different for different securities.

The remaining tasks are to find the particular solutions for the inhomogeneous equations, and solve for the coefficients $w_{k,j}$ through the boundary conditions, which depend on the specific type of security under consideration.

C.4.1 Debt

Suppose the coupon that the firm chooses in state s when $X = 1$ is $C(s)$. Let $D(x, s; s_0)$ be the value of debt after debt is issued, where s_0 shows the dependence of debt value on the state at time 0 through the coupon $C(s_0)$. The dividend rate, default payment, and restructuring payment are specified as:

$$F(X, s; s_0) = (1 - \tau_i)C(s_0) \tag{46}$$

$$H(X, s; s_0) = \alpha(s)V(X, s) \tag{47}$$

and

$$K(X, s; s_0) = D(X_0, s_0; s_0). \tag{48}$$

When $X \in \mathcal{D}_k$ $(k = 1, \cdots, n-1)$, for any state $i \in \mathcal{I}_k$, it follows from the ODE (34) that:

$$\begin{aligned} r^n(i)D(x, i; s_0) &= \mathcal{A}^i D(x, i; s_0) + \tilde{\lambda}_{i,1}D(x, 1; s_0) + \cdots + \tilde{\lambda}_{i,k}D(x, k; s_0) \\ &\quad + \tilde{\lambda}_{i,k+1}H(x, k+1; s_0) + \cdots + \tilde{\lambda}_{i,n}H(x, n; s_0) + (1 - \tau_i)C(s_0). \end{aligned} \tag{49}$$

Define vector $\mathbf{D}(X; s_0) = [D(X, 1; s_0), \cdots, D(X, n; s_0)]'$. We obtain the solution to the homogeneous equations from (38),

$$\mathbf{D}(x; s_0)_{[\mathcal{I}_k]} = \sum_{j=1}^{2k} w_{k,j}^D(s_0) \mathbf{g}_{k,j} \exp(\beta_{k,j} x),$$

(50)

where $\mathbf{g}_{k,j}$ and $\beta_{k,j}$ are characterized in the eigenvalue problem (42).

The inhomogeneous equation has the additional term that is linear in e^x:

$$\tilde{\lambda}_{i,k+1} D(x, k+1; s_0) + \cdots + \tilde{\lambda}_{i,n} D(x, n; s_0) + (1 - \tau_i) C(s_0) = \sum_{j=k+1}^{n} \tilde{\lambda}_{ij} \alpha(j) v(j) e^x + (1 - \tau_i) C(s_0).$$

(51)

It is easy to verify that a particular solution is:

$$\mathbf{D}(x; s_0)_{[\mathcal{I}_k]} = \xi_k^D(\mathcal{I}_k; s_0) e^x + \zeta_k^D(\mathcal{I}_k; s_0),$$

(52)

where

$$\xi_k^D(\mathcal{I}_k; s_0) = \left(\mathbf{r}^n - \tilde{\mathbf{\Lambda}} - \tilde{\theta}_X \right)_{[\mathcal{I}_k, \mathcal{I}_k]}^{-1} \left(\tilde{\mathbf{\Lambda}}_{[\mathcal{I}_k, \mathcal{I}_k^c]} (\alpha \odot \mathbf{v})_{[\mathcal{I}_k^c]} \right)$$

$$\zeta_k^D(\mathcal{I}_k; s_0) = (1 - \tau_i) C(s_0) \left(\mathbf{r}^n - \tilde{\mathbf{\Lambda}} \right)_{[\mathcal{I}_k, \mathcal{I}_k]}^{-1} \mathbf{1}_k$$

(53)

The symbol $\odot$ denotes element-by-element multiplication; α is an $n \times 1$ vector of recovery rates; $\mathbf{v}$ is given in (32); $\xi_k^D(\mathcal{I}_k^c; s_0)$ and $\zeta_k^D(\mathcal{I}_k^c; s_0)$ are equal to zero.

In the regions $\mathcal{D}_n$, the solution to the homogeneous equation is:

$$\mathbf{D}(x; s_0) = \sum_{j=1}^{2n} w_{n,j}^D(s_0) \mathbf{g}_{n,j} \exp(\beta_{n,j} x),$$

(54)

and a particular solution in this region is:

$$\mathbf{D}(x; s_0) = \zeta_n^D = (1 - \tau_i) C(s_0) \mathbf{b}.$$

(55)

When $X \in \mathcal{D}_{n+k}$ $(k = 1, \cdots, n-1)$, the solution to the homogeneous equation is

$$\mathbf{D}\left(x; s_0\right)_{[\mathcal{I}_{n+k}]} = \sum_{j=1}^{2(n-k)} w_{n+k,j}^D(s_0)\mathbf{g}_{n+k,j} \exp\left(\beta_{n+k,j}x\right). \tag{56}$$

The ODE is

$$r^n(i)D\left(x, i; s_0\right) = \mathcal{A}^i D\left(x, i; s_0\right) + \tilde{\lambda}_{i,u(1)}K\left(x, u\left(1\right); s_0\right) + \cdots + \tilde{\lambda}_{i,u(k)}K\left(x, u(k); s_0\right)$$
$$+ \tilde{\lambda}_{i,u(k+1)}D\left(x, u\left(k+1\right); s_0\right) + \cdots + \tilde{\lambda}_{i,u(n)}D\left(x, u\left(n\right); s_0\right) + \left(1 - \tau_i\right)C(s_0). \tag{57}$$

Since $K(x, s)$ only depends on the initial value of debt and not x, we can guess that a particular solution is:

$$\mathbf{D}\left(x; s_0\right)_{[\mathcal{I}_{n+k}]} = \zeta_{n+k}^D\left(\mathcal{I}_{n+k}; s_0\right). \tag{58}$$

Plug the particular solution into the ODE gives

$$r^n(i)\zeta_{n+k}^D\left(i; s_0\right) = \tilde{\lambda}_{i,u(1)}D\left(X_0, s_0; s_0\right) + \cdots + \tilde{\lambda}_{i,u(k)}D\left(X_0, s_0; s_0\right)$$
$$+ \tilde{\lambda}_{i,u(k)}\zeta_{n+k}^D\left(u\left(k\right); s_0\right) + \cdots + \tilde{\lambda}_{i,u(n)}\zeta_{n+k}^D\left(u\left(n\right); s_0\right) + \left(1 - \tau_i\right)C(s_0). \tag{59}$$

The solution can be written as

$$\zeta_{n+k}^D(\mathcal{I}_{n+k}; s_0) = \left(\mathbf{r}^n - \tilde{\mathbf{\Lambda}}\right)_{[\mathcal{I}_{n+k}, \mathcal{I}_{n+k}]}^{-1}\left[(1 - \tau_i)C(s_0)\mathbf{1}_{n-k} + \tilde{\mathbf{\Lambda}}_{[\mathcal{I}_{n+k}, \mathcal{I}_{n+k}^c]}\mathbf{1}_k D(X_0, s_0; s_0)\right], \tag{60}$$

where

$$D\left(X_0, s_0; s_0\right) = \sum_{j=1}^{2n} w_{n,j}^D \mathbf{g}_{n,j}(s_0)X_0^{\beta_{n,j}} + (1 - \tau_i)C(s_0)b(s_0). \tag{61}$$

The fix-point problem for D_0 can be solved by plugging (61) into (60). Since equation (61) is linear in the coefficients $w_{n,j}^D$, so will ζ_{n+k}^D.

Next, I specify the boundary conditions that determine the coefficients $w_{k,j}^D$. First, there are n conditions specifying the value of debt at the n default boundaries:

$$D\left(X_D^i, i; s_0\right) = \alpha(i)V\left(X_D^i, i\right), \quad i = 1, \cdots, n. \tag{62}$$

Another n conditions specify the value of debt at the restructuring boundaries:

$$D\left(X_U^{u(i)}, u(i); s_0\right) = D\left(X_0, s_0\right), \quad i = 1, \cdots, n. \tag{63}$$

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Because the payoff function F and terminal payoffs H, K are bounded and piecewise-continuous in X, while the discount rate r is constant in each state, $D(X, s)$ must be piecewise $\mathcal{C}^2$ with respect to X (see Karatzas and Shreve (1991)), which implies that D is continuous and smooth at all the boundaries for which neither default or restructure has occurred. Thus, for $k = 1, \cdots, n-1$,

$$
\begin{aligned}
\lim_{X \uparrow X_D^{k+1}} D(X, i; s_0) &= \lim_{X \downarrow X_D^{k+1}} D(X, i; s_0) \\
\lim_{X \uparrow X_D^{k+1}} D_X(X, i; s_0) &= \lim_{X \downarrow X_D^{k+1}} D_X(X, i; s_0) \quad i \in \mathcal{I}_k
\end{aligned}
\tag{64}
$$

and

$$
\begin{aligned}
\lim_{X \uparrow X_U^{u(k)}} D(X, u(i); s_0) &= \lim_{X \downarrow X_U^{u(k)}} D(X, u(i); s_0) \\
\lim_{X \uparrow X_U^{u(k)}} D_X(X, u(i); s_0) &= \lim_{X \downarrow X_U^{u(k)}} D_X(X, u(i); s_0) \quad i \in \mathcal{I}_{n+k}
\end{aligned}
\tag{65}
$$

There are $2n^2$ unknown coefficients in $\{w^D(s_0)\}$ and $2n^2$ boundary conditions for each s_0. Importantly, the boundary conditions in $(62-65)$ are all linear in the unknowns, which gives us a system of $2n^2$ linear equations that can be easily solved in closed form.

C.4.2 Equity

Let $E(x, s; s_0)$ be the value of equity after debt is issued, where s_0 again represents the dependence of equity value on coupon chosen at time 0. The dividend rate, default payment, and restructuring payment for equity are

$$
F(X, s; s_0) = (1 - \tau_d)(1 - \tau_c)(X - C(s_0))
\tag{66}
$$

$$
H(X, s; s_0) = 0
\tag{67}
$$

and

$$
K(X, s; s_0) = (1 - q) D(X, s; s) - D(X_0, s_0; s_0) + E(X, s; s).
\tag{68}
$$

For simplicity, I have left out the considerations for partial loss offset and equity issuance costs when determining the dividend rate in (66) for the case when $X < C(s_0)$, i.e., earnings net of interest are negative. Chen (2007) investigates these effects by assuming that when $X < C(s_0)$,

$$
F(X, s; s_0) = \frac{1 - \tau_{c^-}}{1 - e}(X - C(s_0)),
\tag{69}
$$

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where $\tau_{c^-} < \tau_c$ captures the effect of partially lost tax shield due to net operating losses, and e is the proportional cost of equity issuance. Partial loss offset reduces the tax benefits of debt, leading the firm to issue less debt.

When restructuring occurs in state s with cash flow X, applying the scaling property to the restructuring payment, we get:

$$K\left(X, s; s_0\right) = \left((1-q)D\left(X_0, s; s\right) + E\left(X_0, s; s\right)\right)\frac{X}{X_0} - D(X_0, s_0; s_0) \tag{70}$$

Since payoffs are linear in X in all regions, the solutions for $E(X, s; s_0)$ take a similar form as debt $D(X, s; s_0)$:

$$\mathbf{E}(X; s_0)_{[\mathcal{I}_k]} = \sum_j w_{k,j}^E(s_0)\mathbf{g}_{k,j}X^{\beta_{k,j}} + \xi_k^E(\mathcal{I}_k; s_0)X + \zeta_k^E(\mathcal{I}_k; s_0). \tag{71}$$

The first term is the solution to the homogeneous equation, which is identical to debt (except for the coefficients $w_{k,j}^E$). Next, I focus on the particular solutions in each region $\mathcal{D}_k$.

When $X \in \mathcal{D}_k$ $(k = 1, \cdots, n)$, for $i \in \mathcal{I}_k$, the firm is not in default yet. The ODE is:

$$r^n(i)E\left(x, i; s_0\right) = \mathcal{A}^i E\left(x, i; s_0\right) + \tilde{\lambda}_{i,1}E\left(x, 1; s_0\right) + \cdots + \tilde{\lambda}_{i,k}E\left(x, k; s_0\right) + (1-\tau_d)(1-\tau_c)\left(e^x - C(s_0)\right) \tag{72}$$

A particular solution is:

$$\mathbf{E}\left(x; s_0\right)_{[\mathcal{I}_k]} = \xi_k^E(\mathcal{I}_k; s_0)e^x + \zeta_k^E(\mathcal{I}_k; s_0), \tag{73}$$

where

$$\begin{aligned} \xi_k^E(\mathcal{I}_k; s_0) &= (1-\tau_c)(1-\tau_d)\left(\mathbf{r}^n - \tilde{\theta}_X - \tilde{\mathbf{\Lambda}}\right)^{-1}_{[\mathcal{I}_k, \mathcal{I}_k]}\mathbf{1}_k \\ \zeta_k^E(\mathcal{I}_k; s_0) &= -(1-\tau_c)(1-\tau_d)C(s_0)\left(\mathbf{r}^n - \tilde{\mathbf{\Lambda}}\right)^{-1}_{[\mathcal{I}_k, \mathcal{I}_k]}\mathbf{1}_k \end{aligned} \tag{74}$$

When $X \in \mathcal{D}_{n+k}$ $(k = 1, \cdots, n-1)$, for $i \in \mathcal{I}_{n+k}$, the firm has not restructured yet. The ODE is:

$$\begin{aligned} r^n(i)E\left(x, i; s0\right) = {} &\mathcal{A}^i E\left(x, i; s0\right) + \tilde{\lambda}_{i,u(1)}K\left(x, u\left(1\right); s0\right) + \cdots + \tilde{\lambda}_{i,u(k-1)}K\left(x, u\left(k-1\right); s0\right) \\ &+ \tilde{\lambda}_{i,u(k)}E\left(x, u\left(k\right); s0\right) + \cdots + \tilde{\lambda}_{i,u(n)}E\left(x, u\left(n\right); s0\right) + (1-\tau_d)(1-\tau_c)\left(e^x - C(s0)\right) \end{aligned} \tag{75}$$

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The particular solution is:

$$\mathbf{E}\left(x;s_0\right)_{[\mathcal{I}_{n+k}]} = \xi_{n+k}^E\left(\mathcal{I}_{n+k};s_0\right)e^x + \zeta_{n+k}^E\left(\mathcal{I}_{n+k};s_0\right). \tag{76}$$

Plug the particular solution and the expression of K in (70) into the ODE,

$$r^n(i)\left(\xi_{n+k}^E\left(i;s_0\right)e^x + \zeta_{n+k}^E\left(i;s_0\right)\right)$$

$$= \xi_{n+k}^E\left(i;s_0\right)\tilde{\theta}\left(i\right)e^x + \sum_{j=1}^{k-1}\tilde{\lambda}_{i,u(j)}\left(\frac{(1-q)D\left(X_0,s;s\right)+E\left(X_0,s;s\right)}{X_0}e^x - D(X_0,s_0;s_0)\right)$$

$$+ \sum_{j=k}^{n}\tilde{\lambda}_{i,u(j)}\left(\xi_{n+k}^E\left(u\left(j\right);s_0\right)e^x + \zeta_{n+k}^E\left(u\left(j\right);s_0\right)\right) + (1-\tau_d)(1-\tau_c)\left(e^x - C(s_0)\right) \tag{77}$$

which gives

$$r^n(i)\xi_{n+k}^E\left(i;s_0\right) = \tilde{\theta}\left(i\right)\xi_{n+k}^E\left(i;s_0\right) + \sum_{j=1}^{k-1}\tilde{\lambda}_{i,u(j)}\frac{(1-q)D\left(X_0,s;s\right)+E\left(X_0,s;s\right)}{X_0}$$

$$+ \sum_{j=k}^{n}\tilde{\lambda}_{i,u(j)}\xi_{n+k}^E\left(u\left(j\right);s_0\right) + (1-\tau_d)(1-\tau_c)$$

$$r^n(i)\zeta_{n+k}^E\left(i;s_0\right) = -\sum_{j=1}^{k-1}\tilde{\lambda}_{i,u(j)}D\left(X_0,s_0;s_0\right) + \sum_{j=k}^{n}\tilde{\lambda}_{i,u(j)}\zeta_{n+k}^E\left(u\left(j\right);s_0\right) - (1-\tau_d)(1-\tau_c)C(s_0)$$

$$\tag{78}$$

so that

$$\xi_{n+k}^E\left(\mathcal{I}_{n+k};s_0\right) = \left(\mathbf{r}^n - \tilde{\theta}_X - \tilde{\mathbf{\Lambda}}\right)_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}]}^{-1} \tag{79}$$

$$\times \left[(1-\tau_d)(1-\tau_c)\mathbf{1}_{n-k} + \tilde{\mathbf{\Lambda}}_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}^c]}\frac{((1-q)\mathbf{D}\left(X_0\right)+\mathbf{E}\left(X_0\right))_{[\mathcal{I}_{n+k}^c]}}{X_0}\right]$$

$$\zeta_{n+k}^E\left(\mathcal{I}_{n+k};s_0\right) = \left(\mathbf{r}^n - \tilde{\mathbf{\Lambda}}\right)_{[\mathcal{I}_k,\mathcal{I}_k]}^{-1}\left[-\tilde{\mathbf{\Lambda}}_{[\mathcal{I}_{n+k},\mathcal{I}_{n+k}^c]}\mathbf{1}_k D(X_0,s_0;s_0) - (1-\tau_d)(1-\tau_c)C(s_0)\mathbf{1}_{n-k}\right]$$

$$\tag{80}$$

where the value of debt is computed earlier. The initial value of equity $E(X_0,s;s)$ is

$$E\left(X_0,s;s\right) = \sum_{j=1}^{2n}w_{n,j}^E(s)\mathbf{g}_{n,j}(s)X_0^{\beta_{n,j}} + \xi_n^E(s;s)X_0 + \zeta_n^E(s;s). \tag{81}$$

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This can be plugged back into (80). Notice that (81) introduces dependence of $\mathbf{E}(X; s_0)$ on all the $\{w^E(s)\}$, not just $\{w^E(s_0)\}$. It is possible to jointly solve for all $\{w^E(s)\}$. Alternatively, we can solve the fix-point problem through iteration (starting with some guess of $\mathbf{E}(X_0)$), which could be more convenient when n is large.

The boundary conditions for E are similar to those for debt. First, there are the conditions specifying the value of equity at the default and restructuring boundaries. For $i = 1, \cdots, n$,

$$E\left(X_D^i, i; s_0\right) = 0, \tag{82}$$

and

$$E\left(X_U^{u(i)}, u(i); s_0\right) = \frac{X_U^{u(i)}}{X_0}((1-q)D\left(X_0, u(i); u(i)\right) + E\left(X_0, u(i); u(i)\right)) - D(X_0, s_0; s_0). \tag{83}$$

Moreover, we need to ensure that E is C^0 and C^1 at each default/restructuring boundary, which leads to an identical set of value-matching and smooth-pasting conditions as in the case of debt. For $k = 1, \cdots, n-1$,

$$
\begin{aligned}
\lim_{X \uparrow X_D^{k+1}} E(X, i; s_0) &= \lim_{X \downarrow X_D^{k+1}} E(X, i; s_0) \\
\lim_{X \uparrow X_D^{k+1}} E_X(X, i; s_0) &= \lim_{X \downarrow X_D^{k+1}} E_X(X, i; s_0) \quad i \in \mathcal{I}_k
\end{aligned}
\tag{84}
$$

and

$$
\begin{aligned}
\lim_{X \uparrow X_U^{u(k)}} E\left(X, u(i); s_0\right) &= \lim_{X \downarrow X_U^{u(k)}} E\left(X, u(i); s_0\right) \\
\lim_{X \uparrow X_U^{u(k)}} E_X\left(X, u(i); s_0\right) &= \lim_{X \downarrow X_U^{u(k)}} E_X\left(X, u(i); s_0\right) \quad i \in \mathcal{I}_{n+k}
\end{aligned}
\tag{85}
$$

These boundary conditions form a system of linear equations $\left\{w_{k,j}^E\right\}$, which can be solved in closed form.

C.5   Returns on Equity and Debt

To compare the pricing implications of the Markov chain model with that of Bansal and Yaron (2004), I consider a real dividend stream, which is a levered up version of aggregate consumption

(as in BY).

$$\frac{dD_t}{D_t} = \theta_D\left(s_t\right) dt + \sigma_{D,m}\left(s_t\right) dW_t^m, \tag{86}$$

with

$$\theta_D\left(s\right) = \overline{\theta}_m + \phi\left(\theta_m\left(s_t\right) - \overline{\theta}_m\right), \tag{87}$$

$$\sigma_{D,m}\left(s\right) = \varphi_d \sigma_m\left(s_t\right). \tag{88}$$

Thus, the dividend stream has the same expected growth rate as aggregate consumption. Denote the (real) value of the stock as S, which will be a function of current dividend and state, $S\left(D_t, s_t\right)$. Its value can be derived using the same method that determines the present value of unlevered cash flows, but we have to use the real stochastic discount factor m_t instead of n_t. Ignoring taxes,

$$S\left(D, s\right) = D v^D\left(s\right), \tag{89}$$

where $v^D\left(s\right)$ is the pricie-dividend ratio in state s, which is given in a vector,

$$\mathbf{v}^D = \left(\mathbf{r} - \tilde{\theta}_D - \tilde{\mathbf{\Lambda}}\right)^{-1} \mathbf{1}, \tag{90}$$

where $\tilde{\mathbf{\Lambda}}$ is again the generator matrix under risk-neutral measure $\mathcal{Q}$, $\mathbf{r} = \mathbf{diag}\left(\left[r\left(1\right), \cdots, r\left(n\right)\right]\right)$, $\tilde{\theta}_D = \mathbf{diag}\left(\left[\tilde{\theta}_D\left(1\right), \cdots, \tilde{\theta}_D\left(n\right)\right]\right)$, with the risk-neutral growth rates defined as:

$$\tilde{\theta}_D\left(s\right) = \theta_D\left(s\right) - \sigma_{D,m}\left(s\right) \eta\left(s\right). \tag{91}$$

Then, in state i,

$$\frac{dS}{S} = o\left(dt\right) + \sigma_{D,m}\left(i\right) dW_t^m + \sum_{j \neq i}\left(\frac{v^D\left(j\right)}{v^D\left(i\right)} - 1\right) dN^{(i,j)}. \tag{92}$$

The risk premium for S, μ_S, is determined by its covariance with the discount factor. Thus, in state i,

$$\begin{aligned}
\mu_S\left(i\right) &= -\frac{1}{dt} cov_t\left(\frac{dS}{S}, \frac{dm}{m}\right) \\
&= \sigma_{D,m}\left(i\right) \eta\left(i\right) - \sum_{j \neq i} \lambda_{ij}\left(\frac{v^D\left(j\right)}{v^D\left(i\right)} - 1\right)\left(e^{\kappa(i,j)} - 1\right).
\end{aligned} \tag{93}$$

The first term for the risk premium comes from the risk exposure to small shocks. If the small shocks tend to move the stock price and stochastic discount factor in opposite directions (e.g, price

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drops as marginal utility rises), then the stock is risky and demand a positive premium. The same intuition applies to the second term of the risk premium, which comes from the exposure to large shocks.

The total volatility of return consists of two parts, volatility due to Brownian motion and jumps:

$$\sigma_R(i) = \sqrt{\sigma_{D,m}^2(i) + \sum_{j \neq i} \lambda_{ij} \left(\frac{v^D(j)}{v^D(i)} - 1 \right)^2}. \tag{94}$$

Next, I calculate the risk premium for the equity and debt of levered firms. To simplify the notation, I drop the reference to the initial state s_0 when there is no confusion.

The value of equity is given by

$$E(X, s) = \sum_j w_{k,j}^E g_{k,j}(s) X^{\beta_{k,j}} + \xi_k^E(s) X + \zeta_k^E(s). \tag{95}$$

Applying Ito's lemma, we get

$$\frac{dE_t}{E_t} = o \cdot dt + \frac{E_X(X, s)}{E(X, s)} dX + \sum_{s' \neq s} \left(\frac{E(X, s')}{E(X, s)} - 1 \right) dN_t^{(s,s')}. \tag{96}$$

It follows that the risk premium for equity is

$$
\begin{aligned}
\mu_E(X, s) &= \frac{E_X(X, s)}{E(X, s)} \left(\sigma_{X,m}(s) \eta^m(s) + \sigma_{P,2} \eta^P \right) - \sum_{s' \neq s} \lambda_{ss'} \left(\frac{E(X, s')}{E(X, s)} - 1 \right) \left(e^{\kappa(s,s')} - 1 \right) \\
&= \frac{\sum_j w_{k,j}^E g_{k,j}(s) \beta_{k,j} X^{\beta_{k,j}} + \xi_k^E(s) X}{\sum_j w_{k,j}^E g_{k,j}(s) X^{\beta_{k,j}} + \xi_k^E(s) X + \zeta_k^E(s)} \left(\sigma_{X,m}(s) \eta^m(s) + \sigma_{P,2} \eta^P \right) \\
&\quad - \sum_{s' \neq s} \lambda_{ss'} \left(\frac{E(X, s')}{E(X, s)} - 1 \right) \left(e^{\kappa(s,s')} - 1 \right),
\end{aligned} \tag{97}
$$

where

$$\eta^m(s) = \gamma \sigma_m(s) + \sigma_{P,1} \tag{98}$$

$$\eta^P = \sigma_{P,2} \tag{99}$$

$$\sigma_E^2(X, s) = \left(\frac{E_X(X, s)}{E(X, s)} \right)^2 X^2 \sigma_X^2 + \sum_{s' \neq s} \lambda_{ss'} \left(\frac{E(X, s')}{E(X, s)} - 1 \right)^2. \tag{100}$$

Similarly, the conditional risk premium for the corporate bond is

$$
\begin{aligned}
\mu_D \left(X, s \right) &= \frac{\sum_j w_{k,j}^D g_{k,j} \left(s \right) \beta_{k,j} X^{\beta_{k,j}} + \xi_k^D \left(s \right) X}{\sum_j w_{k,j}^D g_{k,j} \left(s \right) X^{\beta_{k,j}} + \xi_k^D \left(s \right) X + \zeta_k^D \left(s \right)} \left(\sigma_{X,m} \left(s \right) \eta^m \left(s \right) + \sigma_{P,2} \eta^P \right) \\
&\quad - \sum_{s' \neq s} \lambda_{ss'} \left(\frac{D \left(X, s' \right)}{D \left(X, s \right)} - 1 \right) \left(e^{\kappa(s,s')} - 1 \right).
\end{aligned}
\tag{101}
$$

D. Calibrating the Continuous-time Markov Chain

The Markov chain for the expected growth rate and volatility of aggregate consumption is calibrated using a two-step procedure. Start with the discrete-time system of consumption and dividend dynamics of Bansal and Yaron (2004) (BY):

$$
\begin{aligned}
g_{t+1} &= \mu_c + x_t + \sqrt{v_t} \eta_{t+1} && \text{(102a)} \\
g_{d,t+1} &= \mu_d + \phi x_t + \sigma_d \sqrt{v_t} u_{t+1} && \text{(102b)} \\
x_{t+1} &= \kappa_x x_t + \sigma_x \sqrt{v_t} e_{t+1} && \text{(102c)} \\
v_{t+1} &= \bar{v} + \kappa_v \left(v_t - \bar{v} \right) + \sigma_v w_{t+1} && \text{(102d)}
\end{aligned}
$$

where g is log consumption growth, g_d is log dividend growth, and $\eta, u, e, w \sim i.i.d. N(0,1)$. I use the parameters from BY, which are at the monthly frequency and are calibrated to the annual consumption data from 1929 to 1998.

The restriction that shocks to consumption, η_{t+1}, and shocks to the conditional moments, e_{t+1}, w_{t+1}, are mutually independent makes it convenient to approximate the dynamics of (x, v) with a Markov chain. I first obtain a discrete-time Markov chain over a chosen horizon Δ, e.g., quarterly, using the quadrature method of Tauchen and Hussey (1991). For numerical reasons, I choose a relatively small number of states ($n = 9$) for the Markov chain, with three different values for v, and three values for x for each v. Next, I convert the grid for (x, v) into a grid for (θ_m, σ_m) as in equation (1). The calibrated values of (θ_m, σ_m) are reported in Table 1. Finally, I transform the discrete-time transition matrix $\mathbf{P} = [p_{ij}]$ into the generator $\mathbf{\Lambda} = [\lambda_{ij}]$ of a continuous-time Markov chain using the method of Jarrow, Lando, and Turnbull (1997) (an approximation based on the assumption that the probability of more than one change of state is close to zero within the period Δ).

Table 2 Panel A reports the parameters for the discrete time consumption model of BY. Panel B compares the statistical properties of consumption growth rates in the data with those of the simulated data from the BY model and the Markov chain model. With just 9 states, the Markov

Table 1: THE CALIBRATED MARKOV CHAIN

This table reports the pairs of values for the conditional volatility $\sigma_m(s_t)$ and expected growth rate of aggregate consumption $\theta_m(s_t)$ for the states $s_t = 1, \cdots, 9$ of the calibrated Markov chain.

	$(\sigma_m(s_t), \theta_m(s_t))$	
1. (0.021, 0.041)	2. (0.027, 0.048)	3. (0.032, 0.053)
4. (0.021, 0.018)	5. (0.027, 0.018)	6. (0.032, 0.019)
7. (0.021, −0.004)	8. (0.027, −0.011)	9. (0.032, −0.016)

chain approximation does a good in matching the mean, volatility, autocorrelation and variance ratio of consumption growth in the BY model. The noticeable differences are that the Markov chain appears to generate a distribution of volatility and variance ratios with lighter right tail, which is likely due to the non-extreme grid points. Figure 5 in the published paper provides more information about the stationary distribution of the Markov chain. Under my calibration, the economy spends about 54% of the time in the "center" state with median expected growth rate and volatility.

E. Investigating State-dependent Default Losses

In this section, I investigate the cyclical variations in default losses. Figure 3 provides further evidence that corporate bond recovery rates covary with macroeconomic variables: GDP, industrial production, consumption, and price-earnings ratio. I evaluate these relations formally with regressions. Altman, Brady, Resti, and Sironi (2005) find that default rates explain a large fraction of the variations in recovery rates, while macro variables appear to have little explanatory power. However, default rates are themselves strongly affected by macroeconomic conditions: Table 3 shows that growth rates of industrial production, GDP, price-earnings ratio, and consumption all have significant explanatory power. The signs of the coefficients are as expected: lower growth rates in industrial production, GDP, price-earnings ratio and consumption are associated with higher default rates. Moreover, squared consumption growth also enters into the regressions significantly. It captures the nonlinear relationship between default rates and consumption growth: default rates rise more rapidly when consumption growth becomes negative.

In Table 4, the univariate regression of recovery rates on default rates confirms the finding of Altman et al. (2005). A regression with only macro variables (PE, g and g^2) can explain 42% of

Table 2: Markov Chain Approximation of the BY Model

The table compares the moments of consumption from the data, the model of Bansal and Yaron (2004), and the Markov chain model in this paper. Parameters in Panel A are from the discrete time model of BY (Table IV). In Panel B, the statistics of the data are from BY (2004) (Table I), based on annual observations from 1929 to 1998. The statistics for the two models are based on 5,000 simulations, each with 70 years of data. The simulations are done at high frequency and then aggregated to get annual growth rates. The symbols $\mu(g)$ and $\sigma(g)$ are mean and standard deviation of growth rates; $AC(j)$ is the jth autocorrelation; $VR(j)$ is the jth variance ratio.

Panel A: Paramters for the BY Model

μ_c	μ_d	ϕ	σ_d	κ_x	σ_x	κ_v	$\bar{v}$	σ_v
0.0015	0.0015	3	4.5	0.979	0.044	0.987	6.08×10^{-5}	0.23×10^{-5}

Panel B: Properties of Annualized Time-Averaged Growth Rates

	Data			BY			Markov Chain	
Variable	Estimate	SE	5%	50%	95%	5%	50%	95%
$\mu(g)$	1.80	-	0.59	1.79	2.99	0.74	1.81	2.90
$\sigma(g)$	2.93	(0.69)	2.26	2.79	3.44	2.19	2.64	3.12
AC(1)	0.49	(0.14)	0.25	0.46	0.63	0.23	0.42	0.58
AC(2)	0.15	(0.22)	-0.04	0.22	0.45	-0.05	0.18	0.39
AC(5)	-0.08	(0.10)	-0.17	0.06	0.30	-0.17	0.05	0.28
AC(10)	0.05	(0.09)	-0.24	-0.03	0.20	-0.24	-0.02	0.19
VR(2)	1.61	(0.34)	1.25	1.46	1.63	1.23	1.42	1.58
VR(5)	2.01	(1.23)	1.36	2.13	2.91	1.33	2.01	2.69
VR(10)	1.57	(2.07)	1.20	2.47	4.21	1.15	2.33	3.85

the variation in recovery rates. This number increases to 50% when the riskfree rate is included. In a two-stage regression (last column of the table), the residuals from the regression of default rates on the other macro variables still have significant explanatory power for recovery rates, suggesting that default rates do contain information about recovery rates not captured by the macro variables.
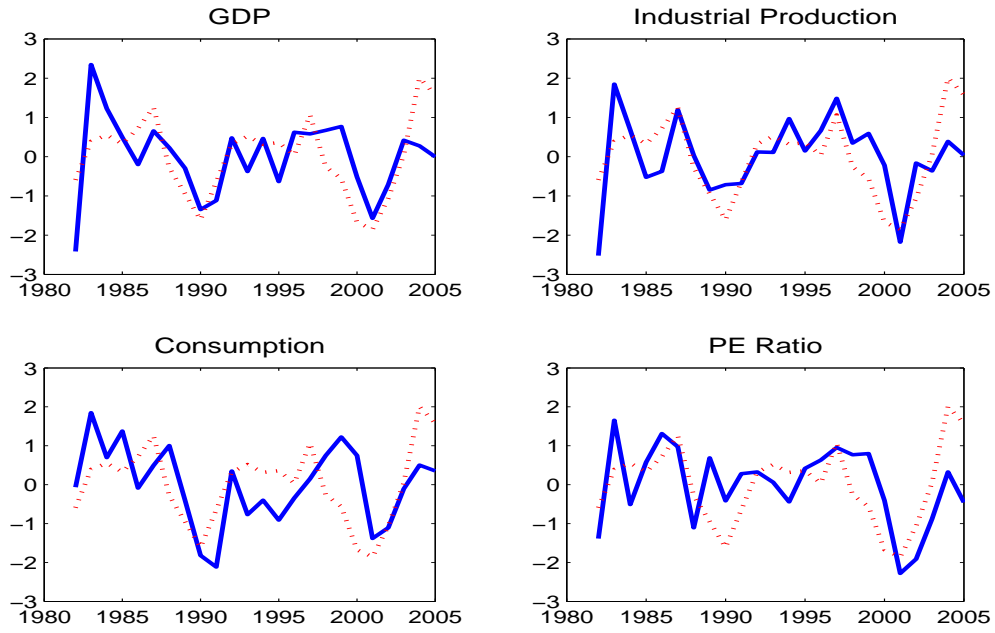
Figure 3: Recovery Rates and Macroeconomic Variables, 1982-2005. All the series are normalized to have mean 0 and standard deviation 1. The dotted line is the normalized recovery rate. GDP, IP and consumption data are from NIPA. Consumption is the sum of nondurables and services deflated with a chain-weighted price indice. Price-Earnings ratios are from Robert Shiller's web site. All macro variables are real annual growth rates.

Table 3: EXPLAINING AGGREGATE DEFAULT RATES

This table reports results from regressions of aggregate default rates on macro variables. ΔIP - real industrial production growth, ΔGDP - real GDP growth, ΔPE - growth rate of Price/Earnings ratio, g - real consumption growth, r_f - real riskfree rate. Numbers in brackets are Newey-West standard errors with lag 3. All variables are annualized, from 1982 to 2005. GDP, IP, consumption and CPI series are from NIPA. PE ratios are from Robert Shiller's web site. Riskfree rates are the 1-month T-bill rates. Default rates and recovery rates are from Moody's.

Intercept	2.01	2.50	1.74	2.75	3.62	3.59	3.74
	(0.38)	(0.64)	(0.27)	(0.54)	(0.40)	(0.31)	(0.33)
ΔIP	-0.14					-0.09	-0.10
	(0.07)					(0.06)	(0.05)
ΔGDP		-0.28					
		(0.15)					
ΔPE			-0.03				
			(0.01)				
g				-0.55	-1.82	-1.70	-1.77
				(0.17)	(0.38)	(0.28)	(0.31)
g^2					0.33	0.34	0.37
					(0.12)	(0.09)	(0.10)
r_f							-0.24
							(0.23)
R^2	0.28	0.29	0.15	0.32	0.50	0.60	0.61
$Adj\ R^2$	0.25	0.26	0.11	0.29	0.45	0.54	0.53

Table 4: EXPLAINING AGGREGATE RECOVERY RATES

This table reports results from regressions of aggregate recovery rates on macro variables. ΔIP - real industrial production growth, ΔGDP - real GDP growth, ΔPE - growth rate of Price/Earnings ratio, g - real consumption growth, r_f - real riskfree rate, DR - default rate. Numbers in brackets are Newey-West standard errors with lag 3. All variables are annualized, from 1982 to 2005. GDP, IP, consumption and CPI series are from NIPA. PE ratios are from Robert Shiller's web site. Riskfree rates are the 1-month T-bill rates. Default rates and recovery rates are from Moody's.

Intercept	52.96	37.05	32.92	39.60	33.58	28.03	31.68	31.79
	(2.69)	(2.61)	(4.09)	(2.27)	(3.47)	(2.47)	(2.90)	(2.37)
DR	-7.36							-6.95
	(1.14)							(1.83)
ΔIP		1.43						
		(0.44)						
ΔGDP			2.55					
			(0.95)					
ΔPE				0.33		0.31	0.35	0.35
				(0.07)		(0.08)	(0.09)	(0.07)
g					3.63	12.86	11.04	10.99
					(1.42)	(3.36)	(2.52)	(2.83)
g^2						-2.83	-2.29	-2.28
						(0.85)	(0.67)	(0.74)
r_f							-5.39	-5.55
							(3.54)	(2.38)
R^2	0.60	0.32	0.27	0.23	0.16	0.42	0.50	0.74
$Adj\ R^2$	0.58	0.29	0.24	0.20	0.12	0.33	0.40	0.67

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