An Elementary Proof of the Zigzag Theorem

STEM Capstone Mini-Project

Holden Mui

Naperville North High School

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Outline

What is a Porism?

Definition

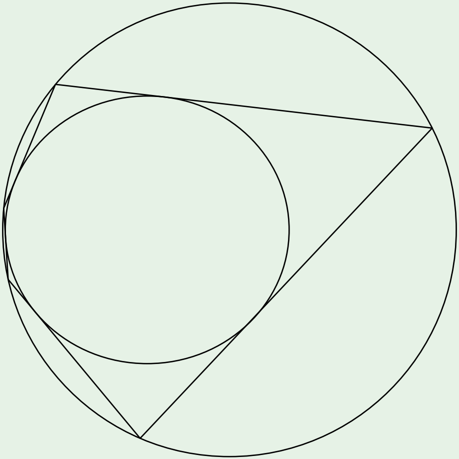
A *porism* of the plane is a statement that holds for an infinite range of values, as long as a certain initial condition is true.

The most famous porism in Euclidean geometry is Poncelet's porism, which states that if there exists a polygon with a given circumscribed conic and inscribed conic, then there exists infinitely many such polygons.

What is a Porism?

Example

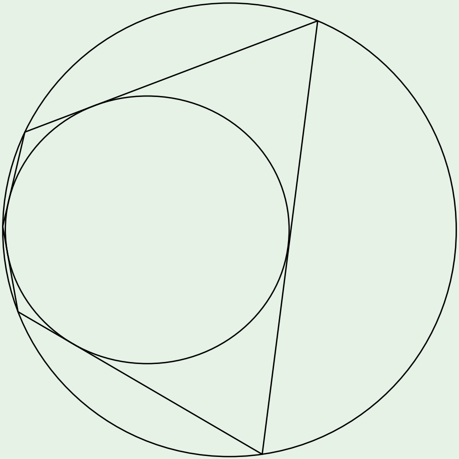
Poncelet's porism:



What is a Porism?

Example

Poncelet's porism:



What is a Porism?

- The most common proof of Poncelet's porism uses algebraic geometry
- However, elementary proofs of Poncelet's porism do exist; one that relies only on Pascal's theorem can be found in Lorenz Halbeisen and Norbert Hungerbuhler's paper "A Simple Proof of Poncelet's Theorem"

What is the Zigzag Theorem?

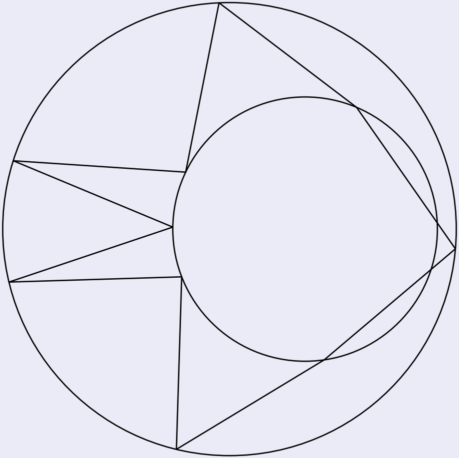
Like Poncelet's porism, the Zigzag theorem is also a porism of the plane.

Theorem (Zigzag Theorem)

Let Γ and Ω be two circles such that there exists a unit equilateral $2n$-gon whose odd-indexed vertices lie on Γ and whose even-indexed vertices lie on Ω. Then there exist infinitely many such $2n$-gons.

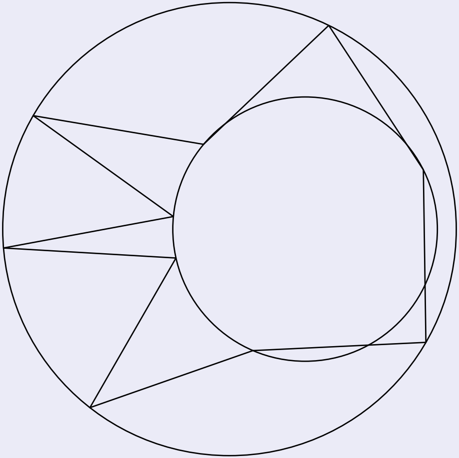
What is the Zigzag Theorem?

Diagram



What is the Zigzag Theorem?

Diagram

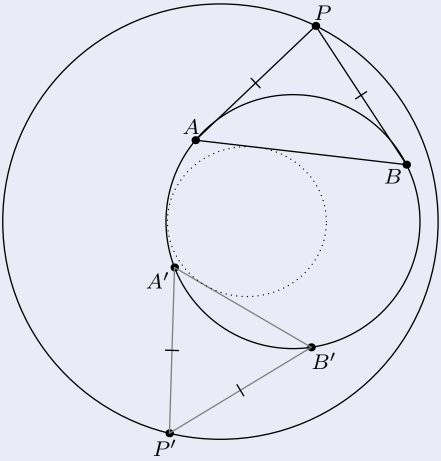


How is the Zigzag Theorem Proved?

- The proof will proceed in two steps.
- First, an intermediate lemma will be proved.
- Then, this lemma will show that the Zigzag theorem is a corollary of Poncelet's porism.

The Intermediate Lemma

Diagram

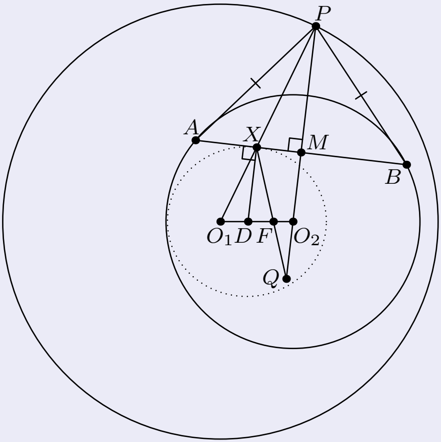


Lemma

Let Γ and Ω be two circles, and let P be a variable point on Γ. Additionally, points A and B lie on Ω such that $AP = BP = 1$. Then as P varies on Γ, $\overline{AB}$ is always tangent to some fixed ellipse.

Proof of Intermediate Lemma

Diagram



Proof

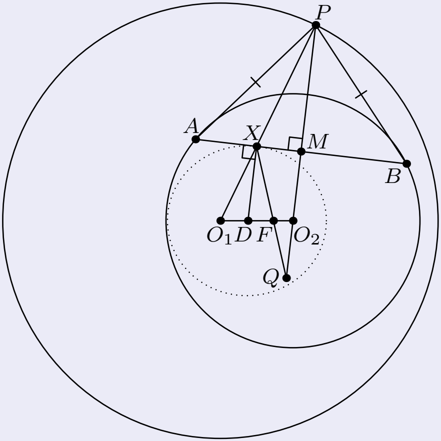
Let:

- O_1 be the center of Γ,
- O_2 be the center Ω,
- $\overline{AB}$ meet $\overline{O_1P}$ at X,
- M be the midpoint of $\overline{AB}$,
- Q be the reflection of P over M, and
- $\overline{DX}$ be an angle bisector of $\triangle O_1FX$.

The goal is to prove that F ix fixed and $O_1X + XF$ is constant.

Proof of Intermediate Lemma

Diagram



Proof

$\overline{DX} \parallel \overline{PQ}$ and $\overline{DX} \perp \overline{AB}$ since

$$\angle AXO_1 = \angle PXM = \angle QXM.$$

By the Angle Bisector Theorem and similar triangles,

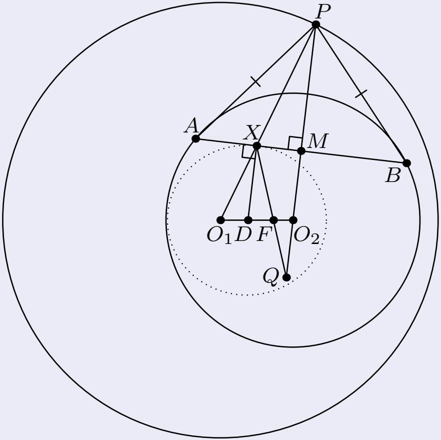
$$\frac{O_1X}{O_1D} = \frac{XF}{DF} = \frac{FQ}{FO_2} = \frac{O_1P}{O_1O_2} = k$$

for some constant k. By Stewart's formula,

$$DX^2 = (k^2 - 1) \cdot O_1D \cdot DF.$$

Proof of Intermediate Lemma

Diagram



Proof

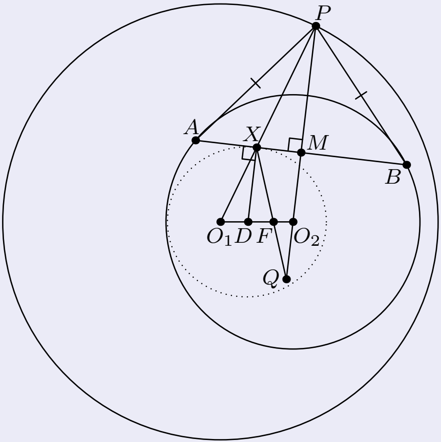
$DX^2 = (k^2 - 1) \cdot O_1D \cdot DF$ from the previous slide, so a computation gives

$$\begin{aligned}
&(k^2 - 1) \cdot FO_2 \cdot O_1O_2 \\
&= \left(\frac{O_1O_2}{O_1D} \cdot DX\right) \cdot \left(\frac{FO_2}{DF} \cdot DX\right) \\
&= O_2P \cdot O_2Q \\
&= MP^2 - MO_2^2 \\
&= AP^2 - AO_2^2
\end{aligned}$$

which is constant.

Proof of Intermediate Lemma

Diagram



Proof

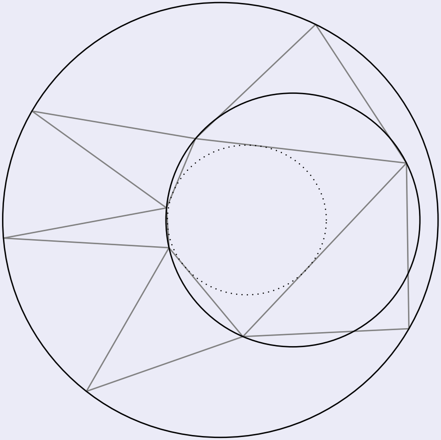
Since $(k^2 - 1) \cdot FO_2 \cdot O_1O_2$ is constant, F is fixed. Lastly, by Angle Bisector Theorem

$$\begin{aligned} O_1X + XF &= k \cdot O_1D + k \cdot DF \\ &= k \cdot O_1F \end{aligned}$$

is also constant. Thus, the intermediate lemma is proved, since the goal was to prove that F is fixed and $O_1X + XF$ is constant.

Reduction to Poncelet's Porism

Diagram



Proof

To finish the problem, let $\mathcal{E}$ be the ellipse guaranteed by the lemma. Using the given condition, there exists an n-gon with circumcircle Ω and inellipse $\mathcal{E}$. Thus, by Poncelet's porism, there are infinitely many such n-gons. By the converse of the lemma, each such n-gon can be used to construct a unit equilateral $2n$-gon whose odd-indexed vertices lie on Γ and whose even-indexed vertices lie on Ω. Therefore, the Zigzag theorem is proved.

Additional Information

- I actually discovered the Zigzag theorem independently while playing around on Geogebra and came up with a proof after a few days
- I would typically save such results so I can submit them as problem proposals to mathematics olympiads, but a nontrivial Google search revealed that this result was already known
- However, none of the papers I looked at contained an elementary proof of the Zigzag theorem, so I am inclined to believe that the proof is new

Thank you!

Thank you for listening!