Generating Symmetric Operations of Every Arity

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What Is an Operation?

Let D be any set, which will be referred to as a **domain**.

Definition

An **operation** f is a function $f : D^k \to D$ for some positive integer k, known as its **arity**.

The output of an operation f with inputs $x_1, x_2, \ldots, x_k$ is denoted

$$f(x_1, x_2, \ldots, x_k).$$

Arity 1 operations are **unary operations**. Arity 2 operations are **binary operations**. Arity 3 operations are **ternary operations**.

Examples of Operations

- Addition
- The ternary majority operation over $D = \{0, 1\}$

$$\mathrm{maj}_3(a, b, c) \overset{\text{def}}{=} \begin{cases} 0 & \text{if the inputs contain more 0's than 1's} \\ 1 & \text{if the inputs contain more 1's than 0's} \end{cases}$$

- The rock-paper-scissors operation over $D = \{\mathrm{rock}, \mathrm{paper}, \mathrm{scissors}\}$

$$\mathrm{rps}(a, b) \overset{\text{def}}{=} \text{the winner of a rock-paper-scissors game between } a \text{ and } b$$

- The **projection operation**

$$\pi_i^k(d_1, \ldots, d_k) \overset{\text{def}}{=} d_i,$$

which takes k inputs and outputs the i^{th} input

Symmetric Operations

Definition

A **symmetric operation** is an operation whose output is invariant under permutation of its inputs.

Examples:

- Addition
- The ternary majority operation
- The rock-paper-scissors operation

Non-examples:

- Most projection operations

Operation Composition

Our goal is to compose specific operations to produce new operations.

Examples:

- If $+(a, b)$ is addition and $\times(a, b)$ is multiplication, then

$$+(x, \times(y, x)) = x + yx.$$

- If $\mathrm{AND}(a, b)$ is logical "and" and $\mathrm{OR}(a, b, c)$ is logical "or," then

$$\mathrm{OR}(\mathrm{AND}(b, c), \mathrm{AND}(c, a), \mathrm{AND}(a, b)) = \mathrm{maj}_3(a, b, c).$$

- The majority operation on $2k + 1$ inputs can be constructed by composing $\mathrm{maj}_3(a, b, c)$ with itself.

Clones

Definition

A **clone** is a set $\mathcal{O}$ of operations that

- contains all the projection operations, and
- is closed under composition.

Examples:

- The set of all operations
- The set of all projection operations
- The union of the set of all projection operations and the set of all constant operations
- The **affine modulo** n clone, the set of all operations over $\mathbb{Z}/n\mathbb{Z}$ that sum a linear combination of their inputs

Non-examples:

- The set of all constant operations

Clones With Symmetric Operations of Every Arity

Definition

A **round clone** is a clone that contains symmetric operations of every arity.

Examples:

- The set of all operations
- The affine modulo n clone, since

$$f_k(x_1, x_2, \ldots, x_k) \stackrel{\text{def}}{=} x_1 + x_2 + \ldots + x_k \pmod{n}$$

  is a symmetric operation with arity k

Non-examples:

- The set of all projection operations, when $|D| \geq 2$

The Research Question

Conjecture

Suppose a clone $\mathcal{O}$ over a finite domain D contains symmetric operations of arities $1, 2, \ldots, |D|$. Then $\mathcal{O}$ is round.

With computer assistance, I

- proved the problem statement for $|D| \leq 4$, and
- obtained strong evidence for $|D| = 5$.

For the remainder, I will present interesting examples for $|D| = 3$.

The SGN Clone

Recall that the function $\text{sgn} : \mathbb{R} \to \mathbb{R}$ is defined as

$$\text{sgn}(x) \stackrel{\text{def}}{=} \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Definition

The **SGN clone** is the clone over $D = \{-1, 0, 1\}$ generated by the symmetric binary operation

$$f_2(a, b) \stackrel{\text{def}}{=} \text{sgn}(a + b).$$

Is The SGN Clone Round?

The SGN clone has a unary and a binary symmetric operation. Since

$$f_3(a,b,c) \stackrel{\text{def}}{=} f_2(f_2(a,b), f_2(f_2(a,c), f_2(b,c))) = \operatorname{sgn}(a+b+c)$$

is ternary and symmetric, the SGN clone satisfies the conjecture's hypothesis. Indeed, the SGN clone contains symmetric operations of every arity, since

$$\begin{aligned} f_{n+1}(x_1,\ldots,x_{n+1}) \stackrel{\text{def}}{=} f_2(f_n(&f_n(\quad x_2, x_3, \ldots, x_n, x_{n+1}),\\ &f_n(x_1, \quad x_3, \ldots, x_n, x_{n+1}),\\ &f_n(x_1, x_2, \quad \ldots, x_n, x_{n+1}),\\ &\vdots,\\ &f_n(x_1, x_2, x_3, \ldots, \quad x_{n+1})), f_n(x_1,\ldots,x_n)) \end{aligned}$$

is the operation $\operatorname{sgn}(x_1 + \ldots + x_{n+1})$, by induction.

The RPS Clone

Recall that the binary operation $\text{rps}(a, b)$ is the winner of a rock-paper-scissors game between a and b:

$$\begin{array}{c|ccc} \text{rps} & r & p & s \\ \hline r & r & p & r \\ p & p & p & s \\ s & r & s & s \end{array}.$$

Definition

The **RPS clone** is the clone generated by $\text{rps}(a, b)$.

Is The RPS Clone Round?

Note that

- rps(a, b) remains the same operation when “rock” is renamed ”paper,” “paper” is renamed ”scissors,” and “scissors” is renamed “rock”, so
- every operation in the RPS clone also has this property.

If there was a ternary symmetric operation f, then $f(r, p, s)$ must be assigned a value, such as r. However, applying the renaming yields

$$f(p, s, r) = p,$$

contradicting symmetry.

Making The RPS Clone Round

If a symmetric ternary operation f_3 is added, then

$$s_3(a, b, c) \overset{\text{def}}{=} \begin{cases} \text{rps}(m, n) & \text{if } \{a, b, c\} = \{m, n\} \\ f_3(r, p, s) & \text{if } a,\ b, \text{ and } c \text{ are distinct} \end{cases}$$

is in the clone. By induction,

$$s_k(x_1, x_2, \ldots, x_k) \overset{\text{def}}{=} \begin{cases} \text{rps}(m, n) & \text{if } \{x_1, x_2, \ldots, x_k\} = \{m, n\} \\ f_3(r, p, s) & \text{if } \{x_1, x_2, \ldots, x_k\} = \{r, p, s\} \end{cases}$$

lies in the clone, so the new clone is round.

Importance and Future Work

If the conjecture is true, then this gives an effective way of determining whether combinatorial puzzles can be solved using systems of linear inequalities.

- My mentor formulated a stronger conjecture, which deals with the connection between the solutions to puzzles and automorphisms without fixed points
- Goal: verify the stronger conjecture for the SGN and RPS clones

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