Gluing Polygons

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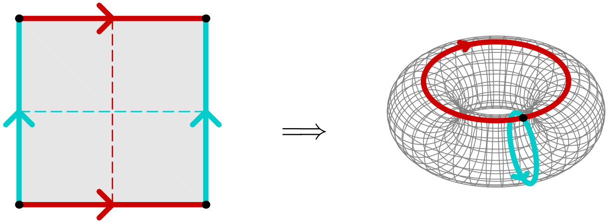
G2 Seminar

31 July 2024

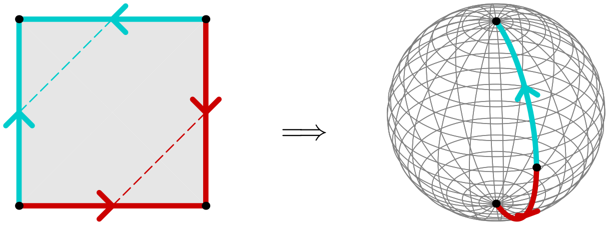
Squares

Imagine a stretchy rubber square.

- Glue opposite sides. What happens?

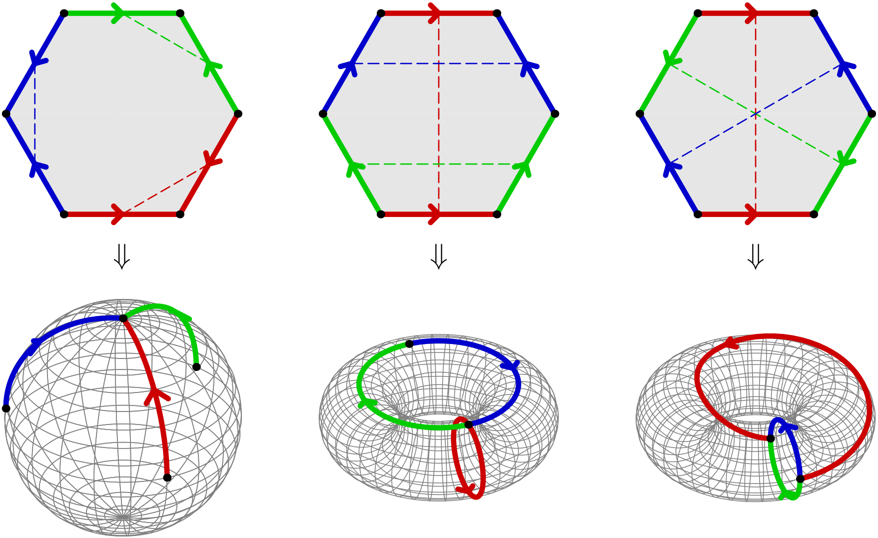


- Glue adjacent sides. What happens?



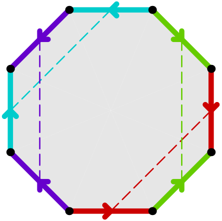
Hexagons

Imagine a stretchy rubber hexagon.



Octagons

Imagine a stretchy rubber octagon.



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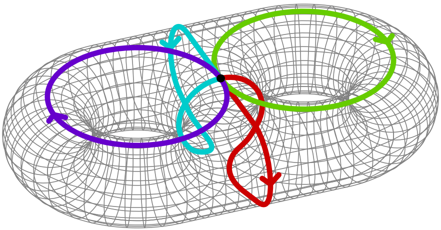
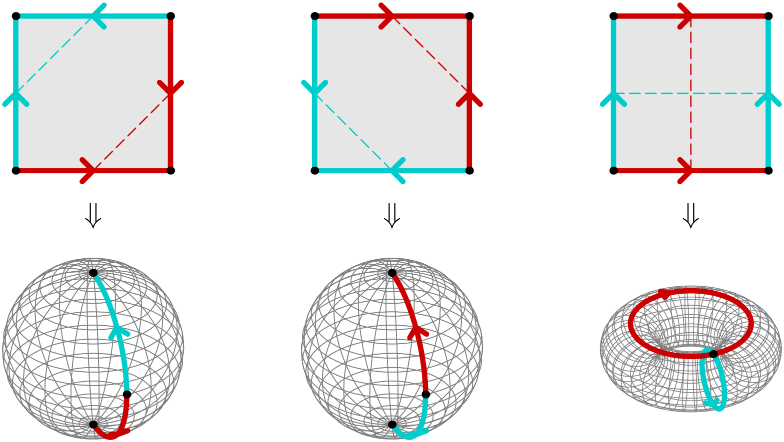


Counting gluings

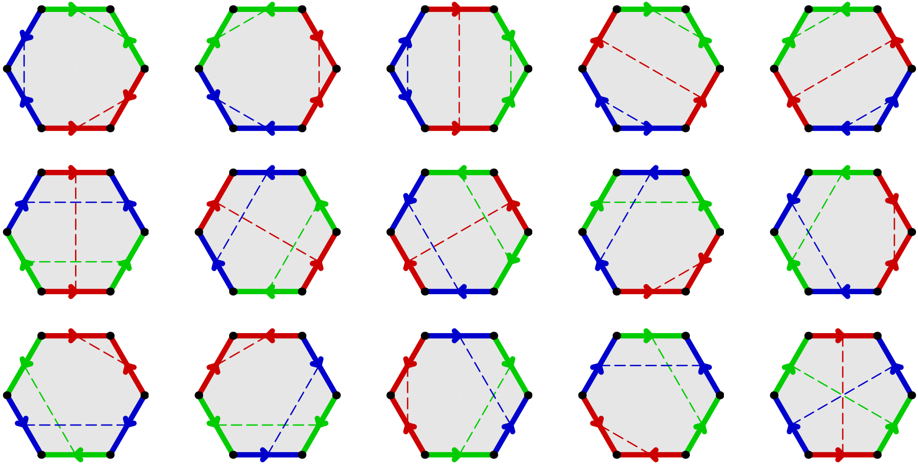
Three ways to glue a square:



Two spheres, one torus.

Counting gluings

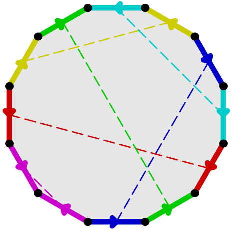
Fifteen ways to glue a hexagon:



Five spheres, ten tori.

Counting gluings

In general, $(2n-1)\cdot(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1 = (2n-1)!!$ ways to glue a $2n$-gon.



Proof.

- $2n-1$ ways to pair first edge
- $2n-3$ ways to pair next unpaired edge
- $\ldots$
- 3 ways to pair fourth-to-last unpaired edge
- 1 way to pair second-to-last unpaired edge $\square$

The Question

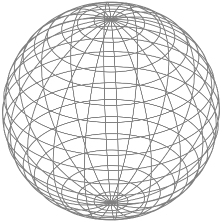
Randomly choose a $2n$-gon gluing.

- Sphere probability?
- Torus probability?

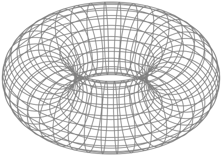
In general:

Question

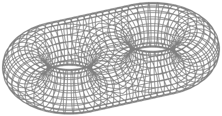
What is the probability that a random $2n$-gon gluing produces a surface with g holes?



zero holes



one hole



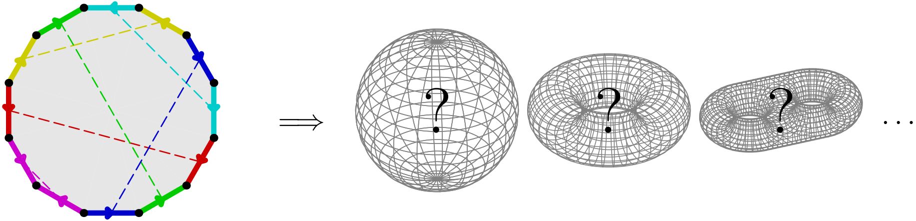
two holes

Determining genus

Definition

The *genus* of a surface is its hole count.

How to determine genus from gluing?

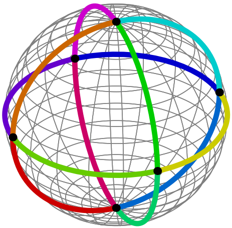


Answer: Euler's formula!

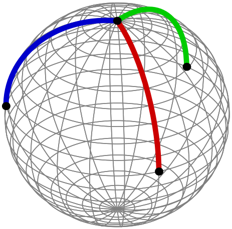
Euler's formula

Theorem

A connected graph (loops and double edges allowed) on a sphere with v vertices, e edges, and f faces satisfies $v - e + f = 2$.



$v - e + f = 6 - 12 + 8 = 2$



$v - e + f = 4 - 3 + 1 = 2$

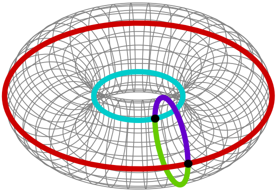
Proof.

Induct on edge count via edge contraction. □

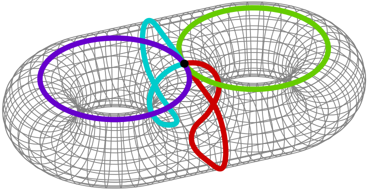
Euler's formula

Theorem

A graph on a genus g surface with v vertices, e edges, and f (simply connected) faces satisfies $v - e + f = 2 - 2g$.



$v - e + f = 2 - 4 + 2 = 0 = 2 - 2g$



$v - e + f = 1 - 4 + 1 = -2 = 2 - 2g$

Proof.

Show $v - e + f$ is invariant, then calculate it for a specific graph. □

Counting vertices

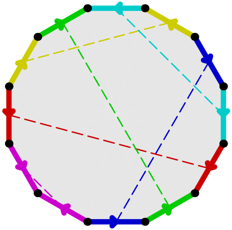
Lemma

A $2n$-gon gluing with v vertices has genus $\frac{1}{2}(n+1-v)$.

Proof.

A gluing of a $2n$-gon produces a graph with n edges and 1 face. Thus

$$v - e + f = v - n + 1 = 2 - 2g \implies g = \frac{1}{2}(n+1-v). \qquad \square$$



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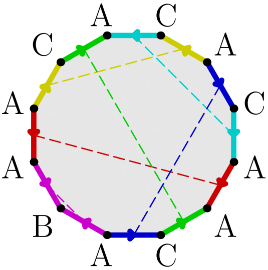


$$\Longrightarrow \quad g = \frac{1}{2}(6+1-3) = 2$$

Data

Question

What is the probability that a random $2n$-gon gluing produces a surface with g holes?

	$g=0$	$g=1$	$g=2$	$g=3$
square ($2n=4$)	$2/3$	$1/3$		
hexagon ($2n=6$)	$1/3$	$2/3$		
octagon ($2n=8$)	$2/15$	$2/3$	$1/5$	
decagon ($2n=10$)	$2/45$	$4/9$	$23/45$	
dodecagon ($2n=12$)	$4/315$	$2/9$	$28/45$	$1/7$

Observations

	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$2n = 4$	$2/3$	$1/3$		
$2n = 6$	$1/3$	$2/3$		
$2n = 8$	$2/15$	$2/3$	$1/5$	
$2n = 10$	$2/45$	$4/9$	$23/45$	
$2n = 12$	$4/315$	$2/9$	$28/45$	$1/7$

- Maximum genus?
- Sphere probability?
- Maximum genus probability?
- Recurrence or closed form?

Maximum genus

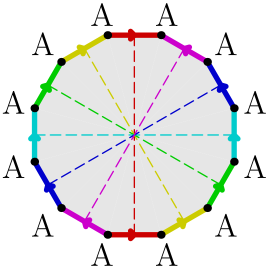
The maximum genus obtainable from a $2n$-gon gluing is $\lfloor \frac{1}{2} n \rfloor$.

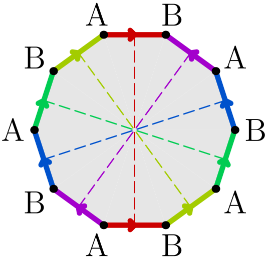
Proof.

- Upper bound: since $v \geq 1$,

$$g = \tfrac{1}{2}(n + 1 - v) \leq \tfrac{1}{2}(n + 1 - 1) = \tfrac{1}{2}n.$$

- Lower bound: gluing opposite edges gives $v = 1$ for even n and $v = 2$ for odd n. □



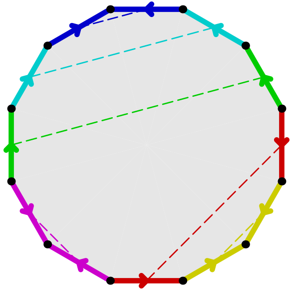


Sphere probability

Sphere-producing gluings are counted by *Catalan numbers*[1] $C_n = \frac{1}{n+1}\binom{2n}{n}$.
To prove this:

Lemma

A gluing produces a sphere if and only if its chord diagram has no crossings.



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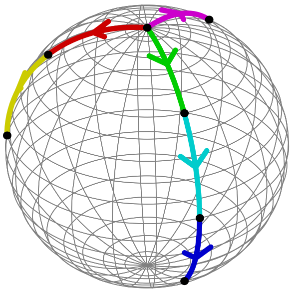
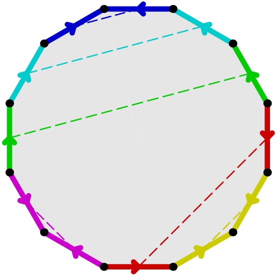


[1]The Catalan numbers begin 1, 2, 5, 14, 42... and count ways to arrange n sets of parentheses. For example, $C_3 = 5$ because of ()()(), ()(()), (())(), (()()), and ((())).

Sphere probability

Lemma

Non-crossing chord diagrams biject to arrangements of n sets of parentheses.



$\iff$

(())((()))()

Thus the sphere probability for a $2n$-gon gluing is $C_n/(2n-1)!!$.

Maximum genus probability

Recall maximum genus is $\lfloor \frac{1}{2} n \rfloor$.

	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$2n = 4$	$2/3$	$1/3$		
$2n = 6$	$1/3$	$2/3$		
$2n = 8$	$2/15$	$2/3$	$1/5$	
$2n = 10$	$2/45$	$4/9$	$23/45$	
$2n = 12$	$4/315$	$2/9$	$28/45$	$1/7$

Observation

For even n, the probability of producing a genus $\frac{1}{2}n$ surface from a $2n$-gon gluing is $\frac{1}{n+1}$.

Recurrence

	$g=0$	$g=1$	$g=2$	$g=3$
$2n=4$	$2/3$	$1/3$		
$2n=6$	$1/3$	$2/3$		
$2n=8$	$2/15$	$2/3$	$1/5$	
$2n=10$	$2/45$	$4/9$	$23/45$	
$2n=12$	$4/315$	$2/9$	$28/45$	$1/7$

Observation

Let $\mathrm{HZ}(n,g)$ *denote the probability that a random* $2n$*-gon gluing produces a genus* g *surface. Then*

$$\mathrm{HZ}(n,g) = \tfrac{2}{n+1}\,\mathrm{HZ}(n-1,g) + \tfrac{n-1}{n+1}\,\mathrm{HZ}(n-2,g-1).$$

For example, $n=6$ and $g=2$ gives $\frac{28}{45} = \frac{2}{7}\cdot\frac{23}{45} + \frac{5}{7}\cdot\frac{2}{3}$.

Recurrence

Let $\mathrm{HZ}(n, g)$ denote the probability that a random $2n$-gon gluing produces a genus g surface.

Theorem (J. Harer and D. Zagier, 1986)

$$\mathrm{HZ}(n, g) = \frac{2}{n+1}\,\mathrm{HZ}(n-1, g) + \frac{n-1}{n+1}\,\mathrm{HZ}(n-2, g-1).$$

Three known proofs:

- analytic
- algebraic
- combinatorial

Corollary

For even n, $\mathrm{HZ}(n, \frac{1}{2}n) = \frac{1}{n+1}$.

No direct bijective proof for this result is known.

Analytic proof

A complex matrix is *unitary* if its inverse is its conjugate transpose.

Lemma

Choose a $k \times k$ unitary matrix uniformly at random. Then

$$\mathbb{E}\left[\operatorname{tr} X^{2n}\right] = (2n-1)!!\sum_{g=0}^{\lfloor \frac{1}{2}n \rfloor} \mathrm{HZ}(n,g)k^{n+1-2g}.$$

For example, $n = 4$ gives $\mathbb{E}\left[\operatorname{tr} X^{8}\right] = 14k^5 + 70k^3 + 21k$.

Proof.

Entries of X are complex normal Gaussians, so expand $\operatorname{tr} X^{2n}$ in terms of its entries and apply Isserlis' theorem. □

It suffices to understand the eigenvalue distribution $\sigma_k(\lambda)$ of X because

$$\mathbb{E}\left[\operatorname{tr} X^{2n}\right] = \mathbb{E}\left[\lambda_1^{2n} + \cdots + \lambda_k^{2n}\right] = k\mathbb{E}\left[\lambda^{2n}\right] = k\int_{-\infty}^{\infty} \lambda^{2n}\sigma_k(\lambda)\mathrm{d}\lambda.$$

Analytic proof

The joint probability distribution for the eigenvalues of a random $k \times k$ unitary matrix is

$$P_k(\lambda_1, \ldots, \lambda_k) = \frac{1}{(2\pi)^{k/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{k} \lambda_i^2\right) \prod_{1 \le a < b \le k} (\lambda_a - \lambda_b)^2.$$

Integrating gives the single eigenvalue distribution $\sigma_k(\lambda)$ as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_k(\lambda, \lambda_2, \ldots, \lambda_k) \mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_k = \frac{1}{k\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2} \sum_{i=0}^{k-1} H_i(\lambda)^2,$$

where H_i is the i^{th} Hermite polynomial. Finally, Hermite polynomial identities give a differential equation satisfied by

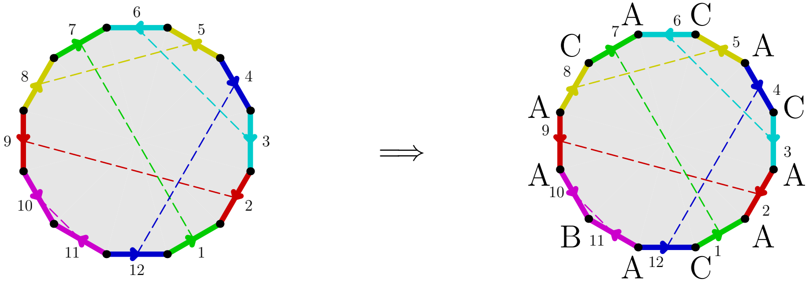
$$u(t) := \int_{-\infty}^{\infty} e^{t\lambda} \sigma_k(\lambda) \mathrm{d}\lambda = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} \lambda^n \sigma_k(\lambda) \mathrm{d}\lambda,$$

which gives a recurrence on coefficients in $u(t)$'s Taylor expansion.

Algebraic proof

Lemma

Label a $2n$-gon's edges from 1 to $2n$, let $\tau : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ be $\tau(x) := x + 1$, and let $\sigma : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ represent a $2n$-gon gluing with v vertices. Then v is the cycle count in $\tau \circ \sigma$.



$$\sigma = (1, 7)(2, 9)(3, 6)(4, 12)(5, 8)(10, 11)$$
$$\tau \circ \sigma = (1, 8, 6, 4)(2, 10, 12, 5, 9, 3, 7)(11)$$

Algebraic proof

Work in $\mathbb{C}[S_{2n}]^{S_{2n}}$. The probability of a random $2n$-gon gluing having v vertices equals the sum of the coefficients on v-cycle permutations in

$$X := \frac{1}{(2n-1)!}\left(\sum_{\tau \in S_{2n}\ 2n\text{-cycle}} \tau\right) \circ \frac{1}{(2n-1)!!}\left(\sum_{\sigma \in S_{2n}\ \text{gluing}} \sigma\right).$$

Irreducible representations of S_{2n} give an orthonormal basis of $\mathbb{C}[S_{2n}]^{S_{2n}}$; most coordinates of X in this basis vanish due to Schur's lemma and the Murnaghan–Nakayama rule. Computing remaining coefficients via the Specht module for hook partitions gives

$$\mathrm{HZ}(n,g) = 2\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}[x^{n+1-2g}]\binom{x+2n-2k-1}{2n},$$

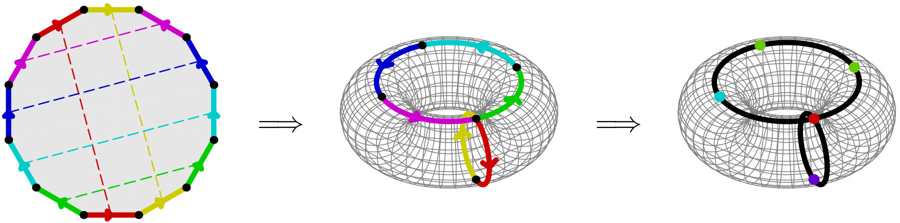
which satisfies the recurrence.

Combinatorial proof

Let $T(n,q)$ count ways to glue a $2n$-gon and color vertices with exactly q colors. Then

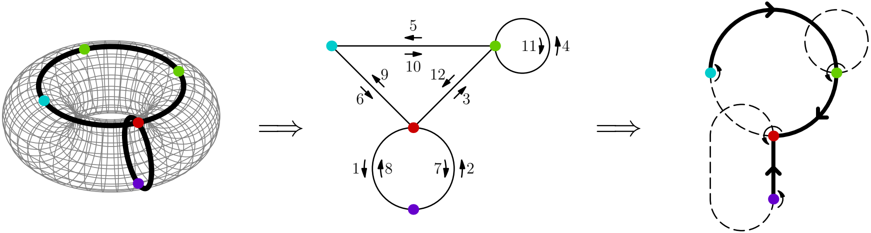
$$\sum_{q=1}^{k} \binom{k}{q} T(n,q) = (2n-1)!! \sum_{g=0}^{\lfloor \frac{1}{2} n \rfloor} \mathrm{HZ}(n,g) k^{n+1-2g}.$$

is the number of ways to glue a $2n$-gon and color its vertices using at most k colors.



Combinatorial proof

Colorings with exactly q colors biject to bi-Eulerian tours on q-vertex graphs. Such tours decompose into trees, rooted rotation systems, and pairings via BEST theorem.

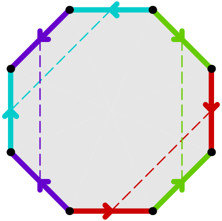


Since Catalan numbers count rooted plane trees,

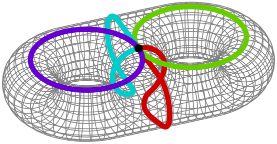
$$T_n(q) = C_{q-1} q! \binom{2n}{2q-2} (2n-2q+1)!!,$$

giving a closed form for $\mathrm{HZ}(n, g)$. This expression satisfies the recurrence.

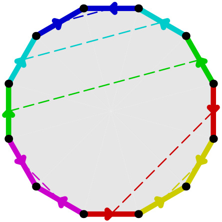
Recap



Randomly glue a $2n$-gon



$1 - 4 + 1 = 2 - 2 \cdot 2$



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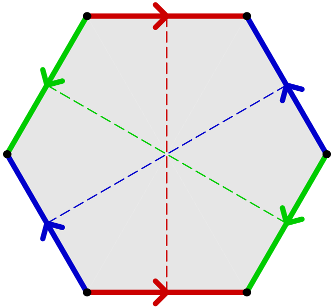
- Polygon gluing and genus (topology)
- Euler's formula $v - e + f = 2 - 2g$ (polyhedral combinatorics)
- Catalan numbers (enumerative combinatorics)
- Analytic recurrence proof (linear algebra, random matrix theory)
- Algebraic recurrence proof (group theory, representation theory)
- Combinatorial recurrence proof (graph theory)

Thank you!

Questions?



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