§ Problem Statement

Let $a_1, b_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of real numbers for which $a_1 > b_1$ and

$$a_{n+1} = a_n^2 - 2b_n$$
$$b_{n+1} = b_n^2 - 2a_n$$

for all positive integers n. Prove that $a_1, a_2, \ldots$ is eventually increasing.

§ Solution

Let r, s, and t be the complex roots of the polynomial $p(\lambda) = \lambda^3 - a_1\lambda^2 + b_1\lambda - 1$. By Vieta's formulas,

$$\begin{aligned} a_1 &= r + s + t \\ b_1 &= 1/r + 1/s + 1/t \\ 1 &= rst. \end{aligned}$$

Claim 1. *For every positive integer n,*

$$a_n = r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}}$$

and

$$b_n = (1/r)^{2^{n-1}} + (1/s)^{2^{n-1}} + (1/t)^{2^{n-1}}.$$

Proof. The base case follows from Vieta's formulas above. For the inductive step, observe that $rst = 1$, so

$$\begin{aligned} a_{n+1} &= a_n^2 - 2b_n \\ &= \left(r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}}\right)^2 - 2\left((1/r)^{2^{n-1}} + (1/s)^{2^{n-1}} + (1/t)^{2^{n-1}}\right) \\ &= \left(r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}}\right)^2 - 2\left((st)^{2^{n-1}} + (tr)^{2^{n-1}} + (rs)^{2^{n-1}}\right) \\ &= r^{2^n} + s^{2^n} + t^{2^n} \end{aligned}$$

and similarly for b_{n+1}. □

Since $p(1) = b_1 - a_1 < 0$, p has a real root greater than 1.

- If all roots are real, then

  $$a_n = r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}} = \Theta\left(\max(|r|, |s|, |t|)^{2^{n-1}}\right)$$

  is eventually increasing.

- If r is real and s, t are not real, they must be complex conjugates of each other, each with magnitude $\frac{1}{\sqrt{r}} < 1$. Therefore

  $$r^{2^{n-1}} - 2 < a_n < r^{2^{n-1}} + 2,$$

  so a_n is eventually increasing.

§ Variants

Variant A. Let $a_1, b_2, \ldots$ and $b_1, b_2, \ldots$ be sequences for which $a_1 > b_1$ and

$$a_{n+1} = a_n^2 - 2b_n$$
$$b_{n+1} = b_n^2 - 2a_n$$

for all positive integers n. Suppose that both sequences are bounded. Find all possible values of (a_1, b_1).

Solution sketch. The answer is $\{(t,t) \mid -1 \le t \le 3\}$. The solution is similar to the one given for the original problem.

§ Metadata

This problem was selected as Problem 2 of the 2025 TST.

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