§ Problem Statement

Let n be a positive integer. Prove that among the first n multiples of three, there are more numbers with an even number of 1s in binary than numbers with an odd number of 1s in binary.

§ Solutions

Solution A

Given a set of positive integers S, define

$$\begin{aligned}\text{even}(S) &= \#\{s \in S \mid s \text{ has an even number of 1s in binary}\}\\ \text{odd}(S) &= \#\{s \in S \mid s \text{ has an odd number of 1s in binary}\}.\end{aligned}$$

The problem will follow from the following lemma.

Lemma 1. *For every positive integer n,*

- $\text{even}(\{3, 6, \ldots, 3n\}) > \text{odd}(\{3, 6, \ldots, 3n\})$
- $\text{even}(\{1, 4, \ldots, 3n-2\}) < \text{odd}(\{1, 4, \ldots, 3n-2\})$
- $\text{even}(\{2, 5, \ldots, 3n-1\}) \le \text{odd}(\{2, 5, \ldots, 3n-1\})$

Proof. Induct on n. The base cases $n \in \{1, 2\}$ are easy to check.

- For $S = \{3, 6, \ldots, 3n\}$, let S_{even} and S_{odd} denote the set of even numbers and odd numbers in S, respectively. Then

$$\begin{aligned}\text{even}(S) &= \text{even}(S_{even}) + \text{even}(S_{odd})\\ &= \text{even}(S_{even}/2) + \text{odd}((S_{odd}-1)/2)\\ &< \text{odd}(S_{even}/2) + \text{even}((S_{odd}-1)/2)\\ &= \text{odd}(S_{even}) + \text{odd}(S_{odd})\\ &= \text{odd}(S)\end{aligned}$$

  since the elements of $S_{even}/2$ are 0 modulo 3 and the elements of $(S_{odd}-1)/2$ are 1 modulo 3.

- For $S = \{1, 4, \ldots, 3n-2\}$, let S_{even} and S_{odd} denote the set of even numbers and odd numbers in S, respectively. Then

$$\begin{aligned}\text{even}(S) &= \text{even}(S_{even}) + \text{even}(S_{odd})\\ &= \text{even}(S_{even}/2) + \text{odd}((S_{odd}-1)/2)\\ &> \text{odd}(S_{even}/2) + \text{even}((S_{odd}-1)/2)\\ &= \text{odd}(S_{even}) + \text{odd}(S_{odd})\\ &= \text{odd}(S)\end{aligned}$$

  since the elements of $S_{even}/2$ are 2 modulo 3 and the elements of $(S_{odd}-1)/2$ are 0 modulo 3.

- For $S = \{2, 5, \ldots, 3n-1\}$, there are two cases.

- Case 1: $2^{2k-1} \le 3n - 1 < 2^{2k}$. Define

$$\begin{aligned} S_{small} &= \{s \in S \mid s < 2^{2k-1}\} \\ S_{big} &= \{s \in S \mid 2 \ge 2^{2k-1}\}. \end{aligned}$$

  Then

$$\begin{aligned} \text{even}(S) &= \text{even}(S_{small}) + \text{even}(S_{big}) \\ &= \text{odd}(S_{small}) + \text{odd}(S_{big} - 2^{2k-1}) \\ &\ge \text{odd}(S_{small}) + \text{even}(S_{big} - 2^{2k-1}) \\ &= \text{odd}(S_{small}) + \text{odd}(S_{big}) \\ &= \text{odd}(S) \end{aligned}$$

  since $\text{even}(\{a, b\}) = \text{odd}(\{a, b\}) = 1$ when a and b are positive integers summing to 2^{2k-1}, and the elements of $S_{big} - 2^{2k-1}$ are 0 mod 3.

- Case 2: $2^{2k} < 3n - 1 < 2^{2k+1}$.

$$\begin{aligned} S_{small} &= \{s \in S \mid s < 2^{2k+1} - (3n-1)\} \\ S_{big} &= \{s \in S \mid 2 \ge 2^{2k+1} - (3n-1)\}. \end{aligned}$$

  Then

$$\begin{aligned} \text{even}(S) &= \text{even}(S_{small}) + \text{even}(S_{big}) \\ &\ge \text{odd}(S_{small}) + \text{even}(S_{big}) \\ &= \text{odd}(S_{small}) + \text{odd}(S_{big}) \\ &= \text{odd}(S) \end{aligned}$$

  since the elements of S_{small} are 2 modulo 3 and $\text{even}(\{a, b\}) = \text{odd}(\{a, b\}) = 1$ when a and b are positive integers summing to 2^{2k+1}.

This completes the triple induction. □

Solution B

First, let's establish the problem statement when $n = \frac{4^d-1}{3}$. Given a positive integer k, let $s(k)$ denote the number of 1s in its binary representation.

Lemma 2. *Let d be a positive integer. Then*

$$\sum_{\substack{0 \le i < 4^d \\ i \equiv 0 \bmod 3}} (-1)^{s(i)} = 2 \cdot 3^{d-1}.$$

Proof. Given a positive integer k, let $s_{\text{even}}(k)$ and $s_{\text{odd}}(k)$ denote the number of 1s in even positions and odd positions of k's binary representation, respectively. Observe that

$$3 \mid k \iff 3 \mid s_{\text{even}}(k) - s_{\text{odd}}(k)$$

and

$$s(k) \text{ even} \iff 2 \mid s_{\text{even}}(k) - s_{\text{odd}}(k).$$

Additionally, observe that the 4^d terms in the expansion of $P(x) = \left((1+x)(1+\frac{1}{x})\right)^d$ correspond to binary expansions with $2d$ digits, and the degree of the term corresponding to k in binary is $s_{\text{even}}(k) - s_{\text{odd}}(k)$. Combining these two facts yields

$$\begin{aligned}\sum_{\substack{0 \le i < 4^d \\ i \equiv 0 \bmod 3}} (-1)^{s(i)} &= \sum_{t \equiv 0 \bmod 6} [x^t]P(x) - \sum_{t \equiv 3 \bmod 6} [x^t]P(x) \\ &= 2 \sum_{t \equiv 0 \bmod 6} [x^t]P(x) - \sum_{t \equiv 0 \bmod 3} [x^t]P(x),\end{aligned}$$

where $[x^t]P(x)$ denotes the x^t coefficient of $P(x)$.

To evaluate this, let $\zeta = e^{2i\pi/6}$. By a roots of unity filter, the first sum equals

$$2 \cdot \frac{P(1) + P(\zeta) + P(\zeta^2) + P(\zeta^3) + P(\zeta^4) + P(\zeta^5)}{6} = \frac{4^d + 3^d + 1 + 0 + 1 + 3^d}{3}$$

and the second term equals

$$\frac{P(1) + P(\zeta^2) + P(\zeta^4)}{3} = \frac{4^d + 1 + 1}{3}.$$

Subtracting these quantities gives $\frac{2}{3} \cdot 3^d$, as desired. □

More generally, we have the following result:

Lemma 3. *Let c and d be positive integers and let $\ell = s_{even}(c) - s_{odd}(c)$. Then*

$$\sum_{\substack{c \cdot 4^d \le i < (c+1) \cdot 4^d \\ i \equiv 0 \bmod 3}} (-1)^{s(i)} = \begin{cases} 2 \cdot 3^{d-1} & \text{if } \ell \equiv 0 \pmod 6 \\ 1 \cdot 3^{d-1} & \text{if } \ell \equiv \pm 1 \pmod 6 \\ -1 \cdot 3^{d-1} & \text{if } \ell \equiv \pm 2 \pmod 6 \\ -2 \cdot 3^{d-1} & \text{if } \ell \equiv 3 \pmod 6. \end{cases}$$

Proof. This is effectively the same as the proof of Lemma 2, with $P(x)$ replaced with

$$Q(x) = x^\ell \left((1+x)(1+\tfrac{1}{x})\right)^d.$$

This gives

$$\sum_{\substack{c \cdot 4^d \le i < (c+1) \cdot 4^d \\ i \equiv 0 \bmod 3}} (-1)^{s(i)} = (\zeta^\ell + \zeta^{-\ell}) \cdot 3^{d-1},$$

implying the cases described above. □

There are many ways to finish from here; one possible way is to write $3n+1$ in the form

$$4^{d_1} + 4^{d_2} + 4^{d_3} + \dots$$

where $d_1 \geq d_2 \geq d_3 \geq \cdots \geq 0$ and no four consecutive d_i's are equal. The goal is to apply Lemma 3 to the intervals

$$[0, 4^{d_1}), [4^{d_1}, 4^{d_1} + 4^{d_2}), [4^{d_1} + 4^{d_2}, 4^{d_1} + 4^{d_2} + 4^{d_3}), \dots$$

and show the result is greater than 1.

To do this, let $D = d_1$ be the maximum exponent. The sum over intervals with length 4^D is at least

$$\begin{cases} 2 \cdot 3^{D-1} & d_1 \neq d_2 \\ 3 \cdot 3^{D-1} & d_1 = d_2 \neq d_3 \\ 4 \cdot 3^{D-1} & d_1 = d_2 = d_3 \neq d_4 \end{cases}$$

by applying Lemma 3 for $\ell \in \{0, 1, 2\}$. The sum over intervals I with length 4^{D-1} is at least

$$(-1 \cdot 3^{D-2}) + 0 + 0$$

by applying Lemma 3 to $\ell \in \{4, 5, \dots, 15\}$ and observing that:

- the third case in Lemma 3 only occurs when $\ell \in \{101_2, 1010_2\} = \{5, 10\}$, and
- the fourth case cannot occur since the smallest ℓ satisfying the fourth case is $\ell = 10101_2 = 21$.

Lastly, the sum over intervals I with length 4^{D-d} for $d \geq 2$ is at least

$$(-2 \cdot 3^{D-d-1}) + (-2 \cdot 3^{D-d-1}) + (-2 \cdot 3^{D-d-1}) = -6 \cdot 3^{D-d-1}.$$

Summing over all exponents gives

$$\begin{aligned} \sum_{i=0}^{n} (-1)^{s(3i)} &\geq 2 \cdot 3^{D-1} - 1 \cdot 3^{D-2} - 6 \cdot (3^{D-3} + 3^{D-4} + \cdots + 3^1 + 3^0) - 3 \cdot \tfrac{1}{3} \\ &> (2 - \tfrac{1}{3} - \tfrac{6}{9}(1 + \tfrac{1}{3} + \tfrac{1}{3^2} + \cdots)) \cdot 3^{D-1} = 2 \cdot 3^{D-2} - 1, \end{aligned}$$

where the $3 \cdot \frac{1}{3}$ accounts for the failure of Lemma 3 when $d = 0$. When $D \geq 2$, this gives

$$\sum_{i=1}^{n} (-1)^{s(3i)} > 2 \cdot 3^{D-2} - 1 > 0.$$

When $D \leq 1$, this amounts to checking $n \leq \frac{1}{3} \cdot 4^2 < 6$, which is easy since $s(3)$, $s(6)$, $s(9)$, $s(12)$, and$s(15)$ are all even.

Solution C

For $r \in \{0, 1, 2\}$, define

$$S_r(a, b) = \sum_{\substack{i \in [a,b) \\ i \equiv r \bmod 3}} (-1)^{s(i)},$$

where $s(i)$ denotes the number of 1s in i's binary representation. The problem is equivalent to proving $S_0(0, 3n + 1) > 1$ for all $n \geq 1$.

Lemma 4. *For any every integer d, $S_r(0, 2^d)$ is given by:*

d	$S_0(0, 2^d)$	$S_1(0, 2^d)$	$S_2(0, 2^d)$
d odd	$3^{\frac{d-1}{2}}$	$-3^{\frac{d-1}{2}}$	0
d even	$2 \cdot 3^{\frac{d-2}{2}}$	$-3^{\frac{d-2}{2}}$	$-3^{\frac{d-2}{2}}$

Proof. Induction; the base cases $d \in \{1, 2\}$ are easy to check. For the inductive step, observe that

$$\begin{aligned} S_0(0, 2^d) &= S_0(0, 2^{d-1}) + S_0(2^{d-1}, 2^d) = S_0(0, 2^{d-1}) - S_{2^{d-1}}(0, 2^{d-1}) \\ S_1(0, 2^d) &= S_1(0, 2^{d-1}) + S_1(2^{d-1}, 2^d) = S_1(0, 2^{d-1}) - S_{1+2^{d-1}}(0, 2^{d-1}) \\ S_2(0, 2^d) &= S_2(0, 2^{d-1}) + S_2(2^{d-1}, 2^d) = S_2(0, 2^{d-1}) - S_{2+2^{d-1}}(0, 2^{d-1}). \end{aligned}$$

Using

$$2^{d-1} \equiv \begin{cases} 1 \bmod 3 & d \text{ odd} \\ 2 \bmod 3 & d \text{ even} \end{cases}$$

and applying the inductive hypothesis gives the desired result. $\square$

This proves the problem for powers of 2. To prove the problem for general n, split $[0, 3n + 1)$ into blocks

$$[0, 3n + 1) = [0, 2^{d_1}) \sqcup [2^{d_1}, 2^{d_1} + 2^{d_2}) \sqcup [2^{d_1} + 2^{d_2}, 2^{d_1} + 2^{d_2} + 2^{d_3}), \ldots$$

whose lengths are decreasing powers of 2. By Lemma 4,

$$S_0(0, 2^{d_1}) \geq \begin{cases} 3^{\frac{d_1-1}{2}} & d_1 \text{ odd} \\ 2 \cdot 3^{\frac{d_1-2}{2}} & d_1 \text{ even} \end{cases}$$
$$S_0(2^{d_1}, 2^{d_1} + 2^{d_2}) = -S_{2^{d_1}}(0, 2^{d_2}) \geq 0$$

and

$$\begin{aligned} S_0(2^{d_1} + \cdots + 2^{d_{i-1}}, 2^{d_1} + \cdots + 2^{d_i}) &= (-1)^{i-1} S_{2^{d_1} + \cdots + 2^{d_{i-1}}}(0, 2^{d_i}) \\ &\geq \begin{cases} -2 \cdot 3^{\frac{d_i-2}{2}} & d_i \text{ even} \\ -1 \cdot 3^{\frac{d_i-1}{2}} & d_i \text{ odd.} \end{cases} \end{aligned}$$

for all i. Let $D = d_1$; assume $D \geq 4$ since checking $D \in \{1, 2, 3\}$ is easy. Summing over all intervals gives

$$S_0(0, 3n+1) \geq 2 \cdot 3^{\frac{D-2}{2}} + 0 - \left(2 \cdot 3^{\frac{D-4}{2}} + 3^{\frac{D-4}{2}} + 2 \cdot 3^{\frac{D-6}{2}} + 3^{\frac{D-6}{2}} + \cdots\right) = 3^{\frac{D-2}{2}} > 1$$

when D is even, and

$$S_0(0, 3n+1) \geq 3^{\frac{D-1}{2}} + 0 - \left(3^{\frac{D-3}{2}} + 2 \cdot 3^{\frac{D-5}{2}} + 3^{\frac{D-5}{2}} + 2 \cdot 3^{\frac{D-7}{2}} + \cdots\right) = \frac{1}{2} \cdot 3^{\frac{D-3}{2}} > 1$$

when D is odd.

§ Variants

Variant A. Find all positive integers k such that for every positive integer n, among the first n multiples of k there are more numbers with an even numbers of 1s in binary than numbers with an odd number of 1s in binary.

Variant B. Given a set of positive integers S, define

$$\begin{aligned} \operatorname{even}(S) &= \#\{s \in S \mid s \text{ has an even number of 1s in binary}\} \\ \operatorname{odd}(S) &= \#\{s \in S \mid s \text{ has an odd number of 1s in binary}\}. \end{aligned}$$

Show that

$$\lim_{n\to\infty} \frac{\operatorname{odd}(\{3, 6, \ldots, 3n\})}{\operatorname{even}(\{3, 6, \ldots, 3n\})} = 1.$$

§ Metadata

This problem was selected as Problem 5 of the 2024 TSTST.

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- Author: Holden Mui
- Subject: number theory
- Description: parity of number of ones in multiples of three in binary
- Keywords: binary, multiples, parity
- Difficulty: IMO 2/5
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